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Stability and instability for periodic solutions of delay equations with "steplike" feedback

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Abstract

We consider the stability of periodic solutions of certain delay equations $x'(t) = f(x_t)$ for which the special structure of the equation allows us to define return maps that are semiconjugate to finite-dimensional maps. We present some general results on assessing stability with the aid of such a semiconjugacy. We then apply our general results to exhibit a stable periodic solution of an equation with two fixed delays and an unstable periodic solution of an equation with state-dependent delay.

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1 Introduction

Given $r > 0$, let $C = C[-r, 0]$ be the set of continuous functions from $[-r, 0]$ to $\mathbb{R}$, equipped with the sup norm. If x is a continuous function whose domain includes the interval $[t-r, t]$, in the usual way we write x_t for the member of C defined by $x_t(s) = x(t+s)$. Throughout, if G is a differentiable function, we shall write $DG[x]$ for the derivative of G at the point x. We shall write $\mathbb{N}$ for the set of natural numbers and $\mathbb{Z}_+$ for the set of nonnegative integers.

Let $\Omega \subset C$ (Ω is endowed with some topology — not necessarily the sup norm), and suppose that $f : \Omega \to \mathbb{R}$ is some function. In this paper we consider autonomous real-valued retarded functional differential equations of the form

$$x'(t) = f(x_t). \tag{1}$$

Various instances of this equation have been intensively studied. By a *solution* of (1) we mean a continuous function $x : [-r, \nu(x_0)) \to \mathbb{R}$ such that $x_t \in \Omega$ for all $t \in [0, \nu(x_0))$ and $x'(t) = f(x_t)$ for all $t \in (0, \nu(x_0))$, where $\nu(x_0) \in (0, \infty]$ is maximal. We regard Ω as the state space and x_0 as the initial condition of x; we call x the *continuation of* x_0 *as a solution of* (1). Any $x_t \in \Omega \subset C$, $t \in [0, \nu(x_0))$, is called a *segment* of x. We also regard as solutions differentiable functions $x : \mathbb{R} \to \mathbb{R}$ with $x_t \in \Omega$ and $x'(t) = f(x_t)$ for all $t \in \mathbb{R}$. Throughout, we shall confine ourselves to situations where a unique solution semiflow

$$F : \cup_{x_0 \in \Omega}[0, \nu(x_0)) \times \{x_0\} \to \Omega$$

is defined.

In this paper we focus on stability of periodic solutions. In particular, our focus is on the local dynamics of return maps (analogs of Poincaré maps). Let $X \subset \Omega$ be some subset, endowed with the subspace topology inherited from Ω. Suppose that there is some relatively open subset $U \subset X$ and a continuous map $\tau : U \to (\hat{\tau}, \infty)$, $\hat{\tau} > 0$, such that for all $x_0 \in U$ we have $F(\tau(x_0), x_0) \in X$. Define the map $R : U \to X$ by $R(x_0) = F(\tau(x_0), x_0)$; R is called a *return map.* Fixed points $p_0 \in U$ of R that are not segments of equilibria are segments of nontrivial periodic solutions p of (1), with period dividing $\tau(p_0)$. Indeed, finding nontrivial fixed points of maps like R is the most prominent technique for proving the existence of periodic solutions of nonlinear equations of the form (1).

The dynamics of R on U often allow us to make assertions about the stability of periodic solutions p. In particular, it often happens that there is some open subset $\mathcal{O}$ in Ω about p_0 all of whose points have continuations that eventually flow into $U \subset X$; in this case, the stability or instability of p_0 as a fixed point of R in U tells us whether solutions of (1) with initial conditions near p_0 in Ω converge to, remain close to, or diverge from p for large time.

Much work has been done, by various authors and for various instances of (1), on the local dynamics of return maps R about fixed points p_0. For example, in [20], [21], and [22] particular periodic solutions are proven to be stable and locally unique by showing that appropriately defined return maps R are contractive. In the case that $f : C \to \mathbb{R}$ is continuously differentiable, many authors (see, for example, [17] and the references therein) have studied the so-called *Floquet multpliers* of p (that is, the spectrum of the *monodromy operator* $D_2F[(\tau(p_0), p_0)]$); this is essentially the same as studying the spectrum of $DR[p_0]$ (see, for example, [24] for a description of the connection). In [23], [24], and [25], a priori estimates on solutions of linear variational equations about certain periodic solutions p lead to bounds on the spectral radius of $DR[p_0]$ directly, thereby yielding stability of p.

Another approach to establishing stability of periodic solutions is to identify problems where the feedback function is simple enough that, on some subset (often intricate) of the phase space, a return map can be defined that is semiconjugate to a much simpler (often finite-dimensional) map. This idea has, by now, a considerable history as applied to equations of the form

$$x'(t) = \mu x(t) + g(x(t-1)) \tag{2}$$

where g is what we shall call "steplike" — that is, constant except on a finite number of small intervals. Under additional assumptions, chaotic behavior of appropriately defined "slowly oscillating" solutions has been proven in [19], [3], and the pair of papers [14], [15]. Stable "rapidly oscillating" periodic solutions are proven to exist in [7] and [18]. In each of the works just cited, the authors consider functions g that are similar to step functions with three discontinuities.

(A distinct family of results on stability of periodic solutions, not focused directly on the dynamics of a return map R, are those using so-called “phase plane” methods. The pioneering work in this direction [8] has been extended by many authors.)

In the current work we take the semiconjugacy approach described above: we consider equations with “steplike” feedback and exploit semiconjugacies between return maps R and simpler maps. Our main goal is to use these semiconjugacies to obtain explicit information about the spectrum of $DR[p_0]$. We present a general framework that seems applicable to a variety of delay equations — where delays are single or multiple, constant or state-dependent — and that captures the essential ideas at play in many of the above-cited works. We include conditions under which, for the “steplike” problems we have in mind, the spectrum of $DR[p_0]$ can be related to the spectrum of a finite-dimensional linear map, and sometimes explicitly computed.

Even for equations with a single constant delay, stability of periodic solutions can be difficult to assess: Floquet multipliers are hard to estimate in general, and the other approaches we have outlined have restricted applicability. In the state-dependent case, even less is known. The methods we present here, to be sure, likewise apply only to very special equations; the novelty of our results lies in the detailed spectral information obtained, and in the application of the general idea to a state-dependent equation. Similarly specific stability results for a particular class of equations, using techniques similar to ours, have recently been obtained by Krisztin and Vas [11].

In Section 2 we present the general framework: hypotheses that can be shown to hold in a variety of different settings when feedback functions are "steplike," and corresponding results relating the dynamics of R near p_0 to the dynamics of a semiconjugating map ρ. There are two main results in the section (Propositions 2.7 and 2.12); for the first we do not assume that the maps are differentiable; for the second we do. In Sections 3 and 4 we apply the general framework to two very specific illustrative examples: we exhibit a stable periodic solution for an equation with two fixed delays in Section 3, and an unstable periodic solution for an equation with state-dependent delay in Section 4.

2 The general framework

2.1 Continuous case

Suppose that X is a metric space, that $U \subset X$ is a relatively open subset, and that $R : U \to X$ is a map. The point of this section is to identify circumstances under which the stability of fixed points of R can be studied via an appropriate semiconjugacy, and which apply to several instances of (1) with steplike feedback.

We fix the following hypotheses for this section.

(I) $R : U \to X$ is continuous.

(II) There is a subset Y of X such that $R(U) \subset Y$. In what follows, we shall always endow Y with the subspace topology inherited from X.

(III) There is a metric space V and a continuous and open map $Z : Y \to V$ such that, for any two $x, y \in U \cap Y$, $Z(x) = Z(y) \implies R(x) = R(y)$.

We write $W = Z(U \cap Y)$. W is open in V.

Lemma 2.1. *There is a continuous map $\rho : W \to V$ such that, for all $x \in U \cap Y$,*

$$\rho Z(x) = ZR(x). \tag{3}$$

PROOF. Let $v \in W$ be given. R is constant on $U \cap Z^{-1}(v)$ by (III), and so given $x \in U \cap Z^{-1}(v)$ we define

$$\rho(v) = ZR(x).$$

Given any $x \in U \cap Y$, the equality $ZR(x) = \rho Z(x)$ obviously holds.

It remains to show that ρ is continuous. Choose $B \subset V$ open. Since Z and R are continuous and Z is open, $A = Z(R^{-1}(Z^{-1}(B)))$ is open. We claim that $A = \rho^{-1}(B)$. Choose $a \in A$. Then $a = Z(x)$ for some $x \in Y \cap R^{-1}(Z^{-1}(b))$ and some $b \in B$, and so $\rho(a) = Z(R(x)) = b \in B$. Thus $A \subset \rho^{-1}(B)$. On the other hand, if $a \in \rho^{-1}(B)$, then any element $x \in U \cap Z^{-1}(a)$ satisfies $Z(R(x)) \in B$, whence $a = Z(x) \in Z(R^{-1}(Z^{-1}(B))) = A$. This proves the claim, and ρ is continuous. □

Remark 2.2. Note that condition (III) is stronger than the conclusion of the above lemma — if we only assumed the existence of a continuous map ρ satisfying (3), we would only be able to assert that R preserved fibers of Z, not that R was constant on fibers of Z. We will comment more on the necessity of this strong condition below.

Lemma 2.3. *Let $x \in U \cap Y$ and let $n \in \mathbb{N}$. If $R^k(x) \in U \cap Y$ for all $k \in \{0, 1, \ldots, n-1\}$, then $\rho^k(Z(x)) \in W$ for all $k \in \{0, 1, \ldots, n-1\}$ also, and in this case*

$$\rho^n(Z(x)) = Z(R^n(x)).$$

PROOF. We proceed by induction. The $n = 1$ case clearly holds. Assume the lemma holds for all natural numbers $n \leq m$, and suppose that $R^k(x) \in U \cap Y$ for all $k \in \{0, 1, \ldots, m\}$. Then clearly $Z(R^k(x)) \in W$ for all such k, and by our inductive hypothesis $Z(R^k(x)) = \rho^k(Z(x))$ for all such k. Thus $\rho^k(Z(x)) \in W$ for all $k \in \{0, 1, \ldots, m\}$ as well.

Now, using (3) and our inductive hypothesis we have

$$Z(R^{m+1}(x)) = \rho(Z(R^m(x))) = \rho(\rho^m(Z(x))) = \rho^{m+1}(Z(x)),$$

as desired. □

The following lemma is an immediate consequence of (III), and uses its full strength (recall Remark 2.2).

Lemma 2.4. *For any subset A of $U \cap Y$, we have $R(U \cap Z^{-1}(Z(A))) = R(A)$.*

PROOF. Given any $x \in U \cap Z^{-1}(Z(A))$, $Z(x) = Z(a)$ for some $a \in A$; thus $R(x) = R(a)$ and $R(U \cap Z^{-1}(Z(A))) \subset R(A)$. On the other hand, $A \subset U \cap Z^{-1}(Z(A))$, and so $R(A) \subset R(U \cap Z^{-1}(Z(A)))$. $\square$

Lemma 2.5. *If $p \in U$ is a fixed point of R, then $\pi := Z(p) \in W$ is a fixed point of ρ.*

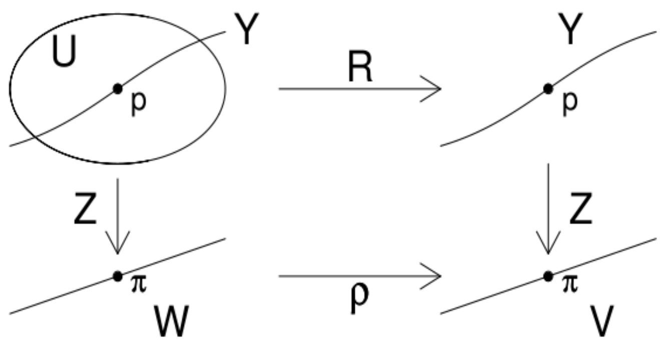
PROOF. Suppose that $p \in U$ is a fixed point of R. Then by (II) we must have $p \in U \cap Y$. Write $\pi = Z(p)$. Then π is a fixed point of ρ:

$$\rho(\pi) = \rho(Z(p)) = Z(R(p)) = Z(p) = \pi. \quad \square$$

Henceforth, we shall assume

(IV) R has a fixed point $p \in U$, and $\pi := Z(p) \in W$ is a fixed point of ρ.

The figure below illustrates the situation.



For the applications we have in mind, X will be a fairly easily-described subspace of the phase space (say, the space of initial conditions that are equal to zero at time 0). p will be a segment of a nontrivial periodic solution. U will typically be a small open subset around the point p, and R will be a return map that advances solutions "far enough" for (III) to hold. (In particular, if p is a segment of a "rapidly oscillating" periodic solution, R might be defined as a power of a more natural-seeming return map Q.) In the examples we consider below, it will be both easy and useful to define R on the closure $\overline{U}$ of U in X.

