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Multiple normalized solutions for $(2, q)$-Laplacian equation problems in whole $\mathbb{R}^N$

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Abstract. This paper considers the existence of multiple normalized solutions of the following $(2, q)$-Laplacian equation:

$$\begin{cases} -\Delta u - \Delta_q u = \lambda u + h(\epsilon x) f(u), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2, \end{cases}$$

where $2 < q < N$, $\epsilon > 0, a > 0$ and $\lambda \in \mathbb{R}$ is a Lagrange multiplier which is unknown, h is a continuous positive function and f is also continuous satisfying L^2-subcritical growth. When ϵ is small enough, we show that the number of normalized solutions is at least the number of global maximum points of h by Ekeland's variational principle.

Keywords: normalized solution, multiplicity, $(2, q)$-Laplacian, variational methods.

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1 Introduction

This paper is devoted to the existence of multiple normalized solutions, with $X := H^1(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$, of the following $(2, q)$-Laplacian equation:

$$-\Delta u - \Delta_q u = \lambda u + h(\epsilon x) f(u), \quad \text{in } \mathbb{R}^N \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2, \tag{1.2}$$

where $\epsilon, a > 0$, $\Delta_q u = \mathrm{div}(|\nabla u|^{q-2} \nabla u)$ is the q-Laplacian of u, $2 < q < N$ and $\lambda \in \mathbb{R}$ is a Lagrange multiplier which is unknown. The continuous function f satisfies the following conditions:

(f_1) f is odd and $\lim_{t \to 0} \frac{|f(t)|}{|t|^{p-1}} = \alpha > 0$ for some $p \in \left(2, 2 + \frac{4}{N}\right)$;

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(f_2) There exist some constants $c_1, c_2 > 0$ and $p_1 \in (q, q+\frac{2q}{N})$ such that $|f(t)| \leq c_1 + c_2|t|^{p_1-1}$, $\forall t \in \mathbb{R}$;

(f_3) the mapping $t \mapsto \frac{f(t)}{t^{q-1}}$ is a non-decreasing function when $t > 0$.

Hereafter, the continuous function h satisfies the following assumptions:

(h_1) $0 < h_0 = \inf_{x\in\mathbb{R}^N} h(x) \leq \max_{x\in\mathbb{R}^N} h(x) = h_{\max}$;

(h_2) $h_\infty = \lim_{|x|\to+\infty} h(x) < h_{\max}$;

(h_3) $h^{-1}(\{h_{\max}\}) = \{e_1, e_2, \ldots, e_l\}$ with $e_1 = 0$ and $e_j \neq e_k$ when $j \neq k$.

In particular, since restriction of (1.2), we are seeking normalized solutions to (1.1), which corresponds to seek critical points of the following functional

$$I_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x)F(u)\mathrm{d}x$$

on the sphere

$$S(a) := \left\{u \in X := H^1(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N) : |u|_2^2 = \int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x = a^2\right\}, \tag{1.3}$$

where $|\cdot|_\tau$ denotes the usual norm on $L^\tau(\mathbb{R}^N)$ for $\tau \in [1, +\infty)$ and $D^{1,q}(\mathbb{R}^N) := \{u \in L^{q^*}(\mathbb{R}^N) : \nabla u \in L^q(\mathbb{R}^N)\}$ with semi-norm $\|u\|_{D^{1,q}(\mathbb{R}^N)} = \|\nabla u\|_q$. Moreover, $\|u\|_X = \|u\|_{H^1(\mathbb{R}^N)} + \|u\|_{D^{1,q}(\mathbb{R}^N)}$. It is well known that $I_\epsilon \in C^1(X, \mathbb{R})$ and

$$\langle I_\epsilon'(u), \varphi\rangle = \int_{\mathbb{R}^N} \nabla u \nabla\varphi \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u\nabla\varphi\mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x) f(u)\varphi\mathrm{d}x$$

for all $u, \varphi \in X$.

The equation (1.1) is related to the general reaction-diffusion system

$$\partial_t u - \Delta_p u - \Delta_q u = f(x, u). \tag{1.4}$$

The system has wide range of applications in physics and related sciences, such as biophysics, chemical reaction and plasma physics. In such applications, the function u describes a concentration, the (p, q)-Laplacian term in (1.4) corresponds to the diffusion as $\operatorname{div}\left[\left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right)\nabla u\right] = \Delta_p u + \Delta_q u$, whereas the term $f(x, u)$ is the reaction and relates to sources and loss processes. Another model related to the (p, q)-Laplacian operator concerns the Lavrentiev gap phenomenon, which involved variational functions with nonstandard (p, q) growth conditions, e.g., in [9,30].

The stationary version of equation (1.4)

$$-\Delta_p u - \Delta_q u = f(x, u), \quad x \in \mathbb{R}^N$$

has been extensively studied. Where $N \geq 3, 1 < p < q < N$, C. J. He et al. in [11] proved the existence of solution by mountain pass theorem and the concentration–compactness principle when f does not satisfy the Ambrosetti–Rabinowitz condition and they derived the regularity of weak solutions in [12]. Furthermore, when nonlinear function f is discontinuous and satisfies the Ambrosetti–Rabinowitz condition, the authors in [31] showed the existence of solution by mountain pass theorem and the concentration-compactness principle. Moreover, some researchers had studied the existence results for the nonlinear function f involving the critical Sobolev exponent in a bounded domain. G. B. Li et al. [21] studied $f = |u|^{p^*-2}u + \mu|u|^{r-2}u$ and obtained infinitely many weak solutions by genus theorem when $1 < r < q < p < N, \mu > 0$. Later on, in [28], the authors proved multiplicity of positive solutions by using the Lusternik–Schnirelman category theorem where $p < r < p^*$. [13] proved some nonexistence results where $N \geq 2, 1 < q < p < N$ and $1 < r < p^*$. Finally, we refer the interested readers works [8,29] for a development of the existence theory for various problems of the (p,q)-Laplacian.

In literature, the following equation

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2 \end{cases} \tag{1.5}$$

has been widely studied by many researchers. In the L^2-subcritical problem, namely $2 < p < 2 + \frac{4}{N}$, it is well konwn that the functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \mathrm{d}x, u \in H^1(\mathbb{R}^N)$$

is bounded from below on the set $\{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x = a^2\}$, so we can found a solution as a global minimizer on the sphere, see [24]. While in the L^2-supercritical problem, namely $2 + \frac{4}{N} < p < \frac{2N}{N-2}$, $E|_{S(a)}$ is unbounded from below. One of the main difficulties in dealing with normalized solutions is proving the Palais–Smale condition, as a compactness property. Jeanjean in [14] got one normalized solution by a mountain pass structure for an auxiliary functional. Furthermore, in [5], the authors obtained infinitely many normalized solutions by using linking geometry for a stretched functional. More results about L^2-supercritical problem can be found in [6,15]. Regarding the critical case, we cite the articles [7,23]. Furthermore, in a recent paper, Yang and Baldelli [27] considered the following equation

$$\begin{cases} -\Delta u - \Delta_q u + \lambda u = |u|^{p-2}u, \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2 \end{cases}$$

in all the possible cases, where $2 < p < \min\{2^*, q^*\}$ and $1 < q < N$. They showed a ground state solution by using Ekeland's variational principle in L^2-subcritical case, while in L^2-critical case, they proved existence and nonexistence results, at last, they get a solution by using a natural constraint approach in L^2-supercritical case.

In addition, the multiplicity of normalized solutions has been wildly researched. For example, Jeanjean and Lu [18] studied the following problem

$$\begin{cases} -\Delta u = \lambda u + h(u), \text{ in } \mathbb{R}^N, \\ u > 0, \quad \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2, \end{cases}$$

they obtained multiple normalized solutions by the variational methods and genus theory. More information about multiplicity of normalized solutions by using genus theory and deformation arguments, see [2,16,17]. Particularly, without use of the genus theory, the authors [19] studied the following problem

$$\begin{cases} -\Delta u + \lambda u = (I_\alpha * [h(\epsilon x)|u|^{\frac{N+\alpha}{N}}])h(\epsilon x)|u|^{\frac{N+\alpha}{N}-2}u + \mu|u|^{q-2}u, \quad x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2. \end{cases}$$

They showed multiple normalized solutions by Ekeland's variational principle when ϵ small enough, $\mu, a > 0$, $2 < q < 2 + \frac{4}{N}$, $\lambda \in \mathbb{R}$ and h is a continuous positive function satisfying (h_1)–(h_3).

This paper is devoted to study the problem (1.1)–(1.2), which has not been studied in our knowledge. In order to get the existence of multiple normalized solutions for (1.1), we will follow the variational methods in [19]. Moreover, since the workspace is $X = H^1(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$, it will be more complicated to obtain the strong $L^2(\mathbb{R}^N)$ convergence of the selected Palais-Smale sequence in X.

The main result of this paper is the following:

Theorem 1.1. *Assume that f satisfies (f_1)–(f_3) and h satisfies (h_1)–(h_3). Then, there exists ϵ_0 such that (1.1) has at least l couples weak solutions $(u_j, \lambda_j) \in X \times \mathbb{R}$ for $0 < \epsilon < \epsilon_0$. Moreover, $\lambda_j < 0$ and $I_\epsilon(u_j) < 0$ for $j = 1, 2, \ldots, l$.*

Now, we will give the outline about this paper. In Section 2, we prove a compactness theorem in the autonomous case. In Section 3, we use the compactness theorem to study the non-autonomous case. Finally, we give the proof of Theorem 1.1 in Section 4.

2 The autonomous case

Firstly, we consider the existence of normalized solution $(u, \lambda) \in X \times \mathbb{R}$, where $X = H^1(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$, for the problem below

$$\begin{cases} -\Delta u - \Delta_q u = \lambda u + \mu f(u), \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2, \end{cases} \tag{2.1}$$

where $a, \mu > 0$, $\lambda \in \mathbb{R}$ and f satisfies (f_1)–(f_3). It is well known that the critical point of the functional

$$J_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x - \int_{\mathbb{R}^N} \mu F(u) \mathrm{d}x$$

is a solution to the problem (2.1), which is restricted to the sphere $S(a)$, where $F(t) = \int_0^t f(s)\mathrm{d}s$. Next, we will show that problem (2.1) has a normalized solution.