The set Y will be some highly restricted subset into which R must map. And here is where the "steplike" nature of the feedback function for (1) enters the picture: for the applications we have in mind, the map $Z : Y \to V$ will be finite-dimensional, and $Z(x)$ — together with the fact that $x \in Y$ — will represent all the information required to continue x uniquely as a solution of (1). (We often find the sets U and Y by examining highly simplified "limiting" problems with discontinuous feedback.)

In practice, hypotheses (II) and (III) will typically require laborious verification. On the other hand, ρ — and its derivative — will often be (relatively) practical to compute, and will allow us to make fairly precise statements about the spectrum of $DR[p]$; we discuss this below. The first main result of this section, however (Proposition 2.7 below), is more elementary and follows from hypotheses (I) – (IV), without any assumptions about differentiability. Such a result is useful, for example, if the differentiability of R cannot be established, or if p is stable but not exponentially stable.

We begin by recalling the following standard definitions.

Definition 2.6. With notation as above,

- p is *stable (with respect to R)* if, given any open subset $A \subset U$ about p, there is a some open $\tilde{A} \subset U$ such that $x \in \tilde{A}$ implies that $R^k(x) \in A$ for all $k \in \mathbb{Z}_+$.

- p is *asymptotically stable (with respect to R)* if p is stable with respect to R and, furthermore, there is some open neighborhood $B \subset U$ of p with the feature that $x \in B$ implies $R^k(x) \to p$ as $k \to \infty$.

- p is *unstable (with respect to R)* if it is not stable with respect to R.

Proposition 2.7. *Suppose that (I) – (IV) hold. Then*

- *π is stable with respect to ρ if and only if p is stable with respect to R;*
- *π is asymptotically stable with respect to ρ if and only if p is asymptotically stable with respect to R.*

Remark 2.8. The main point of difficulty in the proof below is that, if O is a "small" open set about π, $U \cap Z^{-1}(O)$ is not necessarily a "small" open set about p. It is essentially to overcome this difficulty that we need the relatively strong version of (III) that we have given above (more specifically, Lemma 2.4) rather than just assuming the existence of continuous map ρ satisfying the semiconjugacy property (3). Indeed, Proposition 2.7 does not hold if (III) is replaced by the weaker hypothesis that such a ρ exists; an example is furnished by taking V to be a one-point set.

PROOF. Suppose that π is stable with respect to ρ. Choose an open set $A \subset U$ about p. Since R is continuous, there is an open set $B \subset A$ about p such that $R(B) \subset A$. By (II) we actually have that $R(B) \subset A \cap Y$. $Z(B \cap Y)$ is open in W. Now choose an open subset O about π in W such that $\rho^k(O) \subset Z(B \cap Y)$ for all $k \in \mathbb{Z}_+$. Write $Z^{-1}(O) \cap B = \tilde{B} \cap Y$ for some open $\tilde{B} \subset B$.

Given $x \in \tilde{B} \cap Y$, we claim that $R^n(x) \in A \cap Y$ for all $n \in \mathbb{N}$. The $n = 1$ case is clear since $R(\tilde{B}) \subset R(B) \subset A \cap Y$. We now proceed by induction. Suppose that $R^k(x) \in A \cap Y$ for all $k \in \{1, \ldots, n\}$. By our assumption that π

is stable, $\rho^k(Z(x)) \subset \rho^k(O) \subset Z(B \cap Y)$ for all such k. By the second part of Lemma 2.3, we have

$$Z(R^n(x)) = \rho^n(Z(x)) \in Z(B \cap Y) \implies$$
$$R^n(x) \in A \cap Z^{-1}(Z(B \cap Y)) \subset U \cap Z^{-1}(Z(B \cap Y)).$$

By Lemma 2.4, though, $R(U \cap Z^{-1}(Z(B \cap Y))) = R(B \cap Y) \subset A \cap Y$, and so $R^{n+1}(x) \in A \cap Y$. This proves the claim.

Again invoking the continuity of R, we choose an open $\tilde{A} \subset \tilde{B}$ such that $R(\tilde{A}) \subset \tilde{B} \cap Y$. Applying the claim of last paragraph, we see that $R^n(\tilde{A}) \subset A$ for all $n \in \mathbb{Z}_+$. We have shown that p is stable if π is.

Conversely, suppose that π is not stable with respect to ρ. This means that there is some open set $O \subset W$ about π such that, for any open $\tilde{O} \subset W$, there is some $v \in \tilde{O}$ and some $m \in \mathbb{N}$ such that $\rho^m(v) \notin O$. Now $Z^{-1}(O)$ is an open set with respect to Y and so $Z^{-1}(O) \cap U = A \cap Y$ for some open subset $A \subset U$. Now choose any open subset $\tilde{A} \subset A$ and write $\tilde{O} = Z(\tilde{A} \cap Y)$; $\tilde{O}$ is an open subset of O. Choose some $v \in \tilde{O}$ and some $m \in \mathbb{N}$ such that $\rho^m(v) \notin O$. Now let $x \in Z^{-1}(v) \cap \tilde{A}$. Imagine that $R^k(x) \in A \cap Y$ for every $k \in \{1, \ldots, m\}$. Then by Lemma 2.3 we have that $Z(R^m(x)) = \rho^m(v) \notin O$, and so $R^m(x) \notin Z^{-1}(O) \cap U = A \cap Y$ — a contradiction. Thus p is unstable if π is.

Assume now that π is asymptotically stable. Let $O \subset W$ be an open set about π such that $v \in O$ implies that $\rho^k(v) \in W$ for all $k \in \mathbb{N}$ and $\rho^k(v) \to \pi$

as $k \to \infty$. Write $Z^{-1}(O) = A \cap Y$, and choose $x \in U \cap A \cap Y$. Since we have already shown that p is stable if π is, we may assume (shrinking O and A if needed) that $R^k(x) \in U \cap Y$ for all $k \in \mathbb{N}$. We write $v = Z(x)$. Then by Lemma 2.3 we have $Z(R^k(x)) = \rho^k(v)$ for all $k \in \mathbb{N}$. Let an open set B about p be given, and (using the continuity of R) choose an open set D about p with $R(D) \subset B$. Since Z is open, $Z(D \cap Y)$ is open in W. Thus there is some $k_0 \in \mathbb{N}$ such that $\rho^k(v) \subset Z(D \cap Y)$ for all $k > k_0$, and so $R^k(x) \in U \cap Z^{-1}(Z(D \cap Y))$ for all such k. Thus $R^{k+1}(x) \in R(U \cap Z^{-1}(Z(D \cap Y))) = R(D \cap Y) \subset B \cap Y$ for all such k. Thus p is asymptotically stable too. We omit the proof of the similar converse. $\square$

Remark 2.9. As already mentioned, for the applications we have in mind p will be a segment of a periodic solution and R will be a return map. X will typically have no interior in the phase space Ω. Therefore, to use the results in this section to draw conclusions about the stability of the periodic solution, it must be further verified that solutions beginning in some neighborhood $\mathcal{O}$ of p in Ω eventually flow into U, whence the dynamics of R capture the behavior of the solution. This additional requirement will be easy to verify for the examples we consider here.

2.2 Differentiable case

We can enhance the detail of the results above if we impose some additional hypotheses on differentiability. These are as follows.

(D1) V is a Banach space. X and Y are subsets of a Banach space $\mathcal{B}$ that, sufficiently close to p, have the structure of Banach manifolds. Write $T_p(X)$ and $T_p(Y)$ for the tangent spaces at p of X and Y, respectively.

(D2) R, Z, and ρ are continuously differentiable. R and ρ are completely continuous, and R is Lipschitz with Lipschitz constant M.

(D3) $DZ[p] : T_p(Y) \to V$ is surjective.

(D4) $\ker DZ[p] \subset \ker DR[p]$.

The following lemma is an immediate consequence of (I)–(IV), (D1)–(D2), and standard results on differentiation.

Lemma 2.10. *Assume (I)–(IV), (D1)–(D2). Then*

i) $DR[p](T_p(X)) \subset T_p(Y) \subset T_p(X)$, *and* $\|DR[p]\| \leq M$*;*

ii) $DZ[p]DR[p]u = D\rho[\pi]DZ[p]u$ *for all* $u \in T_p(Y)$*;*

iii) $DR[p]$ *and* $D\rho[\pi]$ *are compact continuous linear operators.*

Hypotheses (D3) and (D4), though intended to be natural in light of (III), do not hold automatically given (I)–(IV) and (D1)–(D2). (Consider the simple example obtained by taking $X = Y = V = \mathbb{R}$, $p = 0$, $R(x) = x$, and $Z(x) = x^3$.) We will discuss the interplay of these hypotheses in more detail below.

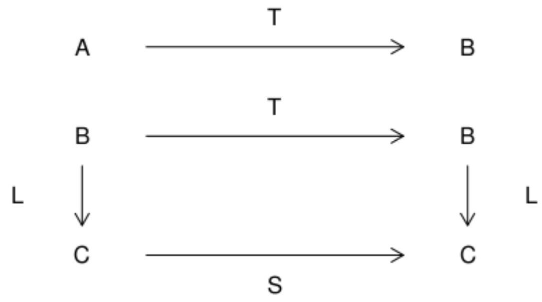
The linear algebra lemma below paves the way for the second main proposition of the current section (Proposition 2.12 below). Though the lemma is stated generally, A, B, and C should be thought of as complexifications of $T_p(X)$, $T_p(Y)$, and V respectively; T, S, and L should be thought of as $DR[p]$, $D\rho[\pi]$, and $DZ[p]$, respectively, appropriately extended over A, B, and C.

Lemma 2.11. *Suppose that A and C are Banach spaces and that B is a linear subspace of A. Suppose that $T : A \to B$, $S : C \to C$, and $L : B \to C$ are continuous linear maps with S and T compact, and that*

$$SLy = LTy \quad \textit{for all} \quad y \in B.$$

Suppose also that L is surjective and that $\ker L \subset \ker T$. Then $\sigma(T) \setminus \{0\} = \sigma(S) \setminus \{0\}$.

Refer to the figure below.



PROOF. Since S and T are compact, the nonzero spectra of T and S consist of eigenvalues.

Suppose that $\lambda \in \sigma(T)$ with $\lambda \neq 0$. This means that there is some nonzero $y \in A$ such that $Ty = \lambda y$. Since $T(y) = \lambda y \in B$, we in fact have $y \in B$. Therefore

$$SLy = LTy = L\lambda y = \lambda Ly.$$

$Ly \neq 0$ since $\ker L$ is contained in $\ker T$. Thus λ is an eigenvalue of S with eigenvector Ly.

On the other hand, suppose that $Sv = \lambda v$, $v \neq 0$, $\lambda \neq 0$. Since L is surjective, there is some $y \in B$ such that $Ly = v$. Thus we have

$$L\lambda y = \lambda Ly = \lambda v = Sv = SLy = LTy.$$

This means that $Ty - \lambda y \in \ker L \subset \ker T$, and so there is some $u \in \ker T$ such that $Ty = \lambda y + u$. Write $\tilde{y} = y + u/\lambda$ and compute:

$$T(\tilde{y}) = Ty = \lambda y + u = \lambda \tilde{y}.$$

This completes the proof. □

The application of the above lemma to our particular situation yields the following.

Proposition 2.12. *If (I)–(IV) and (D1)–(D4) hold, then the nonzero spectrum of $DR[p]$ is equal to the nonzero spectrum of $D\rho[\pi]$.* □

In Sections 3 and 4 we shall find the following lemma useful, which says, roughly speaking, that if X and Y are locally affine and V is finite-dimensional, then (D3) imlies (D4). We suspect that a similar result holds more generally; the version below (which avoids any differential geometry apparatus) is sufficient for our needs. Recall that we are writing $\mathcal{B}$ for a Banach space containing X and Y.

Lemma 2.13. *Suppose that (I)–(IV) and (D1)–(D2) hold, and further assume that there is some open set $\mathcal{U} \subset \mathcal{B}$ about p such that*

$$X \cap \mathcal{U} = (p + A) \cap \mathcal{U} \quad \textit{and} \quad Y \cap \mathcal{U} = (p + B) \cap \mathcal{U},$$

where A and B are closed linear subspaces of $\mathcal{B}$. Suppose moreover that V is finite-dimensional. Then (D3) implies (D4).

PROOF. Note that, under the above hypotheses, $T_p(X) = A$ and $T_p(Y) = B$.

For simplicity we write $L = DZ[p]$.