Lemma 2.1 ([20, Lemma 2.7]). *Assume that $k > 1$, Ω is an open set in $\mathbb{R}^N$, $\alpha, \beta > 0$ and $\Theta \in C(\Omega \times \mathbb{R}^N, \mathbb{R}^N)$ satisfying*

(1) $\alpha|\xi|^k \le \Theta(x,\xi)\xi$, $\forall (x,\xi) \in \Omega \times \mathbb{R}^N$,

(2) $|\Theta(x,\xi)| \le \beta|\xi|^{k-1}$, $\forall (x,\xi) \in \Omega \times \mathbb{R}^N$,

(3) $(\Theta(x,\xi) - \Theta(x,\eta))(\xi - \eta) > 0$, $\forall (x,\xi) \in \Omega \times \mathbb{R}^N$ with $\xi \ne \eta$,

(4) $\Theta(x,\gamma\xi) = \gamma|\gamma|^{k-2}\Theta(x,\xi)$, $\forall (x,\xi) \in \Omega \times \mathbb{R}^N$ and $\gamma \in \mathbb{R} \setminus \{0\}$.

Consider $(u_n), u \in W^{1,k}(\Omega)$, then $\nabla u_n \to \nabla u$ in $L^k(\Omega)$ if and only if

$$\lim_{n\to\infty} \int_\Omega \left(\Theta(x,\nabla u_n(x)) - \Theta(x,\nabla u(x))\right)\left(\nabla u_n(x) - \nabla u(x)\right)\mathrm{d}x = 0.$$

Lemma 2.2. *The functional J_μ restricts to $S(a)$ is bounded from below.*

Proof. From the conditions (f_1)–(f_2), we can infer that there exist some constants $C_1, C_2 > 0$ such that

$$|F(t)| \leq C_1|t|^p + C_2|t|^{p_1}, \quad \forall t \in \mathbb{R}.$$

By the L^q-Gagliardo–Nirenberg inequality [1, Theorem 2.1], we get that

$$|u|_l \leq C|\nabla u|_q^{\nu_{l,q}} |u|_2^{(1-\nu_{l,q})}, \quad \forall u \in D^{1,q}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \tag{2.2}$$

for some positive constant $C > 0$, where $\nu_{l,q} = \frac{Nq(l-2)}{l[Nq-2(N-q)]}$, $l \in (2, q^* = \frac{Nq}{N-q})$. Hence,

$$\begin{aligned} J_\mu(u) \geq \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x - CC_1 a^{(1-\nu_{p,q})p} \left(\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x\right)^{\frac{\nu_{p,q}p}{q}} \\ - CC_2 a^{(1-\nu_{p_1,q})p_1} \left(\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x\right)^{\frac{\nu_{p_1,q}p_1}{q}}. \end{aligned} \tag{2.3}$$

As $p \in (2, 2+\frac{4}{N})$, $p_1 \in (q, q+\frac{2q}{N})$, clearly $\nu_{p,q}p, \nu_{p_1,q}p_1 < q$, which ensures the boundedness of J_μ from below. If J_μ is not bound from below, then there is u such that

$$\frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x - C\left(\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x\right)^{\frac{\nu_{p,q}p}{q}} - C\left(\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x\right)^{\frac{\nu_{p_1,q}p_1}{q}} \to -\infty,$$

which is a contradiction since $\nu_{p,q}p, \nu_{p_1,q}p_1 < q$. □

This lemma ensures that $m_\mu(a) := \inf_{u \in S(a)} J_\mu(u)$ is well defined.

Lemma 2.3. *Let $\mu, a > 0$, then $m_\mu(a) < 0$.*

Proof. By (f_1), we can deduce $\lim_{t\to 0} \frac{pF(t)}{t^p} = \alpha > 0$, which implies that, for some $\delta > 0$,

$$\frac{pF(t)}{t^p} \geq \frac{\alpha}{2} \tag{2.4}$$

for all $t \in [0, \delta]$. Let $0 < u_0 \in S(a) \cap L^\infty(\mathbb{R}^N)$, we set

$$H(u_0, r)(x) = e^{\frac{Nr}{2}} u_0(e^r x), \quad \forall x \in \mathbb{R}^N, \ \forall r \in \mathbb{R}.$$

It is well known that

$$\int_{\mathbb{R}^N} |H(u_0, r)(x)|^2 \mathrm{d}x = a^2.$$

Furthermore, by a direct calculation, we have

$$\int_{\mathbb{R}^N} F(H(u_0, r)(x)) \mathrm{d}x = e^{-Nr} \int_{\mathbb{R}^N} F(e^{\frac{Nr}{2}} u_0(x)) \mathrm{d}x.$$

Then, for $r < 0$ and $|r|$ big enough, we have

$$0 \leq e^{\frac{Nr}{2}} u_0(x) \leq \delta, \quad \forall x \in \mathbb{R}^N.$$

Furthermore, by (2.4), we derive

$$\int_{\mathbb{R}^N} F(H(u_0, r)(x)) \mathrm{d}x \geq \frac{\alpha}{2p} e^{\frac{(p-2)Nr}{2}} \int_{\mathbb{R}^N} |u_0(x)|^p \mathrm{d}x,$$

so,

$$J_\mu(H(u_0,r)) \le \frac{e^{2r}}{2}\int_{\mathbb{R}^N}|\nabla u_0|^2\mathrm{d}x + \frac{e^{\frac{Nqr}{2}+rq-rN}}{2}\int_{\mathbb{R}^N}|\nabla u_0|^q\mathrm{d}x - \frac{\mu\alpha e^{\frac{(p-2)Nr}{2}}}{2p}\int_{\mathbb{R}^N}|u_0(x)|^p\mathrm{d}x.$$

Since $q>2, p\in\left(2,2+\frac{4}{N}\right)$, increasing $|r|$ if necessary, we get that

$$\frac{e^{2r}}{2}\int_{\mathbb{R}^N}|\nabla u_0|^2\mathrm{d}x + \frac{e^{\frac{Nqr}{2}+rq-rN}}{2}\int_{\mathbb{R}^N}|\nabla u_0|^q\mathrm{d}x - \frac{\mu\alpha e^{\frac{(p-2)Nr}{2}}}{2p}\int_{\mathbb{R}^N}|u_0(x)|^p\mathrm{d}x = A_r < 0,$$

then

$$J_\mu(H(u_0,r)) \le A_r < 0,$$

showing that $m_\mu(a)<0$. □

Lemma 2.4. *If $\mu>0$, $a>0$, then*

(i) $a\mapsto m_\mu(a)$ is a continuous mapping;

(ii) if $a_1\in(0,a)$ and $a_2=\sqrt{a^2-a_1^2}$, we have $m_\mu(a)<m_\mu(a_1)+m_\mu(a_2)$.

Proof. (i) Let $a>0$ and $(a_n)\subset(0,+\infty)$ such that $a_n\to a$, we need to prove that $m_\mu(a_n)\to m_\mu(a)$. There exists $u_n\in S(a_n)$ such that $m_\mu(a_n)\le J_\mu(u_n)<m_\mu(a_n)+\frac{1}{n}$ for every $n\in\mathbb{N}^+$. Firstly, we deduce from Lemma 2.3 that $m_\mu(a_n)<0$. Then by Lemma 2.2, we can get that (u_n) is bounded in X. Now considering $v_n:=\frac{a}{a_n}u_n\in S(a)$, since the boundedness of (u_n) and $a_n\to a$, we have

$$\begin{aligned} m_\mu(a) &\le J_\mu(v_n)\\ &= J_\mu(u_n)+\frac{1}{2}\left(\frac{a^2}{a_n^2}-1\right)\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x+\frac{1}{q}\left(\frac{a^q}{a_n^q}-1\right)\int_{\mathbb{R}^N}|\nabla u_n|^q\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}\Big(\mu F(u_n)\mathrm{d}x-\mu F(\frac{a}{a_n}u_n)\Big)\mathrm{d}x\\ &= J_\mu(u_n)+o_n(1). \end{aligned}$$

Let $n\to+\infty$, we can get $m_\mu(a)\le\lim_{n\to+\infty}\inf m_\mu(a_n)$. In the same manner, let (w_n) be a bounded minimizing sequence of $m_\mu(a)$ and $z_n:=\frac{a_n}{a}w_n\in S(a_n)$, then we have

$$m_\mu(a_n)\le J_\mu(z_n)=J_\mu(w_n)+o_n(1)\Longrightarrow\lim_{n\to+\infty}\sup m_\mu(a_n)\le m_\mu(a),$$

so we get $m_\mu(a_n)\to m_\mu(a)$.

(ii) For any fix $a_1\in(0,a)$, we first claim that

$$m_\mu(\theta a_1)<\theta^2 m_\mu(a_1),\ \forall\theta>1. \tag{2.5}$$

Let $(u_n)\subset S(a_1)$ be a minimizing sequence for $m_\mu(a_1)$, then $u_n(\theta^{-\frac{2}{N}}x)\in S(\theta a_1)$. Since $\theta>1$ and $\frac{2(N-q)}{N}<\frac{2(N-2)}{N}<2$, we have

$$\begin{aligned} m_\mu(\theta a_1)-\theta^2 J_\mu(u_n) &\le J_\mu(u_n(\theta^{-\frac{2}{N}}x))-\theta^2 J_\mu(u_n)\\ &=\frac{\theta^{\frac{2(N-2)}{N}}-\theta^2}{2}|\nabla u_n|_2^2+\frac{\theta^{\frac{2(N-q)}{N}}-\theta^2}{q}|\nabla u_n|_q^q\le 0. \end{aligned}$$

As a consequence $m_\mu(\theta a_1) \le \theta^2 m_\mu(a_1)$. If $m_\mu(\theta a_1) = \theta^2 m_\mu(a_1)$, we will have $|\nabla u_n|_2^2 \to 0$ and $|\nabla u_n|_q^q \to 0$ as $n \to +\infty$, which can indicates that $\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \to 0$ by inequality (2.2). Then,

$$\begin{aligned}
0 &> m_\mu(a_1)\\
&= \lim_{n\to+\infty} J_\mu(u_n) = \frac{1}{2}\lim_{n\to+\infty}\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x + \frac{1}{q}\lim_{n\to+\infty}\int_{\mathbb{R}^N} |\nabla u_n|^q \mathrm{d}x - \lim_{n\to+\infty}\int_{\mathbb{R}^N} \mu F(u_n)\mathrm{d}x\\
&= 0,
\end{aligned}$$

which is a contradiction. So we get $m_\mu(\theta a_1) < \theta^2 m_\mu(a_1)$. In the same manner, we can get

$$m_\mu(\theta a_2) < \theta^2 m_\mu(a_2), \quad \forall \theta > 1. \tag{2.6}$$

Finally, apply (2.5) with $\theta = \frac{a}{a_1} > 1$ and (2.6) with $\theta = \frac{a}{a_2} > 1$ respectively, we get

$$m_\mu(a) = \frac{a_1^2}{a^2} m_\mu\left(\frac{a}{a_1} a_1\right) + \frac{a_2^2}{a^2} m_\mu\left(\frac{a}{a_2} a_2\right) < m_\mu(a_1) + m_\mu(a_2). \qquad \square$$

Next, we will show the compactness theorem on $S(a)$ which is useful for studying the autonomous and the nonautonomous case.