Claim: There is a neighborhood $N \subset U \cap Y$ about p in Y and a constant $\gamma := \gamma(N)$ such that, for all $x \in N$, there is some $y \in U \cap Y$ such that $Z(x) = Z(y)$ and

$$\|y - p\| \leq \gamma |Z(y) - \pi| = \gamma |Z(x) - \pi|.$$

Proof of Lemma, given claim: suppose that $v \in B$ belongs to $\ker L$. This means that, given any $\epsilon > 0$, $|Z(p + hv) - \pi| \leq \epsilon \|hv\|$ for all scalars h with $|h|$ sufficiently small. Taking $|h|$ smaller if necessary, we may assume that $p + hv \in N$; by our claim there is some $y \in U \cap Y$ such that $Z(y) = Z(p + hv)$ and $\|y - p\| \leq \gamma |Z(p + hv) - \pi| \leq \gamma\epsilon \|hv\|$. Now, using (III) and the fact that R has Lipshitz constant M we see that, for all $|h|$ sufficiently small,

$$\|R(p + hv) - p\| = \|R(y) - p\| \leq M \|y - p\| \leq M\gamma\epsilon \|hv\|.$$

Since ϵ was arbitrary, it follows that $v \in \ker DR[p]$ also.

Proof of Claim: $L : B \to V$ is a finite-dimensional continuous linear map, and so $\ker L$ is closed and has finite codimension. Since closed finite-codimensional subspaces are complemented, B splits into two closed complementary subspaces: $B = \ker L \oplus E$. The restriction of L to E is invertible.

Now consider the map

$$H : E \times E \to V \ \text{ given by } \ H(v, u) = \pi + L(v) - Z(p + u).$$

$H(0, 0) = 0$, H is C^1, and $DH_2[0, 0] = L|_E$ is an isomorphism, so by the implicit function theorem there is an open neighborhood $\mathcal{O}$ about 0 in E and a C^1 function $\alpha : \overline{\mathcal{O}} \to E$ such that $\alpha(0) = 0$ and, for each v in $\overline{\mathcal{O}}$,

$$Z(p + \alpha(v)) = \pi + Lv.$$

Now, $\pi + L(\mathcal{O})$ is an open neighborhood of π in V. Shrinking $\mathcal{O}$ if necessary, we may assume that $\pi + L(\mathcal{O}) \subset W$. Let N be some neighborhood of p (open relative to $U \cap Y$) contained in $Z^{-1}(\pi + L(\mathcal{O}))$. Given $x \in N$, since L is surjective $Z(x) = \pi + Lv = Z(p + \alpha(v))$ for some $v \in \mathcal{O} \subset E$. Since L is invertible on E, there is some κ such that $\|v\| \leq \kappa |Lv| = \kappa |Z(x) - \pi|$. Thus

$$\|\alpha(v)\| \leq \ell_\alpha \|v\| \leq \ell_\alpha \kappa |Z(x) - \pi|,$$

where ℓ_α is the Lipschitz constant of α on $\overline{\mathcal{O}}$. Writing $y = p + \alpha(v)$ and $\gamma = \ell_\alpha \kappa$ proves the claim. □

As we have already stated, for the applications we have in mind the feedback function in (1) will be smooth, but extremely restricted. Accordingly, once we have determined the stability of a periodic solution for such a restricted equation, we would ideally like to perturb the equation to less restrictive forms while preserving the existence and stability of the periodic solution.

In certain cases where the feedback function f in (1) is smooth on $C[-r, 0]$ and smoothly parameterized in an appropriate sense, in [12] it is proven that $DR[p]$ varies continuously under perturbations in f, and so such preservation is possible. (Indeed, in [12] this result is applied to perturbations of "steplike" feedback functions to less restrictive ones.) It sometimes happens that knowledge of $D\rho[\pi]$ can provide explicit bounds on operator-norm perturbations of $DR[p]$ that preserve spectral properties, and so can help generate bounds on perturbations of (1); we hope to address this point in future work. Since estimating how perturbations of f in (1) change $DR[p]$ seems difficult in general, however, these bounds may ultimately be of limited practical applicability in continuation arguments.

3 Example — an equation with two fixed delays

As in Section 1, we write $C = C[-r, 0]$.

Let us first consider the equation

$$y'(t) = \sum_{i=1}^{D} h_i(y(t - d_i)), \tag{4}$$

where the $h_i(y)$ are step functions. We assume $0 < d_1 < \cdots < d_D = r$.

We write $\mathcal{K}_i$ for the points of discontinuity of h_i — that is, h_i is constant on any connected component of the complement of $\mathcal{K}_i$. We assume that each set $\mathcal{K}_i$ is finite, and write $\mathcal{K} = \cup_{i=1}^D \mathcal{K}_i$.

By a *solution* of (4) we mean either a continuous function $y : [-r, \infty) \to \mathbb{R}$ that satisfies the integral equation

$$y(t) = y(0) + \int_0^t \sum_{i=1}^D h_i(y(s - d_i))\, ds$$

for all $t \geq 0$ or a continuous function $y : \mathbb{R} \to \mathbb{R}$ that satisfies the above integral equation everywhere. We state the following proposition, which is an easy extension of the $\mathcal{K} = \{0\}$ case discussed in Proposition 2.3 of [10]:

Proposition 3.1. *Suppose that $\phi \in C$. Then ϕ has a unique continuation $x : [-r, \infty) \to \mathbb{R}$ as a solution of (4). x is differentiable for almost all $t > 0$, and $x'(t)$ satisfies (4) as written wherever $x'(t)$ exists.*

The solution semiflow $G : \mathbb{R}_+ \times C \to C$ for (4) is *not* continuous.

It is often possible to find periodic solutions of (4) explicitly (especially if D and the cardinality of $\mathcal{K}$ are small). In some cases — for example, the model equation $x'(t) = -\mathrm{sign}(x(t-1))$ — the global dynamics are very well understood (see [5], [1], [13], and Section XVI.2 of [4]).

The primary motivation for studying equations like (4) is, of course, to shed light on equations of the form

$$x'(t) = \sum_{i=1}^D f_i(x(t - d_i)), \tag{5}$$

where the $f_i : \mathbb{R} \to \mathbb{R}$ are bounded C^1 functions that are, in some sense, "similar" to the step functions h_i. By standard theory, equation (5) defines a unique continuous solution semiflow $F : \mathbb{R}_+ \times C \to C$ that is continuously differentiable on $(r, \infty) \times C$ and such that $F(\tau, \cdot)$ is completely continuous for all $\tau \geq r$.

We now define the class of equations that we wish to consider. We use the notation of (4) and (5).

Definition 3.2. Let $\eta \geq 0$. The C^1 feedback function f_i is *η-steplike (with respect to h_i)* if $f_i(x) = h_i(x)$ for all x not in the set

$$\cup_{c \in \mathcal{K}_i} (c - \eta, c + \eta).$$

We say that equation (5) is *η-steplike with respect to* (4) if f_i is η-steplike with respect to h_i for all $i \in \{1, \ldots, D\}$.

We now define the periodic solutions of (4) that we use to guide our investigations of (5).

Definition 3.3. Suppose that $q : \mathbb{R} \to \mathbb{R}$ is a periodic solution of (4). We say that q is *simple* if $q(t) \in \mathcal{K}$ implies that $q'(t) \neq 0$ and $q(t - d_i) \notin \mathcal{K}_i$ for all $i \in \{1, \ldots, D\}$.

Informally, a periodic solution $q(t)$ is simple if it always crosses points of $\mathcal{K}$ (discontinuities of the step functions) transversally with locally constant slope. The usefulness of some condition of this kind for transferring results about q to results about periodic solutions of delay equations with the feedback functions "smoothed out" has long been recognized.

The approach we take here to defining our sets X, U, and Y, and the maps R, Z, and ρ, bears strong affinity to the earlier work with "steplike" feedback functions mentioned in Section 1, as well as the recent work of Krisztin and Vas [11]. We also mention [10], where periodic solutions of equations (4) are studied in the special case that $\mathcal{K} = \{0\}$. In that work, it is shown (via a fixed-point index argument) that if q is a simple periodic solution of (4) that satisfies a further technical condition (called *nondegeneracy* in [10]) then (5) has a similar periodic solution provided that the f_i are close enough to the h_i. (In the language of Section 2, the nondegeneracy condition is that the appropriate linear map $D\rho[\pi]$ — a specific instance of which we shall study in this section — does not have eigenvalue 1.) The class of functions f_i considered in [10] is somewhat more general than the "steplike" feedback functions we consider here: in particular, it is only required that $|f_i(x) - h_i(x)| \leq \epsilon$ for $|x| \geq \eta$, where η and ϵ are sufficiently small. On the other hand, only a very weak stability result is obtained in [10]. The basic approach used in [10] is, again, similar to the approach that we use here.

In this section, for the sake of brevity and clarity, we choose a particular simple periodic solution q of a particular equation (4), and obtain a similar periodic solution p for an appropriately related problem with steplike feedback. We then apply the apparatus developed Section 2 to show that p is stable. We refrain in places from using the specifics of the equation to make sharp estimates that might obscure how to generalize the example; our goal is to make the main ideas as transparent as possible.

The step-feedback problem we consider is

$$y'(t) = h_1(y(t-1)) + h_2(y(t-5)), \tag{6}$$

where

$$h_1(y) = \begin{cases} 3, & y < 0; \\ 0, & y = 0; \\ -2, & y > 0; \end{cases} \quad \text{and} \quad h_2(y) = \begin{cases} -1, & y < 0; \\ 0, & y = 0; \\ 1, & y > 0. \end{cases}$$

We shall suppose that g_1 and g_2 are C^1 and bounded, and that the equation

$$x'(t) = g_1(x(t-1)) + g_2(x(t-5)) \tag{7}$$

is η-steplike with respect to (6). We fix the notation

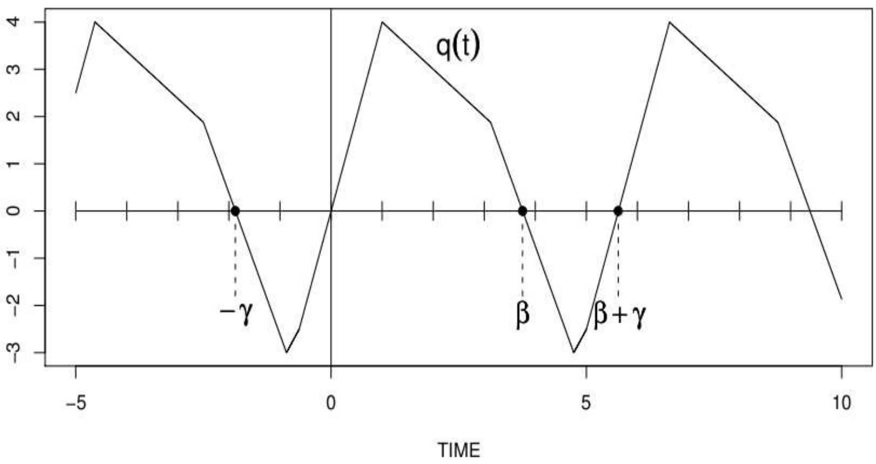
$$\mu \geq |g_1| + |g_2| \geq |h_1| + |h_2| = 4$$

and observe that any solution $x : [-5, \infty) \to \mathbb{R}$ of (7) satisfies $|x'(t)| \leq \mu$ for all $t > 0$. We shall assume throughout that μ is fixed, even as η varies.

Direct computation verifies that (6) has a periodic solution q with two zeros per minimal period, $q(0) = 0$, $q'(0) = 4$, and positive zeros

$$\beta < \beta + \gamma < 2\beta + \gamma < 2\beta + 2\gamma < \cdots,$$

where $\beta = 15/4$ and $\gamma = 15/8$. This solution is illustrated in the figure below.



Remark 3.4. Some detailed information about q on $[0, \beta + \gamma]$ is as follows.

- $q'(t) = 4$ $(q(t-1) < 0$ and $q(t-5) > 0)$ for $t \in [0, 1)$;
- $q'(t) = -1$ $(q(t-1) > 0$ and $q(t-5) > 0)$ for $t \in (1, 25/8)$;
- $q'(t) = -3$ $(q(t-1) > 0$ and $q(t-5) < 0)$ for $t \in (25/8, 15/4 = \beta]$;
- $q'(t) = -3$ $(q(t-1) > 0$ and $q(t-5) < 0)$ for $t \in [15/4 = \beta, 19/4)$;
- $q'(t) = 2$ $(q(t-1) < 0$ and $q(t-5) < 0)$ for $t \in (19/4, 5)$;
- $q'(t) = 4$ $(q(t-1) < 0$ and $q(t-5) > 0)$ for $t \in (5, 45/8 = \beta + \gamma]$.

$q(t)$ is a simple periodic solution of (6). To make the generalization of our approach to simple periodic solutions of (4) more apparent, we introduce the notation $\sigma = 3$; σ should be thought of as chosen so that $|q'(z)| \geq \sigma$ for all zeros z of q. The following lemma is clear.

Lemma 3.5. *There is a number* $\alpha \in (0, 1/6)$ *such that the following hold.*

- *if* z *is any zero of* q*, then* $|q(t-1)| \geq 2\sigma\alpha$ *and* $|q(t-5)| \geq 2\sigma\alpha$ *for all* $t \in [z-3\alpha, z+3\alpha]$*; in particular,* $q'(t)$ *is constant on* $[z-3\alpha, z+3\alpha]$.
- $|q(t)| \leq 3\sigma\alpha$ *only if* t *is contained in an interval of the form* $[z-3\alpha, z+3\alpha]$*, where* z *is a zero of* q*.* □

The above lemma says that q has constant slope on intervals $[z-3\alpha, z+3\alpha]$ of radius 3α about each of its zeros z; that the delayed absolute values of q are greater than or equal to $2\sigma\alpha$ on these intervals; and that these intervals are the only places where $|q(t)|$ can possibly attain values less than or equal to $3\sigma\alpha$. (Of course, if $|q'(z)| > \sigma$, the interval about z where $|q(t)| \leq 3\sigma\alpha$ will be strictly contained in $[z-3\alpha, z+3\alpha]$.) We henceforth regard α as fixed.

We now introduce the spaces X and Y that we need. We take X to be the hyperplane of initial conditions whose values are equal to 0 at time zero:

$$X = \{\ x_0 \in C \ : \ x_0(0) = 0\ \}.$$

We define the set $Y \subset X$ as follows:

$$Y = \left\{ y_0 \in X \ : \ \begin{array}{c} \|y_0 - q_0\| < \sigma\alpha; \\ y_0'(s) = 4,\ \ s \in (-2\alpha, 0); \\ y_0'(s) = -3,\ \ s \in (-\gamma-2\alpha, -\gamma+2\alpha) \end{array} \right\}.$$

That is, elements of Y lie in X, are uniformly within $\sigma\alpha$ of q_0, and have the same constant slope as q_0 on the intervals $(-\gamma-2\alpha, -\gamma+2\alpha)$ and $(-2\alpha, 0)$. We endow Y with the subspace topology inherited from X. (For a simple periodic solution of a general equation (4), α and Y can be defined analogously.)

Since $\alpha < 1/6$ by assumption, -5 is not within 3α of any zero of q. Therefore, by Lemma 3.5, the only places on $[-5, 0]$ where $|q(t)| \le 2\sigma\alpha$ are subsets of the intervals $[-2\alpha, 0]$ and $[-\gamma - 2\alpha, -\gamma + 2\alpha]$. Now suppose that $y_0 \in Y$. Since $\|y_0 - q_0\| < \sigma\alpha$, we see that $y_0(t) < \sigma\alpha$ for $t \in [-\gamma + 2\alpha, -2\alpha]$ and $y_0(t) > \sigma\alpha$ for $t \in [-5, -\gamma - 2\alpha]$. Since y_0 is of constant slope on $[-2\alpha, 0]$ and on $[-\gamma - 2\alpha, -\gamma + 2\alpha]$, is uniformly within $\sigma\alpha$ of q_0, and satisfies $y_0(0) = 0$, we see in particular that y_0 must in fact have a single zero on $[-5, 0)$, and that this zero must lie in in the interval $(-\gamma - \alpha, -\gamma + \alpha)$. Let us write $-Z(y_0)$ for the location of this zero.

Lemma 3.6. *The map $Z : Y \to \mathbb{R}$ is continuous, open, and differentiable. Given $y_0 \in Y$, the map $DZ[y_0] \in \mathcal{L}(\mathcal{T}_{y_0}(Y), \mathbb{R})$ is surjective.*

PROOF. Every element in Y can be written in the form $q_0 + u_0$, where $\|u_0\| < \sigma\alpha$ and u_0 belongs to the closed linear subspace B of C consisting of initial conditions that are equal to 0 on $[-2\alpha, 0]$ and are constant on $[-\gamma - 2\alpha, -\gamma + 2\alpha]$. ($B$ is the tangent space to Y.) Write $\bar{u}$ for the value of u_0 on $[-\gamma - 2\alpha, -\gamma + 2\alpha]$. A simple calculation shows that Z is in fact affine on Y, and is given by the formula $Z(q_0 + u_0) = \gamma - \bar{u}/3$. The lemma follows. □

Suppose that $y_0 \in Y$. Then $y_0(s) > 0$ for $s \in [-5, -Z(y_0))$ and $y_0(s) < 0$ for $s \in (-Z(y_0), 0)$, and it is clear that the one-dimensional information $Z(y_0)$

uniquely determines the continuation of y_0 as a solution of (6). The main idea is that, since y_0 is of a narrowly specified form near its zeros, the number $Z(y_0)$ (together with the fact that $y_0 \in Y$) also determines the continuation of y_0 as a solution of (7), provided that η is small enough.

Let us now outline the rest of the section. We write U for an open ball in X about q_0 of radius δ. We first prove that, for δ and η small enough, the results of Section 2 are applicable. In particular, we shall show the following.

i) There is a continuously differentiable, compact return map $R : \overline{U} \to X$ for the equation (7). (Since $\mathbb{R}$ is C^1 on $\overline{U}$, it has a global Lipschitz constant on U.)

ii) $R(\overline{U}) \subset Y$;

iii) Given $x_0, y_0 \in \overline{U} \cap Y$, $Z(x_0) = Z(y_0)$ implies $R(x_0) = R(y_0)$;

iv) R has a fixed point $p_0 \in U$.

Once (i) – (iv) are established, since X, U, and Y are intersections of open subsets of C with affine closed subsets of C, we can apply Lemma 2.13 and Lemma 3.6 to conclude that $\ker DZ[p_0] \subset \ker DR[p_0]$ (this is also easy to see directly in this case, since Z is affine).

We will then show that the semiconjugating map ρ is affine, and that $D\rho[Z(p_0)]$ has spectrum entirely inside the unit circle. We can therefore apply

Propositions 2.7 and 2.12 to conclude that p_0 is an asymptotically stable fixed point of R, and that the spectrum of $DR[p_0]$ lies entirely within the unit circle.

Standard theory also shows that a neighborhood of p_0 in C flows into X under the solution semiflow for (7), and so the stability of p_0 as a fixed point of R does indeed tell us that the continuation p of as a solution of (7) is a stable periodic solution (recall Remark 2.9). This is the main result of this section:

Proposition 3.7. *For all η sufficiently small,* (7) *has a stable periodic solution p with $p_0 \in Y$.* □

In fact, as $\eta \to 0$ (as long as the bound μ remains fixed), the periodic solution p approaches q in the sense that the period of p approaches the period $\beta + \gamma$ of q, and $|p(t) - q(t)|$ is uniformly small for $t \in [0, \beta + \gamma]$. We do not formulate this result in detail here (though it will be fairly evident from the work we do below).

Lemma 3.8. *Let U be an open ball in X about q_0 of radius δ. There are positive numbers η_0 and δ_0 such that the following hold for all $(\eta, \delta) \in (0, \eta_0] \times (0, \delta_0]$. Given $x_0 \in \overline{U}$ with continuation x as a solution of* (7)*, the first two positive zeros $z_1 < z_2$ of x are well-defined and isolated, and*

$$x_{z_2} \in Y.$$

PROOF. We begin with the following observation about the restriction of q to $[0, \beta + \gamma + 3\alpha]$. Over this interval, given any $\triangle \leq 2\sigma\alpha$, there are precisely two subintervals where $|q(t-1)| < \triangle$: these are

$$I_1^1(\triangle) := \left(1 - \frac{\triangle}{4}, 1 + \frac{\triangle}{4}\right)$$

and

$$I_2^1(\triangle) := \left(\beta + 1 - \frac{\triangle}{3}, \beta + 1 + \frac{\triangle}{3}\right).$$

Similarly, there are precisely two subintervals where $|q(t-5)| < \triangle$: these are

$$I_1^5(\triangle) := \left(5 - \gamma - \frac{\triangle}{3}, 5 - \gamma + \frac{\triangle}{3}\right)$$

and

$$I_2^5(\triangle) := \left(5 - \frac{\triangle}{4}, 5 + \frac{\triangle}{4}\right).$$

For $i \in \{1, 5\}$ and $j \in \{1, 2\}$, write

$$n_j^i(\triangle) = \inf(I_j^i(\triangle)) \text{ and } m_j^i(\triangle) = \sup(I_i^j(\triangle)).$$

The following facts follow from computation and Lemma 3.5 (together with the facts that $\sigma = 3$ and $\alpha < 1/6$). For any $\triangle \leq 2\sigma\alpha$, we have

$$m_j^i(\triangle) - n_j^i(\triangle) \leq 2\triangle/3 < 1$$

and

$$3\alpha < n_1^1 < m_1^1 < n_1^5 < m_1^5 < \beta - 3\alpha \tag{8}$$

$$< \beta + 3\alpha < n_2^1 < m_2^1 < n_2^5 < m_2^5 < \beta + \gamma - 3\alpha.$$

We henceforth assume that $\delta < \sigma\alpha$ and that $\eta < \sigma\alpha$.

Given $x_0 \in \overline{U}$, let us write $x(t)$ for the continuation of x_0 as a solution of (7). Let us also write

$$D(t) = \delta,\ t \leq 0;\quad D(t) = \max\{\delta, \sup_{s\in[0,t]} |x(s) - q(s)|\ \},\ t \geq 0$$

and set

$$\tau = \min\{\ t > 0\ :\ D(t) = \sigma\alpha\ \}$$

(if τ does not exist, set $\tau = \infty$). Observe that, for all $t \in [0, \tau]$, $\|x_t - q_t\| \leq D(t) \leq \sigma\alpha$. Notice too that $D(t)$ is nondecreasing, and is constant on any interval where $x'(t) = q'(t)$.

The main observation is the following: for all $t \in [0, \tau] \cap [0, \beta + \gamma + 3\alpha]$, we have that $|x(t-1)| \geq \eta$ and $x(t-1)q(t-1) > 0$ (and so $g_1(x(t-1)) = h_1(q(t-1)))$ whenever $|q(t-1)| > D(t-1) + \eta$ — that is, whenever $t \notin I_1^1(D(t-1)+\eta) \cup I_2^1(D(t-1)+\eta)$. Similarly, for $t \in [0, \tau] \cap [0, \beta + \gamma + 3\alpha]$ we also have that $|x(t-5)| \geq \eta$ and $x(t-5)q(t-5) > 0$ whenever $t \notin I_1^5(D(t-5)+\eta) \cup I_2^5(D(t-5)+\eta)$. Thus, in particular (since $D(t)$ is increasing, and so $D(t-1) \geq D(t-5)$), we have that, for all $t \in [0, \tau] \cap [0, \beta + \gamma + 3\alpha]$, $x'(t) = q'(t)$ whenever

$$t \notin I_1^1(D(t-1)+\eta) \cup I_2^1(D(t-1)+\eta) \cup I_1^5(D(t-1)+\eta) \cup I_2^5(D(t-1)+\eta);$$

otherwise, we have only the crude bound $|x'(t) - q'(t)| \leq 2\mu$.

Let us assume for the moment that $\tau \geq \beta + \gamma + 3\alpha$. The above paragraph, together with (8), tells us the following: as t runs from 0 to $\beta + \gamma + 3\alpha$,

t will go through 4 disjoint intervals (subintervals of the disjoint intervals $I_1^1(2\sigma\alpha)$, $I_1^5(2\sigma\alpha)$, $I_2^1(2\sigma\alpha)$, and $I_2^5(2\sigma\alpha)$, respectively) where $x'(t) = q'(t)$ is not guaranteed. None of these intervals will intersect the set

$$[0, 3\alpha] \cup [\beta - 3\alpha, \beta + 3\alpha] \cup [\beta + \gamma - 3\alpha, \beta + \gamma + 3\alpha].$$

We now bound the size of these intervals.