Proposition 2.5. *Assume that $(u_n) \subset S(a)$ is a minimizing sequence of $m_\mu(a)$. Then, for some subsequence, either*

(i) (u_n) is strongly convergent,

or

(ii) there exists a sequence $v_n(\cdot) = u(\cdot + y_n)$ with $|y_n| \to +\infty$ and $(y_n) \subset \mathbb{R}^N$, which is strongly convergent to a function $v \in S(a)$ with $J_\mu(v) = m_\mu(a)$.

Proof. It is easy to obtain the boundedness of sequence (u_n) by Lemma 2.2, then there is a subsequence $u_n \rightharpoonup u$ in X, which is still denoted as itself. For the case of $u \neq 0$ and $|u|_2 = b$, by the Brézis–Lieb lemma in [26], we can deduce that $b \in (0, a)$ and

$$\begin{aligned}
|u_n|_2^2 &= |u|_2^2 + |u_n - u|_2^2 + o_n(1),\\
|\nabla u_n|_2^2 &= |\nabla u|_2^2 + |\nabla(u_n - u)|_2^2 + o_n(1).
\end{aligned}$$

Moreover, according to the assumption of f, we can deduce

$$\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^N} F(u)\mathrm{d}x + \int_{\mathbb{R}^N} F(u_n - u)\mathrm{d}x + o_n(1).$$

Now, we will prove $\nabla u_n \to \nabla u$ a.e. on $\mathbb{R}^N$, up to subsequences. Choose $\psi \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \le \psi \le 1$ in $\mathbb{R}^N$, $\psi(x) = 1$ for every $x \in B_1(0)$ and $\psi(x) = 0$ for every $x \in \mathbb{R}^N \setminus B_2(0)$. Take $R > 1$ and define $\psi_R(x) = \psi(x/R)$. Using the $\langle J'_\mu(u), \phi\rangle$ with $u = u_n$ and $\phi = (u_n - u)\psi_R$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[\nabla u_n - \nabla u + |\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u\right](\nabla u_n - \nabla u)\psi_R \mathrm{d}x\\
&\quad = \langle J'_\mu(u_n), (u_n - u)\psi_R\rangle - \int_{\mathbb{R}^N} \nabla u_n u_n \nabla\psi_R \mathrm{d}x - \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n u_n \nabla\psi_R dx\\
&\qquad + \int_{\mathbb{R}^N} \mu f(u_n) u_n \psi_R \mathrm{d}x + \int_{\mathbb{R}^N} \nabla u_n u \nabla\psi_R + \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n u \nabla\psi_R \mathrm{d}x\\
&\qquad - \int_{\mathbb{R}^N} \mu f(u_n) u \psi_R \mathrm{d}x - \int_{\mathbb{R}^N} \nabla u_n \nabla u \psi_R \mathrm{d}x - \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \nabla u_n \psi_R \mathrm{d}x.\\
&\qquad + \int_{\mathbb{R}^N} |\nabla u|^q \psi_R \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^2 \psi_R \mathrm{d}x.
\end{aligned}$$

Since, $(u_n) \subset S(a)$ and $(J_\mu|_{S(a)})'(u_n) \to 0$, we have $\langle J_\mu'(u_n), (u_n - u)\psi_R \rangle \to 0$ as $n \to \infty$. Moreover, combining with the definition of ψ_R and $u_n \rightharpoonup u$ in X, we can get, as $n \to \infty$,

$$\int_{\mathbb{R}^N} \nabla u_n u_n \nabla \psi_R \mathrm{d}x - \int_{\mathbb{R}^N} \nabla u_n u \nabla \psi_R \mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^N} |\nabla u_n|^{q-2} \nabla u_n u_n \nabla \psi_R \mathrm{d}x - \int_{\mathbb{R}^N} |\nabla u_n|^{q-2} \nabla u_n u \nabla \psi_R \mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^N} \mu f(u_n) u_n \psi_R \mathrm{d}x - \int_{\mathbb{R}^N} \mu f(u_n) u \psi_R \mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^N} |\nabla u|^{q-2} \nabla u \nabla u_n \psi_R \mathrm{d}x \to \int_{\mathbb{R}^N} |\nabla u|^q \psi_R \mathrm{d}x,$$

$$\int_{\mathbb{R}^N} \nabla u_n \nabla u \psi_R \mathrm{d}x \to \int_{\mathbb{R}^N} |\nabla u|^2 \psi_R \mathrm{d}x.$$

So,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[\nabla u_n - \nabla u + |\nabla u_n|^{q-2} \nabla u_n - |\nabla u|^{q-2} \nabla u \right] (\nabla u_n - \nabla u) \psi_R \mathrm{d}x = 0,$$

which is equivalent to

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (\nabla u_n - \nabla u)^2 \psi_R dx = 0, \ \lim_{n\to\infty} \int_{\mathbb{R}^N} (\nabla u_n - \nabla u)^q \psi_R \mathrm{d}x = 0.$$