Write $\triangle_0 = \delta$. Then $D(t) = \triangle_0$ for all $t \in [0, n_1^1(\triangle_0 + \eta)]$. Since

$$\text{measure}(I_1^1(\triangle_0 + \eta)) \leq \frac{2(\triangle_0 + \eta)}{\sigma},$$

we have

$$D(m_1^1(\triangle_0 + \eta)) \leq \triangle_0 + 2\mu\frac{2(\triangle_0 + \eta)}{\sigma} =: \triangle_1.$$

Let us assume that δ and η are chosen small enough that $\triangle_1 < \sigma\alpha$. In this case, the intervals $I_j^i(\triangle_1 + \eta)$ are all disjoint and, of course, $I_j^i(\triangle_1 + \eta) \supset I_i^j(\triangle_0 + \eta)$. Since $I_1^1(\triangle_1 + \eta)$ has length less than 1 and contains $I_1^1(\triangle_0 + \eta)$ we have that, for all $t \in [m_1^1(\triangle_0 + \eta), m_1^1(\triangle_1 + \eta)]$, $t - 1 < n_1^1(\triangle_0 + \eta)$ and so $D(t-1) < \triangle_0$. Therefore, for all $t \in [m_1^1(\triangle_0 + \eta), m_1^1(\triangle_1 + \eta)]$ we have

$$\begin{aligned} &t \notin I_1^1(\triangle_0 + \eta) \cup I_2^1(\triangle_0 + \eta) \cup I_1^5(\triangle_0 + \eta) \cup I_2^5(\triangle_0 + \eta) \\ &\supset I_1^1(D(t-1) + \eta) \cup I_2^1(D(t-1) + \eta) \cup I_1^5(D(t-1) + \eta) \cup I_2^5(D(t-1) + \eta). \end{aligned}$$

Thus $x'(t) = q'(t)$ — and $D(t) \leq \triangle_1$ — for all $t \in [m_1^1(\triangle_0 + \eta), m_1^1(\triangle_1 + \eta)]$.

On the other hand, for $t \in [m_1^1(\triangle_1+\eta), n_1^5(\triangle_1+\eta)]$, we certainly have

$$
\begin{aligned}
& t_1 \notin I_1^1(\triangle_1+\eta) \cup I_2^1(\triangle_1+\eta) \cup I_1^5(\triangle_1+\eta) \cup I_2^5(\triangle_1+\eta) \\
& \supset I_1^1(D(t)+\eta) \cup I_2^1(D(t)+\eta) \cup I_1^5(D(t)+\eta) \cup I_2^5(D(t)+\eta) \\
& \supset I_1^1(D(t-1)+\eta) \cup I_2^1(D(t-1)+\eta) \cup I_1^5(D(t-1)+\eta) \cup I_2^5(D(t-1)+\eta)
\end{aligned}
$$

and so $x'(t) = q'(t)$ on this interval too, and we have $D(n_1^5(\triangle_1+\eta)) \leq \triangle_1 < \sigma\alpha$.

A similar argument to the one we just gave shows that, for η and δ small enough,

$$
D(m_1^5(\triangle_1+\eta)) \leq \triangle_1 + 2\mu\frac{2(\triangle_1+\eta)}{\sigma} =: \triangle_2,
$$

where $\triangle_2 < \sigma\alpha$; and that $D(t)$ is constant on $[m_1^5(\triangle_1+\eta), n_2^1(\triangle_2+\eta)]$.

Reasoning similarly as t passes through the remaining two subintervals of $[0, \beta+\gamma+3\alpha]$ where the equality $x'(t) = q'(t)$ is not guaranteed, we arrive at the following conclusion. For δ and η small enough, the following hold:

- $x'(t) = q'(t)$ for $t \in [0, 3\alpha] \cup [\beta-3\alpha, \beta+3\alpha] \cup [\beta+\gamma-3\alpha, \beta+\gamma+3\alpha]$;
- $|x(t) - q(t)| < c := \frac{\sigma\alpha}{1+2\mu/\sigma}$ for $t \in [0, \beta+\gamma+3\alpha]$.

In this case, the first and second positive zeros $z_1 < z_2$ of x must occur within $c/\sigma < \alpha$ units of β and $\beta+\gamma$, respectively. Thus, very loosely speaking, x_{z_2} is obtained by advancing $\beta+\gamma$ units along x and then shifting by less than α. Therefore (using the fact that $|x'(t) - q'(t)| \leq 2\mu$ everywhere) we have

$$
\|x_{z_2} - q_{\beta+\gamma}\| = \|x_{z_2} - q_0\| < c + 2\mu|z_2 - (\beta+\gamma)| \leq c + 2\mu c/\sigma \leq \sigma\alpha.
$$

Moreover, since $x'_{\beta+\gamma}(s) = q'_0(s)$ for $s \in [-\gamma-3\alpha, -\gamma+3\alpha] \cup [-3\alpha, 0]$ and $|z_2 - (\beta+\gamma)| < \alpha$, we have that $x'_{z_2}(s) = q'_0(s)$ for $s \in [-\gamma-2\alpha, -\gamma+2\alpha] \cup [-2\alpha, 0]$; thus $x_{z_2} \in Y$. $\square$

We henceforth fix $\delta \le \delta_0$, where δ_0 is as in Lemma 3.8. We will also assume henceforth that $\eta \le \eta_0$, though we shall need to impose some further smallness conditions on η below. We shall use the facts, shown in the above proof, that given $x_0 \in \overline{U}$ with continuation x as a solution of (7), we have $|x(t)-q(t)| < \sigma\alpha$ for all $t \in [-r, \beta+\gamma+3\alpha]$, and $x'(t) = q'(t)$ on $[0, 3\alpha] \cup [\beta-3\alpha, \beta+3\alpha] \cup [\beta+\gamma-3\alpha, \beta+\gamma+3\alpha]$.

We write $R : \overline{U} \to Y \subset X$ for the map $x_0 \mapsto x_{z_2}$. This is our return map of interest. Standard arguments show that R is C^1 (and hence Lipschitz) on $\overline{U}$ and (since $\beta+\gamma-3\alpha > 5 = r$) is compact, and that given any $x_0 \in U$ there is a neighborhood $\mathcal{O}$ of x_0 in C from which solutions of (7) flow into U (recall Remark 2.9).

We now show that condition (III) of Section 2 holds (that is, that $Z(x_0)$ determines $R(x_0)$) and give a formula for the semiconjugating map ρ.

Lemma 3.9. *Let all notation be as established above.*

1. *Given $x_0, y_0 \in \overline{U} \cap Y$, $Z(x_0) = Z(y_0)$ implies that $R(x_0) = R(y_0)$.*

2. *Write $\rho : \mathbb{R} \to \mathbb{R}$ for the map such that $\rho(Z(x_0)) = Z(R(x_0))$ for all $x_0 \in \overline{U} \cap Y$. Then there is a constant k such that*

$$\rho(v) = \frac{v}{3} + k.$$

Moreover, $k \to 5/4$ *as* $\eta \to 0$.

PROOF. Suppose that $x_0 \in \overline{U} \cap Y$. We write x for the continuation of x_0 as a solution of (7) and y for the continuation of x_0 as a solution of (6). We write $y_0 = x_0$ and $Z(x_0) = v$. We write $z_1 < z_2$ for the first two positive zeros of x and $\zeta_1 < \zeta_2$ for the first two positive zeros of y.

We first compute y. Recalling the proof of Lemma 3.8, we write $I^i_j := I^i_j(2\sigma\alpha)$. The same argument as in the proof of Lemma 3.8 shows that $y_{\zeta_2} \in Y$, that $|y(t) - q(t)| < \sigma\alpha$ for all $t \in [-r, \beta + \gamma + 3\alpha]$, and that $y'(t) = q'(t)$ on $[0, 3\alpha] \cup [\beta - 3\alpha, \beta + 3\alpha] \cup [\beta + \gamma - 3\alpha, \beta + \gamma + 3\alpha]$. In fact, considering that $y(t - i)$ can have zeros only on the intervals I^i_j and that $y(t - i)$ is of constant slope on the intervals I^i_j, we see that $y(t - i)$ has exactly one zero on each of the intervals I^i_j and that the sequence of derivatives $y'(t)$ will be the same as for q. In particular (recall Remark 3.4):

- $y'(t) = 4$ for $t \in [0, 1)$;
- $y'(t) = -1$ for $t \in (1, 5 - v)$;
- $y'(t) = -3$ for $t \in (5 - v, \zeta_1]$.

Thus, computing, we arrive at $y(1) = 4$; $y(5 - v) = 4 - 1(5 - v - 1) = v$; and

$$\zeta_1 = \frac{-2}{3}v + 5.$$

Next we have

- $y'(t) = -3$ for $t \in [\zeta_1, \zeta_1 + 1)$;
- $y'(t) = 2$ for $t \in (\zeta_1 + 1, 5)$;
- $y'(t) = 4$ for $t \in (5, \zeta_2]$.

Thus, computing, we have $y(\zeta_1 + 1) = -3$; $y(5) = -3 + 2(5 - \zeta_1 - 1) = 5 - 2\zeta_1$; and

$$\zeta_2 = \frac{2\zeta_1 - 5}{4} + 5 = \frac{25}{4} - \frac{v}{3}.$$

Thus

$$Z(y_{\zeta_2}) = \zeta_2 - \zeta_1 = \frac{5}{4} + \frac{v}{3}.$$

(Observe that $\gamma = 15/8$ is the fixed point of this map, as should be the case.)

Now we turn to x. Our work in the proof of Lemma 3.8 — in particular, that x is of constant slope near all its zeros in $[-r, \beta + \gamma + 3\alpha]$ — shows that there are four disjoint intervals $J_j^i \subset I_j^i(2\sigma\alpha)$ ($i \in \{1, 5\}$ and $j \in \{1, 2\}$) such that, for $t \in [0, \beta + \gamma + 3\alpha]$, $|x(t - i)| < \eta$ precisely on $J_1^i \cup J_2^i$. Using the notation $A < B$ to mean that the supremum of interval A is less than the infimum of interval B, our work in the last lemma in fact tells us that

$$(0, 3\alpha) < J_1^1 < J_1^5 < (\beta - 3\alpha, \beta + 3\alpha) < J_2^1 < J_2^5 < (\beta + \gamma - 3\alpha, \beta + \gamma + 3\alpha).$$

Furthermore, $z_1 \in (\beta - \alpha, \beta + \alpha)$ and $z_2 \in (\beta + \gamma - \alpha, \beta + \gamma + \alpha)$.

Off of the intervals J_j^i, x follows the same sequence of slopes as does q. In particular:

- $x'(t) = 4$ $(x(t-1) < -\eta$ and $x(t-5) > \eta)$ for $t \in [0, \inf(J_1^1))$;

- $x'(t) = -1$ $(x(t-1) > \eta$ and $x(t-5) > \eta)$ for $t \in (\sup(J_1^1), \inf(J_1^5))$;

- $x'(t) = -3$ $(x(t-1) > \eta$ and $x(t-5) < -\eta)$ for $t \in (\sup(J_1^5), z_1]$;

- $x'(t) = -3$ $(x(t-1) > \eta$ and $x(t-5) < -\eta)$ for $t \in [z_1, \inf(J_2^1))$;

- $x'(t) = 2$ $(x(t-1) < -\eta$ and $x(t-5) < -\eta)$ for $t \in (\sup(J_2^1), \inf(J_2^5))$;

- $x'(t) = 4$ $(x(t-1) < -\eta$ and $x(t-5) > \eta)$ for $t \in (\sup(J_2^5), z_2]$.

We now come to the main reason for designing the space Y as we have. For all $t \in J_j^i \subset I_j^i(2\sigma\alpha)$, $|q(t-i)| < 2\sigma\alpha$ and so $t - i$ is within 2α units of a zero of q. By the definition of Y (if $t - i \leq 0$) and our work in the last lemma (if $t - i \geq 0$) we can conclude that the function $x(t-i)$ is linear on J_j^i. In particular, using the fact that the zeros of x on $[-5, \beta + \gamma + 3\alpha]$ occur at $-v < 0 < z_1 < z_2$, we see that the intervals J_j^i satisfy the following.

- J_1^1 is centered at 1, has length $2\eta/4 = \eta/2$, and $x(t-1) = 4(t-1)$ for $t \in J_1^1$;

- J_1^5 is centered at $5 - v$, has length $2\eta/3$, and $x(t-5) = -3(t-(5-v))$ for $t \in J_1^5$;

- J_2^1 is centered at $z_1 + 1$, has length $2\eta/3$, and $x(t-1) = -3(t-(z_1+1))$ for $t \in J_2^1$;

- J_2^5 is centered at 5, has length $2\eta/4 = \eta/2$, and $x(t-5) = 4(t-5)$ for $t \in J_2^5$.