Then, by Lemma 2.1 for $\Theta(x, \xi) = |\xi|^{k-2}\xi$ with $k = 2, k = q$, we have $\nabla u_n \to \nabla u$ in $L^2(B_2(0))$ and $L^q(B_2(0))$, which ensures that $\nabla u_n \to \nabla u$ a.e. on $\mathbb{R}^N$, up to subsequence. Now, applying Brézis–Lieb lemma in [26] again, we obtain

$$|\nabla u_n|_q^q = |\nabla u|_q^q + |\nabla (u_n - u)|_q^q + o_n(1).$$

Let $v_n = u_n - u$ and $|v_n|_2 = d_n \to d$, we can get that $a^2 = b^2 + d^2$ and $d_n \in (0, a)$ for n big enough. So,

$$m_\mu(a) + o_n(1) = J_\mu(u_n) = J_\mu(u) + J_\mu(v_n) + o_n(1) \geq m_\mu(d_n) + m_\mu(b) + o_n(1).$$

By the continuity of $a \mapsto m_\mu(a)$ (see Lemma 2.4(i)), we have

$$m_\mu(a) \geq m_\mu(d) + m_\mu(b),$$

which is contradicted to the conclusion of Lemma 2.4(ii), where $a^2 = b^2 + d^2$. This asserts that $|u|_2 = a$.

Combining with $|u_n|_2 = |u|_2 = a$, $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ is reflexive, we can get

$$u_n \to u \text{ in } L^2(\mathbb{R}^N). \tag{2.7}$$

Combining with the inequality (2.2) and $(f_1) - (f_2)$, we get

$$\int_{\mathbb{R}^N} F(u_n) \mathrm{d}x \to \int_{\mathbb{R}^N} F(u) \mathrm{d}x. \tag{2.8}$$

So

$$m_\mu(a) = J_\mu(u_n) + o_n(1) = J_\mu(u) + J_\mu(v_n) + o_n(1) \geq \frac{1}{2} |\nabla v_n|_2^2 + \frac{1}{q} |\nabla v_n|_q^q + m_\mu(a) + o_n(1),$$

which indicates $|\nabla v_n|_2^2, |\nabla v_n|_q^q \leq o_n(1)$. So we have $v_n \to 0$ in X, which means $u_n \to u$ in X.

Let us assume $u = 0$, i.e., $u_n \rightharpoonup 0$ in X. Then, for some $\varsigma, r > 0$ and $\{y_n\} \subset \mathbb{R}^N$, we have

$$\int_{B_r(y_n)} |u_n|^2 \mathrm{d}x \geq \varsigma, \ \forall y_n \in \mathbb{R}^N. \tag{2.9}$$

Otherwise we must have $u_n \to 0$ in $L^k(\mathbb{R}^N), \forall k \in (2, 2^*)$, which implies $F(u_n) \to 0$ in $L^1(\mathbb{R}^N)$. But it contradicts to the fact that

$$0 > m_\mu(a) + o_n(1) = J_\mu(u_n) \geq -\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x.$$

Then (2.9) holds. Since $u = 0$, combining with the inequality (2.9) and the Sobolev embedding, we can infer that (y_n) is unbounded. Then we consider $v_n(x) = u(x + y_n)$, which is easy to check that (v_n) is also a minimizing sequence of $m_\mu(a)$ and $(v_n) \subset S(a)$. So, there holds $v_n \rightharpoonup v$ in X, where $v \in X \setminus \{0\}$. According to the proof of the first part, we deduce that $v_n \to v$ in X. □

Lemma 2.6. *Assume (f_1)–(f_3) hold, $\mu > 0$. Then, problem (2.1) has a positive radial solution u and $\lambda < 0$.*

Proof. We can assume that there is a bounded minimizing sequence $(u_n) \subset S(a)$ of $m_\mu(a)$ by Lemma 2.2. Then, applying Proposition 2.5, we can deduce $m_\mu(a) = J_\mu(u)$, where $u \in S(a)$. Thus, we can get that there exists a constant $\lambda_a \in \mathbb{R}$ such that

$$J'_\mu(u) = \lambda_a \Psi'(u) \text{ in } X', \tag{2.10}$$

where $\Psi(u) := \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x$. Then, according to (2.10),

$$-\Delta u - \Delta_q u = \lambda_a u + \mu f(u), \quad x \in \mathbb{R}^N,$$

and

$$\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x - \int_{\mathbb{R}^N} \lambda_a u^2 \mathrm{d}x - \int_{\mathbb{R}^N} \mu f(u) u \mathrm{d}x = 0.$$

By (f_3), it is easy to obtain $qF(t) \leq f(t)t$ when $t \geq 0$, furthermore, since $m_\mu(a) = J_\mu(u) < 0$, we get

$$\begin{aligned}
0 &> J_\mu(u) - \frac{1}{q}\Big(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x - \int_{\mathbb{R}^N} \lambda_a u^2 \mathrm{d}x - \int_{\mathbb{R}^N} \mu f(u) u \mathrm{d}x \Big) \\
&= (\frac{1}{2} - \frac{1}{q}) \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} \lambda_a u^2 \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} \mu f(u) u \mathrm{d}x - \int_{\mathbb{R}^N} \mu F(u) \mathrm{d}x \\
&\geq \frac{1}{q} \int_{\mathbb{R}^N} \lambda_a u^2 \mathrm{d}x,
\end{aligned}$$

which implies that $\lambda_a < 0$.