Thus we have that $x(t) = 4t$ for $t \in [0, \inf(J_1^1)]$ and that

$$\begin{aligned} x(t) &= 4\inf(J_1^1) + \int_{\inf(J_1^1)}^{t} g_1(x(u-1))\, du + \int_{\inf(J_1^1)}^{t} g_2(x(u-5))\, du \\ &= 4\inf(J_1^1) + \frac{1}{4}\int_{-\eta}^{4(t-1)} g_1(s)\, ds + \int_{\inf(J_1^1)}^{t} 1\, du \end{aligned}$$

for $t \in [\inf(J_1^1), \sup(J_1^1)]$. In particular,

$$x(\sup(J_1^1)) - x(\inf(J_1^1)) = \frac{1}{4}\int_{-\eta}^{\eta} g_1(u)\, du + \frac{\eta}{2} =: \kappa_1^1.$$

Since we are assuming that $|g_2| \le \mu$, we have that $\kappa_1^1 \to 0$ as $\eta \to 0$.

From $t = \sup(J_1^1)$ to $t = \inf(J_1^5) = 5 - v - \eta/3$, x has constant slope -1. We can now give a formula for x on $[\inf(J_1^5), \sup(J_1^5)]$ similar to the the formula above, and write

$$x(\sup(J_1^5)) - x(\inf(J_1^5)) = -\frac{4\eta}{3} + \frac{1}{3}\int_{-\eta}^{\eta} g_2(u)\, du =: \kappa_1^5.$$

We then compute

$$x(\sup(J_1^5)) = x(5-v+\eta/3) = 4(1-\eta/4) - ((5-v-\eta/3) - (1+\eta/4)) + \kappa_1^1 + \kappa_1^5,$$

and so

$$z_1 = 5 - \frac{2}{3}v + \frac{7}{36}\eta + \frac{1}{3}\left(\kappa_1^1 + \kappa_1^5\right).$$

(Compare with the formula for ζ_1, and note that $|z_1 - \zeta_1| \to 0$ as $\eta \to 0$.) Similar computations show that the restriction of x to $[0, z_2]$ is completely determined by v, and that $Z(x_{z_2})$ and $Z(y_{\zeta_2})$ differ by a constant that approaches 0 as $\eta \to 0$. Since $z_2 > \beta + \gamma - \alpha > 5$, to say that the restriction of x to $[0, z_2]$ is completely determined by v implies that $R(x_0) = x_{z_2}$ is completely determined by v; this is the first part of the lemma. The second part of the lemma follows follows from our calculation of $Z(y_{\zeta_2})$ and the fact that $Z(x_{z_2})$ and $Z(y_{\zeta_2})$ differ by a constant that approaches 0 as $\eta \to 0$. □

The map ρ, extended to all of $\mathbb{R}$, has a unique fixed point π. For η small enough, π lies in $W := Z(U \cap Y)$, for $\pi \to \gamma$ as $\eta \to 0$. Now, the proof of Lemma 3.8 (specifically, the bounds on the quantities $\triangle_k$ in terms of η and $\triangle_{k-1}$) shows that there is some $\delta_1 < \delta$ such that, for $y_0 \in X$ and all η small enough, $\|y_0 - q_0\| < \delta_1$ implies that $R(y_0) \in U \cap Y$. Let η be so small, and also small enough that $Z^{-1}(\pi)$ has an element $y_0 \in Y$ with $\|y_0 - q_0\| < \delta_1$ (if $\pi \in W$, it is easy to construct a member y_0 of $Z^{-1}(\pi)$ with $\|y_0 - q_0\| \leq 3|\pi - \gamma|$). Then we have that $Z(R(y_0)) = \rho(\pi) = \pi = Z(y_0)$, and so we conclude that $R(R(y_0)) = R(y_0)$. Thus $p_0 := R(y_0)$ is a fixed point of R, and condition (IV) of Section 2 holds.

The fixed point of π of ρ is asymptotically stable, and $D\rho[\pi]$ has the single eigenvalue $1/3$. We can now use the results of Section 2 to conclude that p_0 is an asymptotically stable fixed point of R, and that $DR[p_0]$ has nonzero spectrum equal to $\{1/3\}$. The continuation p of p_0 as a solution of (7) is periodic; the fact that initial conditions in a neighborhood of p_0 in C flow into

X allows us to conclude Proposition 3.7.

Remark 3.10. Observe that the nonzero spectrum of $DR[p_0]$ does not depend on η or μ, assuming that these parameters are such that our basic conclusions hold.

4 Example — a threshold delay equation

We consider a scalar-valued state-dependent delay equation

$$x'(t) = g(x(t - d(x_t))), \tag{9}$$

where $d : C[-r, 0] \to [0, r]$ is given by the threshold condition

$$\int_{-d(\phi)}^{0} \theta(\phi(s))\, ds = 1.$$

We assume the following throughout:

(TD1) g is C^1, bounded with bounded derivative, and satisfies the negative feedback condition $xg(x) < 0$ for all $x \neq 0$;

(TD2) $\theta : \mathbb{R} \to (r^{-1}, N) \subset (0, \infty)$ is C^1, with bounded derivative, and even.

If θ is constant, of course, (9) is a constant-delay equation.

Particular periodic solutions of somewhat more general versions of (9) have been proven to exist in [2] and [16]; see [16] for some discussion of threshold-type delays in mathematical modeling.

Below we will impose the further condition that g is η-steplike relative to $-$sign — that is, that $g(x) = -\text{sign}(x)$ for $|x| \geq \eta$ (recall Definition 3.2). Some of the material in this section is taken from [9], where equation (9) is shown to have several periodic solutions under a slightly more general condition of the type

$$|g(x) + \text{sign}(x)| \leq \epsilon \text{ for } |x| \geq \eta, \quad \epsilon, \eta \text{ sufficiently small.}$$

We collect some properties of d in the lemma below. The differentiability of d follows from the implicit function theorem. The Lipschitz constant for d and the formula for the derivative of $(t - d(x_t))$ are given in [9].

Lemma 4.1. *The delay functional $d : C \to [0, r]$ is continuously differentiable, with bounded derivative.*

In fact, d has Lipschitz constant (with respect to the uniform norm on C) less than or equal to $r^2 \ell_\theta$, where ℓ_θ is the Lipschitz constant of θ.

If $x(t)$ is any continuous function, then

$$\frac{d}{dt}(t - d(x_t)) = \frac{\theta(x(t))}{\theta(x(t - d(x_t)))} \geq \frac{1}{rN}.$$

□

We use the framework for state-dependent delay equations described, for example, in Section 3 of [6]. Let us define the following "solution manifold":

$$\mathcal{D} = \{\phi \in C^1 \ : \ \phi'(0) = g(\phi(-d(\phi)))\ \}.$$

We give $\mathcal{D}$ the subspace topology inherited from C^1: $\|\phi\| = \sup|\phi(s)| + \sup|\phi'(s)|$. $\|\cdot\|$ shall refer to this norm henceforth. By Section 3 of [6], Lemma 4.1 is enough to establish the following (except for the fact that solutions are defined for all positive time, which can be established with a little bit of additional work using the boundedness of g and the negative feedback condition):

Proposition 4.2 (The solution semiflow for (9))**.** *There is a unique continuous solution semiflow* $F : \mathbb{R}_+ \times \mathcal{D} \to \mathcal{D}$*. This solution semiflow is continuously differentiable on* $(r, \infty) \times \mathcal{D}$*. For all* $t \geq r$*, the solution map* $F(t, \cdot)$ *is completely continuous.* □

We henceforth impose the following additional assumptions:

(TD3) There is no finite interval on which θ' has infinitely many zeros.

(TD4) g is odd, $|g(x)| \leq \mu$ for all x, and there is some $\eta > 0$ such that $g(x) = -\text{sign}(x)$ for $|x| \geq \eta$.

Assumption (TD3) is included only to simplify the proof of Lemma 4.9 below; we do not believe it to be necessary. We view θ as fixed henceforth. We assume that g satisfies (TD4) but will need to impose further size conditions on η — keeping μ fixed — as we proceed (η should be thought of as small).

We shall need the following fact, which is a consequence of the evenness of θ and the oddness of g.

Lemma 4.3. *Let x and y be the continuations of $x_0 \in \mathcal{D}$ and $-x_0 \in \mathcal{D}$, respectively, as solutions of* (9)*. Then $x(t) = -y(t)$ for all $t \geq 0$.* $\square$

It will be convenient to introduce the following notation: given $s \geq 0$, we write

$$\Theta(s) = \int_0^s \theta(u)\, du.$$

Θ is strictly increasing on $\mathbb{R}_+$. Observe that, since θ is even, we have $\int_{-s}^0 \theta(u)\, du = \Theta(s)$ as well.

As in Section 3, we are going to define a set Y that contains a periodic solution of (9) and whose members have constant slope near their zeros. The next lemma shows why this constant slope condition will be useful.

Lemma 4.4. *Suppose that $x_0 \in \mathcal{D}$ with continuation x as a solution of* (9)*. Suppose moreover that $x(-d(x_0)) = -\eta$, and moreover that*

$$x(-d(x_0) + s) = -\eta + s, \quad s \in [0, 2\eta].$$

Write $\sigma = \min\{t > 0 \;:\; x(t - d(x_t)) = 0\}$. Then $x(\sigma + t) = x(\sigma - t)$ for $t \in [0, \sigma]$, $x(t - d(x_{2\sigma})) = \eta$, and

$$\int_0^\sigma \theta(x(s))\, ds = \int_\sigma^{2\sigma} \theta(x(s))\, ds = \Theta(\eta).$$

A similar statement holds, mutatis mutandis, *if*

$$x(-d(x_0) + s) = \eta - s, \quad s \in [0, 2\eta].$$

This lemma says that, if $x(s - d(x_0))$ traverses the interval $[-\eta, \eta]$ at constant slope ± 1, then the solution $x(t)$ traces out a symmetric arc on $[0, 2\sigma]$. The evenness of θ and the oddness of g play central roles.

PROOF. Write $y(t) = t - d(x_t) + d(x_0)$ for $t \geq 0$, and observe that $y'(t) = \frac{d}{dt}(t - d(x_t))$. Since $y'(t) \geq (rN)^{-1}$ by Lemma 4.1, there are unique positive times $\sigma < \sigma^*$ such that $y(\sigma) = \eta$ and $y(\sigma^*) = 2\eta$. For all times $t \in (0, \sigma^*)$, using our assumed form for x on $[-d(x_0), -d(x_0) + 2\eta]$, we get

$$x(t - d(x_t)) = x(y(t) - d(x_0)) = -\eta + y(t).$$

Using the formula for $(t - d(x_t))'$ in Lemma 4.1 we therefore see that, for all $t \in [0, \sigma^*]$, $(x(t), y(t))$ solves the following ordinary initial value problem:

$$\begin{aligned} x'(t) &= g(-\eta + y(t)); \\ y'(t) &= \frac{\theta(x(t))}{\theta(-\eta + y(t))}; \\ x(0) &= x_0(0); \quad y(0) = 0. \end{aligned}$$

This IVP has a unique solution $(x(t), y(t))$ on $[0, \sigma^*]$.

Given such a solution $(x(t), y(t))$, we now define the following functions $\tilde{x}$ and $\tilde{y}$ on $[0, 2\sigma]$: for $t \in [0, \sigma]$, we define

$$\tilde{x}(t) = x(t); \quad \tilde{x}(\sigma + t) = x(\sigma - t);$$

$$\tilde{y}(t) = y(t); \quad \tilde{y}(\sigma + t) = 2\eta - y(\sigma - t).$$

Direct computation (using the fact that g is odd and θ is even) shows that $(\tilde{x}(t), \tilde{y}(t))$ is a solution of our IVP on $[0, 2\sigma]$. Thus we conclude that $x = \tilde{x}$

and $y = \tilde{y}$, that $\sigma^* = 2\sigma$, and that x is symmetric about σ on $[0, 2\sigma]$. (In fact, x is increasing on $[0, \sigma]$ and decreasing on $[\sigma, 2\sigma]$.)

Finally, observe that, by the threshold condition in our delay equation (9) and our assumptions on x,

$$0 = 1 - 1 = \int_{-d(x_0)+\eta}^{\sigma} \theta(x(s))\ ds - \int_{-d(x_0)}^{0} \theta(x(s))\ ds = \int_0^{\sigma} \theta(x(s))\ ds - \Theta(\eta).$$

The lemma follows. □

Let us define

$$X = \{\phi \in \mathcal{D}\ :\ \phi(0) = 0\ \}.$$

The following lemma will guide our definition of the map Z.

Lemma 4.5. *Suppose that $x_0 \in X$ with $x_0'(0) = 1$ and $x(-d(x_0)) \leq -\eta$. Write*

$$\zeta = \min\{\ t \in [-d(x_0), 0]\ :\ x_0(t) = 0\ \}$$

and assume that $[\zeta - \eta, \zeta + \eta] \subset (-d(x_0), 0)$. Assume moreover that, on $[\zeta - \eta, \zeta + \eta]$, x_0 is given by the formula

$$x_0(\zeta + s) = s, \quad s \in [-\eta, \eta].$$

Finally, writing x for the continuation of x_0 as a solution of (9), assume that x has a first positive zero z, and that $x(t - d(x_t)) \geq \eta$ for all $t \in [0, z]$ such that $t - d(x_t) \geq \zeta + \eta$.