Next, we will show that u is positive. From the definition of $J_\mu(u)$, we have $J_\mu(|u|) = J_\mu(u)$. Moreover we can get $|u| \in S(a)$. Then, we deduce

$$m_\mu(a) = J_\mu(u) = J_\mu(|u|) \geq m_\mu(a).$$

Then we have $J_\mu(|u|) = m_\mu(a)$. Therefore, we replace u by $|u|$. If u^* is the Schwarz's Symmetrization of u [22, Section 3.3], we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x \geq \int_{\mathbb{R}^N} |\nabla u^*|^2 \mathrm{d}x, \quad \int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x \geq \int_{\mathbb{R}^N} |\nabla u^*|^q \mathrm{d}x$$

and
$$\int_{\mathbb{R}^N} F(u)\mathrm{d}x = \int_{\mathbb{R}^N} F(u^*)\mathrm{d}x.$$
It is easy to check that $u^* \in S(a)$ and $J_\mu(u^*) = m_\mu(a)$. Thus, we replace u by u^*.

Next, we prove $u(x)$ is positive for all $x \in \mathbb{R}^N$. Firstly, we assume that the conclusion is false, then there is $x_0 \in \mathbb{R}^N$ satisfying $u(x_0) = 0$. Furthermore, we can assume that there is $x_1 \in \mathbb{R}^N$ satisfying $u(x_1) > 0$ by $u \neq 0$. Thus, we can find a ball with a sufficiently large radius $R > 0$ such that $x_0, x_1 \in B_R(0)$. Then, combining with the Harnack Inequality ([10, Theorem 8.20]), we can infer there is a constant $C > 0$ such that
$$\sup_{y\in B_R(0)} u(y) \leq C \inf_{y\in B_R(0)} u(y),$$
which contradicts to the fact that
$$\sup_{y\in B_R(0)} u(y) \geq u(x_1) > 0 \quad \text{and} \quad \inf_{y\in B_R(0)} u(y) = u(x_0) = 0. \qquad \square$$

The next corollary is obtained by Lemma 2.6.

Corollary 2.7. *Fix $a > 0$ and let $0 \leq \mu_1 < \mu_2$. Then, $m_{\mu_2}(a) < m_{\mu_1}(a) < 0$.*

Proof. Let $u_{\mu_1} \in S(a)$ satisfy $J_{\mu_1}(u_{\mu_1}) = m_{\mu_1}(a)$, then
$$m_{\mu_2}(a) \leq J_{\mu_2}(u_{\mu_1}) < J_{\mu_1}(u_{\mu_1}) = m_{\mu_1}(a). \qquad \square$$

3 The nonautonomous case

Next, we will show some properties of $I_\epsilon : X \to \mathbb{R}$,
$$I_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x)F(u)\mathrm{d}x,$$
which is restricted to $S(a)$.

Firstly, we define $I_{\max}, I_\infty : X \to \mathbb{R}$ as
$$I_{\max}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\mathrm{d}x - \int_{\mathbb{R}^N} h_{\max}F(u)\mathrm{d}x$$
and
$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\mathrm{d}x - \int_{\mathbb{R}^N} h_\infty F(u)\mathrm{d}x.$$
Moreover, Lemma 2.2 guarantees that
$$m_\infty(a) = \inf_{u\in S(a)} I_\infty(u),\ m_\epsilon(a) = \inf_{u\in S(a)} I_\epsilon(u),\ m_{\max}(a) = \inf_{u\in S(a)} I_{\max}(u).$$
Then, according to Corollary 2.7 and $h_\infty < h_{\max}$, we can immediately get
$$m_{\max}(a) < m_\infty(a) < 0. \tag{3.1}$$

Now, we fix $0 < \rho_1 = \frac{1}{2}(m_\infty(a) - m_{\max}(a))$.

Lemma 3.1. $\lim_{\epsilon\to 0^+} m_\epsilon(a) \leq m_{\max}(a)$*. Hence, there exists $\epsilon_0 > 0$ such that $m_\epsilon(a) < m_\infty(a)$ for all $0 < \epsilon < \epsilon_0$.*

Proof. Let $u_0 \in S(a)$ satisfying $I_{\max}(u_0) = m_{\max}(a)$. A simple calculus gives that

$$m_\epsilon(a) \leq I_\epsilon(u_0) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_0|^2 \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u_0|^q \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x) F(u_0) \mathrm{d}x.$$

Letting $\epsilon \to 0^+$ and applying (h_3) we can get

$$\limsup_{\epsilon \to 0^+} m_\epsilon(a) \leq \lim_{\epsilon \to 0^+} I_\epsilon(u_0) = I_{\max}(u_0) = m_{\max}(a).$$

According to (3.1), we obtain $m_\epsilon(a) < m_\infty(a)$ for ϵ small enough. □

The following two lemmas will be used to prove $(PS)_c$ condition for I_ϵ at some levels.