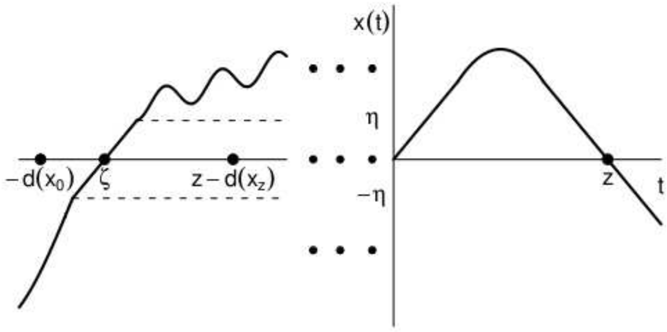
Write $D = \int_\zeta^0 \theta(x(s))\ ds$.

Then (viewing g — and hence η — as fixed) the restriction of x to $[0, z]$ is determined completely by D, and

$$\int_0^z \theta(x(s))\ ds = 2 - 2D.$$

If the hypotheses are satisfied by $-x_0$ rather than by x_0, the same conclusion holds.

See the figure below.



PROOF. Write $\tau = \min\{t > 0\ :\ t - d(x_t) = \zeta - \eta\}$. $x'(t) = 1$ on $[0, \tau]$. We have that

$$1 = \int_{\zeta-\eta}^{\tau} \theta(x(s))\ ds = \Theta(\eta) + D + \Theta(\tau),$$

whence $\Theta(\tau) = 1 - \Theta(\eta) - D$. Since the function Θ is invertible, τ is completely determined by D.

Write $y(t) = t - d(x_t) - (\zeta - \eta)$, and write σ for the unique number such that $y(\tau+\sigma) = \eta$. Lemma 4.4 now applies and tells us that $(x(t), y(t))$ is given on $[\tau, \tau + 2\sigma]$ by the solution of an ODE; the initial conditions of the ODE (namely, $x = \tau$ and $y = 0$) are determined by D. $x(t)$ increases on $[\tau, \tau + \sigma]$ and decreases on $[\tau + \sigma, \tau + 2\sigma]$, and is symmetric on $[\tau, \tau + 2\sigma]$ about its critical point at $\tau + \sigma$ — in particular, $x(\tau + 2\sigma) = x(\tau) > 0$. Furthermore, by the last point of Lemma 4.4,

$$\int_{\tau}^{\tau+2\sigma} \theta(x(s))\ ds = 2\Theta(\eta).$$

According to our hypotheses, after time $\tau + 2\sigma$, x will decrease with slope -1 until time z. By symmetry,

$$\int_{\tau+2\sigma}^{z} \theta(x(s))\ ds = \Theta(\tau).$$

Thus we have

$$\begin{aligned}\int_0^z \theta(x(s))\ ds &= 2\Theta(\tau) + 2\Theta(\eta) \\ &= 2 - 2\Theta(\eta) - 2D + 2\Theta(\eta) = 2 - 2D,\end{aligned}$$

as desired.

The final assertion of the lemma follows from Lemma 4.3. □

The work we have done allows us to assert, given any positive even n and for η sufficiently small, the existence of a periodic solution p of (9) with $p_0 \in X$ and such that p has two zeros per minimal period and n zeros on $(-d(p_0), 0)$.

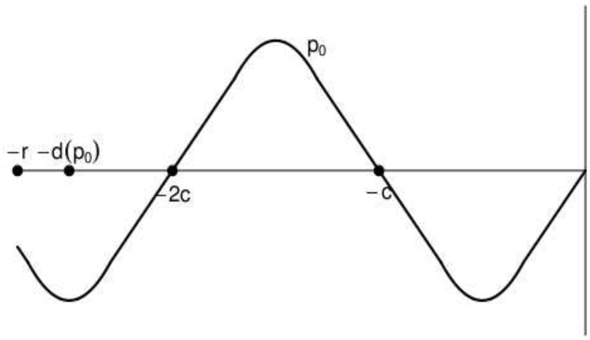
Here we shall focus on the $n = 2$ case. Recall that we are viewing θ and the bound $|g(x)| \leq \mu$ as fixed.

Proposition 4.6. *There is an $\eta_0 > 0$ such that, for $\eta \in (0, \eta_0]$, the following holds. There is a periodic solution p of* (9) *with period $2c$, two zeros per minimal period, $p(0) = 0$, $p'(0) = 1$, and such that:*

- *p has exactly 2 zeros $-2c, -c$ on $(-d(p_0), 0)$.*
- *$p(t)$ is of constant slope ± 1 on intervals of radius 4η about its zeros. Moreover, these intervals are precisely where $|p(t)| \leq 4\eta$, and for t in any such interval we have $|p(t - d(p_t)| \geq 2\eta$.*
- *$-d(p_0) < -2c - 4\eta$.*

In particular, p is of constant slope 1 on $[-4\eta, 0]$; constant slope -1 on $[-c - 4\eta, -c + 4\eta]$; and constant slope 1 on $[-2c - 4\eta, -2c + 4\eta]$.

The initial segment p_0 of p is illustrated below.



PROOF. Given η sufficiently small, there is a unique τ such that

$$\Theta(\eta) + \Theta(\tau) = \frac{1}{5}.$$

Since $\Theta(\eta) \to 0$ as $\eta \to 0$, $\tau \to \Theta^{-1}(1/5)$ as $\eta \to 0$. Let us assume in particular that $\tau > 4\eta$.

Let $(a(t), b(t))$ be the unique solution of the ordinary differential IVP

$$a'(t) = g(-\eta + b(t));$$
$$b'(t) = \frac{\theta(a(t))}{\theta(-\eta + b(t))};$$
$$a(0) = \tau; \quad b(0) = 0.$$

This is just the IVP studied in Lemma 4.4. Write σ for the unique positive time at which $b(t) = \eta$; observe that $\sigma \leq \eta r N$ (since $b'(t) \geq 1/(rN)$). Consider the periodic function p defined as follows:

$$p(t) = t, \quad t \in [0, \tau];$$

$$p(t) = a(t - \tau), \quad t \in [\tau, \tau + \sigma];$$

$$p(-t) = -p(t), \quad t \in \mathbb{R};$$

$$p(t) = -p(t + 2(\tau + \sigma)), \quad t \in \mathbb{R}.$$

On $[0, \tau + 2\sigma]$, $p(t)$ coincides with a solution $x(t)$ as described in Lemma 4.5. Thus we have that

$$\int_0^\tau \theta(p(s))\, ds = \Theta(\tau) \;\text{ and }\; \int_\tau^\sigma \theta(p(s))\, ds = \Theta(\eta).$$

Since $\Theta(\eta) + \Theta(\tau) = 1/5$, we have $d(p_0) = 5\tau + 5\sigma$, and $p(-d(p_0)) = -p(\tau + \sigma) < -\tau < -4\eta$. Observe that $p(\tau + 2\sigma - d(p_{\tau+2\sigma})) = \eta$, that $p(2\tau + 2\sigma) = 0$, and that $p(2\tau + 2\sigma - d(p_{2\tau+2\sigma})) = p(2\tau + 2\sigma - 5\tau - 5\sigma) = p(\tau + \sigma)$. More generally, as t runs from $\tau + 2\sigma$ to $2\tau + 2\sigma$, $p(t - d(p_t))$ increases from η to $p(\tau + \sigma)$. By the work in Lemma 4.5 we now see that the continuation of p_0 as a solution of (9) conincides with p on $[0, 2\tau + 2\sigma]$. Since $p_{2\tau+2\sigma} = -p_0$, the symmetry statement of Lemma 4.5 now shows that p is a periodic solution of (9), with $c = 2\tau + 2\sigma$.

p has constant slope on intervals of radius $\tau > 4\eta$ about each of its zeros. The further condition that $|p(t-d(p_t))| \geq 2\eta$ on these intervals can be satisfied by shrinking η if necessary. (Note too that p' is nonconstant on intervals of length 2σ, that $2\sigma \leq 2\eta r N$, and that the critical points of p occur at the midpoints of these intervals.) This proves the lemma. □

We henceforth view g and p as fixed — in particular, we view $\eta \leq \eta_0$ as fixed such that Proposition 4.6 holds. We shall write ℓ_g for the Lipschitz constant of g and ℓ_d for the Lipschitz constant of d (with respect to the sup norm) given in Lemma 4.1.

Our goal, of course, is to show that the conditions of Section 2 hold and to compute an appropriate semiconjugating map ρ, thereby obtaining the main result of this section:

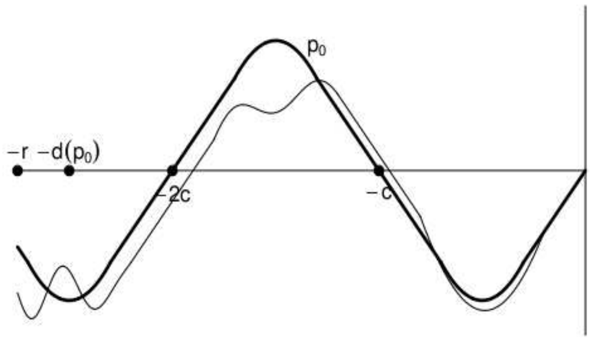
Proposition 4.7. *Assume (TD1)–(TD4). The periodic solution p of* (9) *described in Proposition 4.6 is unstable.*

We begin by defining the sets U and Y. U we shall take simply to be an open ball in X about p_0 of radius δ. We shall obtain specific requirements on δ below.

We now define the set $Y \subset X$ as follows: each $y_0 \in Y$ satisfies $\|y_0 - p_0\| < \eta$, differs from p_0 by a constant on the intervals

$$[-2c - 2\eta, -2c + 2\eta] \text{ and } [-c - 2\eta, -c + 2\eta],$$

and is equal to p_0 on the interval $[-2\eta, 0]$. p_0 (thicker line) and a typical element $y_0 \in Y$ (thinner line) are illustrated below. (The distance between y_0 and p_0 is exaggerated in the figure to emphasize the shape of y_0.)



Let us take a moment to understand the tangent spaces at p_0 to X and to Y; we draw on Section 3 of [6]. The solution manifold $\mathcal{D}$ is the set of points $\phi \in C^1$ such that $g(-d(\phi)) = \phi'(0)$. Since d is continuous and g is η-steplike with respect to $-$sign, though (and since $p(-d(p_0)) < -4\eta$), the map $\phi \mapsto g(-d(\phi))$ is constantly 1 in a C^1 neighborhood about p_0. Thus, locally about p_0, X has the form of an affine subspace, and the tangent space to X at p_0 is the corresponding linear subspace:

$$T_{p_0}(X) = \{\ v \in C^1[-r, 0]\ :\ v(0) = 0,\ v'(0) = 0\ \}.$$

Y is the intersection of X, a ball in C^1, and another affine subspace; specifically, we have

$$T_{p_0}(Y) = \left\{\ v \in C^1[-r, 0]\ :\ \left[\begin{array}{l} v(s) = 0,\ s \in [-2\eta, 0]; \\ v(s) = \text{constant},\ s \in [-c - 2\eta, -c + 2\eta]; \\ v(s) = \text{constant},\ s \in [-2c - 2\eta, -2c + 2\eta]. \end{array}\right.\ \right\}.$$

By Lemma 2.13, then, if we establish condition (D3) for a suitable map Z, we will have established (D4) as well.

The lemma below is clear from facts we have established about p. We sketch the main idea: for $t \in (-d(p_0)), 0)$, $|p(t)| \leq 2\eta$ only on

$$[-2c - 2\eta, -2c + 2\eta] \cup [-c - 2\eta, -c + 2\eta] \cup [-2\eta, 0];$$

since y_0 is of constant nonzero slope on each of these intervals and is within η of p_0 everywhere on $[-d(p_0), 0]$, we have

Lemma 4.8. *Given $y_0 \in Y$, y_0 has exactly two zeros $\zeta_{-2} < \zeta_{-1}$ in $(-d(p_0), 0)$. $\zeta_{-2} \in (-2c - \eta, -2c + \eta)$ and $\zeta_{-1} \in (-c - \eta, -c + \eta)$. y_0 is of constant slope 1 on the interval $[\zeta_{-2} - \eta, \zeta_{-2} + \eta]$ and of constant slope -1 on the interval $[\zeta_{-1} - \eta, \zeta_{-1} + \eta]$; these intervals, along with $[-\eta, 0]$, are precisely the places on $[-d(p_0), 0]$ where $|y_0(s)| \leq \eta$.* □

We now define the map $Z : Y \to \mathbb{R}^2$ by the formula

$$Z(y_0) = (v_1, v_2),$$

where $\zeta_{-2} < \zeta_{-1}$ are the zeros of y_0 on $(-d(p_0), 0)$ and

$$v_2 = \int_{\zeta_{-2}}^{\zeta_{-1}} \theta(y_0(s))\, ds \quad \text{and} \quad v_1 = \int_{\zeta_{-1}}^{0} \theta(y_0(s))\, ds.$$

Observe that $Z(p_0) = (2/5, 2/5)$.

Lemma 4.9. *Z is differentiable and open. $DZ[p_0]$ is surjective.*

PROOF. Suppose that $y_0 \in Y$ with zeros $\zeta_{-2} < \zeta_{-1}$ on $(-d(p_0), 0)$ and that v is a member of the tangent space of Y at y_0 — in particular, v is C^1, equal to 0 at 0, and constant on each of the intervals

$$[-2c - 2\eta, -2c + 2\eta], \ [-c - 2\eta, -c + 2\eta], \ [-2\eta, 0].$$

For $k \in \{-2, -1\}$ write $\hat{v}_k$ for the value of v on $[kc - 2\eta, kc + 2\eta]$. Observe that, for real scalars h with $|h|$ sufficiently small, the zeros of $y_0 + hv$ on $(-d(p_0), 0)$ are

$$\zeta_{-2} - h\hat{v}_{-2} \text{ and } \zeta_{-1} + h\hat{v}_{-1}.$$

We compute the second coordinate of $(Z(y_0 + hv) - Z(y_0))/h$; the computation for the first coordinate is similar.

$$\begin{aligned}
&\frac{1}{h}\left(Z(y_0 + hv)_2 - Z(y_0)_2\right) \\
&= \frac{1}{h}\left(\int_{\zeta_{-2} - h\hat{v}_{-2}}^{\zeta_{-1} + h\hat{v}_{-1}} \theta((y_0 + hv)(s))\, ds - \int_{\zeta_{-2}}^{\zeta_{-1}} \theta(y_0(s))\, ds\right) \\
&= \frac{1}{h}\left(\int_{\zeta_{-2} - h\hat{v}_{-2}}^{\zeta_{-2}} \theta((y_0 + hv)(s))\, ds + \int_{\zeta_{-1}}^{\zeta_{-1} + h\hat{v}_{-1}} \theta((y_0 + hv)(s))\, ds\right) \\
&+ \frac{1}{h}\left(\int_{\zeta_{-2}}^{\zeta_{-1}} \theta((y_0 + hv)(s)) - \theta(y_0(s))\, ds\right).
\end{aligned}$$

Taking the limit as $|h| \to 0$ we arrive at

$$\hat{v}_{-2}\theta(0) + \hat{v}_{-1}\theta(0) + \int_{\zeta_{-2}}^{\zeta_{-1}} \theta'(y_0(s))v(s)\, ds,$$

which is the formula for the second coordinate of $DZ[y_0]v$.

We now show that Z is open. Let $y_0 \in Y$ with zeros $\zeta_{-2} < \zeta_{-1}$ on $(-d(p_0), 0)$. Let us write (a, b) for the range of values assumed by y on the interval $(-2c + 2\eta, -c - 2\eta)$. By assumption (TD3) (this is where we use this assumption), there is an subinterval I of (a, b) where θ' is all of one sign. Perturbing y on a subinterval of $y^{-1}(I) \cap (-2c + 2\eta, -c - 2\eta)$ will result in perturbations (both positive and negative) in the value of the second coordinate of $Z(y_0)$. The same reasoning applies to the first coordinate of $Z(y_0)$. A similar argument applied to the derivative formula above shows that $DZ[p_0]$ is surjective. $\square$

We remind the reader that U is an open (relative to X) ball about p_0 of radius δ. Let us choose and fix a positive even integer m such that $mc - 4\eta > r$.

We now define our return map R.

Lemma 4.10. *There is a δ_0 such that, for all $\delta \in (0, \delta_0]$, the following holds. If $x_0 \in \overline{U}$ has continuation x as a solution of* (9)*, the first m positive zeros $z_1 < z_2 < \cdots < z_m$ of x are defined and isolated, and the map $R : \overline{U} \to X$ given by $R(x_0) = x_{z_m}$ is C^1, compact, and has image in Y.*

PROOF. Since $|p'(t)| \leq \mu := \sup |g|$ for all t, we have the following estimate for all $t \geq 0$. (Recall that we are viewing g as fixed, and writing ℓ_g and ℓ_d for the Lipschitz constants of g and d, respectively. Although the Lipschitz constant for d given in Lemma 4.1 is relative to the sup norm, it is of course valid with respect to the larger C^1 norm as well.)

$$
\begin{aligned}
&|x'(t) - p'(t)| \le |g(x(t - d(x_t))) - g(p(t - d(p_t)))| \\
&\le \ell_g |x(t - d(x_t)) - p(t - d(p_t))| \\
&\le \ell_g \left(|x(t - d(x_t)) - p(t - d(x_t))| + |p(t - d(x_t)) - p(t - d(p_t))|\right) \\
&\le \ell_g \left(|x(t - d(x_t)) - p(t - d(x_t))| + \mu |d(x_t) - d(p_t)|\right) \\
&\le \ell_g (1 + \mu \ell_d) \|x_t - p_t\| =: A \|x_t - p_t\|.
\end{aligned}
$$

It follows that, for all $t \in [0, cm + 4\eta]$, we have

$$
|x(t) - p(t)| \le e^{(cm+4\eta)A} \|x_0 - p_0\| =: B\|x_0 - p_0\| \le B\delta. \tag{10}
$$

For such t, reasoning similar to that in the above equality also yields

$$
|x(t - d(x_t)) - p(t - d(p_t))| \le B\delta(1 + \mu\ell_d) \tag{11}
$$

and

$$
|x'(t) - p'(t)| \le \ell_g B\delta(1 + \mu\ell_d). \tag{12}
$$

Let us choose δ small enough that $B\delta < \eta/2$, $B\delta(1 + \mu\ell_d) < \eta/2$, and $\ell_g B\delta(1 + \mu\ell_d) < \eta/2$. That is:

- $|x(t) - p(t)| < \eta/2$ for all $t \in [0, mc + 4\eta]$;
- $|x(t - d(x_t)) - p(t - d(p_t))| < \eta/2$ for all $t \in [0, mc + 4\eta]$;
- $|x'(t) - p'(t)| < \eta/2$ for all $t \in [0, mc + 4\eta]$.

Since $mc - 4\eta > r$, it follows that $\|x_t - p_t\| < \eta/2$ for all $t \in [mc - 4\eta, mc + 4\eta]$.

Recall that, when $t \in [0, cm + 4\eta]$ is within 4η of any zero of p, $|p(t - d(x_t))| \geq 2\eta$. With δ as we have chosen, then, we have that $|x(t - d(x_t))| > 3\eta/2 > \eta$ for all such t. We conclude that $x'(t) = p'(t) = \pm 1$ for all t in $[0, 4\eta]$ or in $[kc - 4\eta, kc + 4\eta]$, $k \in \{1, \ldots, m\}$. Since $|x(t) - p(t)| < \eta/2$ for $t \in [0, mc + 4\eta]$, we therefore see tha x has precisely m zeros $z_1 < \cdots < z_m$ on $(0, cm + 4\eta)$, with $|z_k - kc| \leq B\delta < \eta/2$ for all $k \in \{1, \ldots, m\}$. Translating now shows that x_{z_m} has constant slope on the intervals $[-2c - 2\eta, -2c + 2\eta]$, $[-c - 2\eta, -c + 2\eta]$, and $[-2\eta, 0]$.

It remains to ensure that $\|x_{z_m} - p_0\| \leq \eta$. Choose K such that $|p'(t)| \leq K$ and $|p''(t)| \leq K$ for all t. A shifting argument like the one at the end of the proof of Lemma 3.8 now shows that

$$\|x_{z_m} - p_0\| \leq \|x_{z_m} - p_{z_m}\| + 2K|z_m - cm| \leq (1 + 2K)B\delta.$$

Shrinking δ as necessary makes the quantity on the right less than η, and we have that $R(x_0) = x_{z_m} \in Y$.

The compactness and continuous differentiability of R follows, by standard arguments, from the corresponding properties of the solution semiflow $F : \mathbb{R}_+ \times \mathcal{D} \to \mathcal{D}$ for equation (9). $\square$

Essentially the same argument as in the above lemma actually shows that an open neighborhood $\mathcal{O}$ of p_0 in $\mathcal{D}$ flows into X, and so the dynamics of R about p_0 do indeed capture the dynamics of the solution semiflow about p

(recall Remark 2.9).

We henceforth assume that $\delta \le \delta_0$ as in Lemma 4.10.

We have two tasks remaining. To establish that we are in the framework of Section 2, it remains to establish condition (III); to exploit this framework and prove Proposition 4.7 (that p is unstable), it remains to compute the semiconjugating map ρ near $Z(p_0) = (2/5, 2/5) =: \pi$.

Let $x_0 \in U \cap Y$, with continuation x as a solution of (9). Write

$$z_{-2} < z_{-1} < 0 = z_0 < z_1 < \cdots < z_m$$

for the zeros of x on $(-d(p_0), cm + 4\eta)$. The work in the proof of the last lemma and Lemma 4.8 (about the properties of points of Y) together show the following:

- for all $t \in [-d(x_0), z_m]$, x will have slope equal to ± 1 on intervals of radius at least η about each of its zeros, and each of these zeros is within η of a corresponding zero of p; and
- for all $t \in [0, cm + 4\eta]$, $|x(t) - p(t)| \le \eta/2$; and
- for all $t \in [0, z_m]$, $|x(t - d(x_t)) - p(t - d(p_t))| \le \eta/2$. In particular, $|x(t - d(x_t)| \ge \eta$ whenever $t \ge 0$ is within η of a zero of x.

It follows that, for all $k \in \{0, \ldots, m-1\}$, x_{z_k} satisfies the hypotheses of Lemma 4.5. For as t goes from z_k to z_{k+1}, $x(t - d(x_t))$ will begin outside

$[-\eta,\eta]$, then cross the interval $[-\eta,\eta]$ in the manner described in Lemma 4.5, and then remain outside this interval through time z_{k+1}.

Moreover, for all $k \in \{0,\dots,m\}$, x_{z_k} will have precisely two zeros

$$-(z_k - z_{k-2}) \text{ and } -(z_k - z_{k-1}) \text{ on } (-d(p_0), 0).$$

We extend the definition of Z to all the points x_{z_k} in the obvious way:

$$Z(x_{z_k}) = \left(\int_{z_{k-1}}^{z_k} \theta(x(s))\, ds, \int_{z_{k-2}}^{z_{k-1}} \theta(x(s))\, ds \right).$$

We now have that the number

$$D_k := Z(x_{z_k})_1 + Z(x_{z_k})_2$$

— this is the number called D in the statement of Lemma 4.5 — will completely determine the restriction of x to $[z_k, z_{k+1}]$.

We can now find and analyze the semiconjugating map $\rho : \mathbb{R}^2 \to \mathbb{R}^2$. We write $Z(x_0) = (v_1, v_2)$ (and so the quantity called D in Lemma 4.5 is $v_1 + v_2$). By Lemma 4.5 we have

$$Z(x_{z_1}) = (2 - 2(v_1 + v_2), v_1),$$

and similarly

$$Z(x_{z_{k+1}}) = (2 - 2(Z(x_{z_k})_1 + Z(x_{z_k})_2), Z(x_{z_k})_1).$$

Let us consider, therefore, the map $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ given by the formula

$$\psi(v_1, v_2) = (2 - 2(v_1 + v_2), v_1).$$

This is an affine map with fixed point $(2/5, 2/5)$ and derivative

$$S = \begin{pmatrix} -2 & -2 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of S are $-1 \pm i$.

We can now write down our semiconjugating map ρ:

$$Z(R(x_0)) = \psi^m(Z(x_0)) = \psi^m(v_1, v_2) =: \rho(v_1, v_2).$$

Since ρ is equal to the mth power of ψ, it has derivative S^m, the spectrum of which lies outside the unit circle.

By repeated application of Lemma 4.5, we see that (v_1, v_2) determines the restriction of x to $[0, z_m]$. Since z_m is guaranteed to be greater than $cm - 4\eta > r$, we see that $Z(x_0) = (v_1, v_2)$ determines all of $x_{z_m} = R(x_0)$. This establishes (III).

Since (D1)–(D4) also hold, we have found the nonzero eigenvalues of $DR[p_0]$ as well, and we conclude that p is unstable.

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