Lemma 3.2. *Assume that $(u_n) \subset S(a)$ is a minimizing sequence with $I_\epsilon(u_n) \to c$ and $c < m_{\max}(a) + \rho_1 < 0$. If $u_n \rightharpoonup u$ in X, then $u \neq 0$.*

Proof. Firstly, we assume the conclusion is false, i.e., $u \equiv 0$. Then, we have

$$c = m_\epsilon(a) = I_\epsilon(u_n) + o_n(1) = I_\infty(u_n) + \int_{\mathbb{R}^N} (h_\infty - h(\epsilon x)) F(u_n) \mathrm{d}x + o_n(1).$$

According to (h_2), there exist some constants $\xi, R > 0$ such that

$$h_\infty \geq h(x) - \xi, \quad |x| > R.$$

Thus, we have the following estimate

$$c = I_\epsilon(u_n) + o_n(1) \geq I_\infty(u_n) + \int_{B_{R/\epsilon}(0)} (h_\infty - h(\epsilon x)) F(u_n) \mathrm{d}x - \xi \int_{B^c_{R/\epsilon}(0)} F(u_n) \mathrm{d}x + o_n(1).$$

Recalling that (u_n) is bounded in X, then for some constant $C > 0$, there holds

$$\int_{\mathbb{R}^N} F(u_n) \mathrm{d}x \leq C_1 \left(\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x \right)^{\frac{\nu_{p,q}p}{q}} + C_2 \left(\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x \right)^{\frac{\nu_{p_1,q}p_1}{q}} \leq C.$$

By the fact of $u_n \to 0$ in $L^l(B_{R/\epsilon}(0))$ when $l \in [1, 2^*)$, one has

$$c = I_\epsilon(u_n) + o_n(1) \geq I_\infty(u_n) - \xi C > m_\infty(a) - \xi C + o_n(1),$$

which combines with the arbitrariness of $\xi > 0$, we can get

$$c \geq m_\infty(a),$$

which contradicts to the fact that $c < m_{\max}(a) + \rho_1 < m_\infty(a)$. So, we can get that $u \neq 0$. □

Lemma 3.3. *Assume that $(u_n) \subset S(a)$ is a $(PS)_c$ sequence of I_ϵ satisfying $u_n \rightharpoonup u_\epsilon$ in X when $c < m_{\max}(a) + \rho_1 < 0$, that is, as $n \to +\infty$,*

$$I_\epsilon(u_n) \to c \quad \text{and} \quad \|I_\epsilon|'_{S(a)}(u_n)\| \to 0.$$

Then there holds

$$\liminf_{n \to +\infty} |u_n - u_\epsilon|_2^2 \geq \beta,$$

where $u_n \nrightarrow u_\epsilon$ in X and $\beta > 0$ independent of $\epsilon \in (0, \epsilon_0)$.

Proof. Firstly, defining the functional $\Psi : X \to \mathbb{R}$ with $\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x$, we can see $S(a) = \Psi^{-1}(\{a^2/2\})$. According to [26, Proposition 5.12], there exist $(\lambda_n) \subset \mathbb{R}$ such that

$$\|I_\epsilon'(u_n) - \lambda_n \Psi'(u_n)\|_{X'} \to 0 \quad \text{as } n \to +\infty.$$

(u_n) is bounded in X since I_ϵ is bounded from below and coercive as J_μ, which ensures that (λ_n) is bounded, then there exists λ_ϵ such that $\lambda_n \to \lambda_\epsilon$ as $n \to +\infty$. Thus, we have

$$I_\epsilon'(u_\epsilon) - \lambda_\epsilon \Psi'(u_\epsilon) = 0 \quad \text{in } X',$$

and

$$\|I_\epsilon'(v_n) - \lambda_\epsilon \Psi'(v_n)\|_{X'} \to 0 \quad \text{as } n \to +\infty,$$

where $v_n := u_n - u_\epsilon$. According to (f_3), we can get $qF(t) \le f(t)t$ when $t \ge 0$. Then we have

$$0 > \rho_1 + m_{\max}(a) > c = \liminf_{n\to+\infty} I_\epsilon(u_n) = \liminf_{n\to+\infty}\left(I_\epsilon(u_n) - \frac{1}{q}\langle I_\epsilon'(u_n), u_n\rangle + \frac{1}{q}\lambda_n a^2\right) \ge \frac{1}{q}\lambda_\epsilon a^2,$$

which implies that

$$\limsup_{\epsilon\to 0} \lambda_\epsilon \le \frac{q(\rho_1 + m_{\max}(a))}{a^2} < 0.$$

Then, there is a constant λ^* satisfying $\lambda_\epsilon < \lambda^* < 0$, which is independent of ϵ. Therefore,

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v_n|^q \mathrm{d}x - \lambda_\epsilon \int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x = \int_{\mathbb{R}^N} h(\epsilon x) f(v_n) v_n \mathrm{d}x + o_n(1),$$

and

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v_n|^q \mathrm{d}x - \lambda^* \int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x \le \int_{\mathbb{R}^N} h(\epsilon x) f(v_n) v_n \mathrm{d}x + o_n(1).$$

According to (f_1), we get $f(t) < \varepsilon t, \forall \varepsilon > 0$ if t small enough, which combines with (f_2) to give

$$\int_{\mathbb{R}^N} f(v_n) v_n \mathrm{d}x \le C_2 \int_{\mathbb{R}^N} |v_n|^{p_1} \mathrm{d}x + \varepsilon \int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x \le C_2 \int_{\mathbb{R}^N} |v_n|^{p_1} \mathrm{d}x.$$

So, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x &+ \int_{\mathbb{R}^N} |\nabla v_n|^q \mathrm{d}x + C_0 \int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x \\ &\le h_{\max} \int_{\mathbb{R}^N} f(v_n) v_n \mathrm{d}x \le C_2 h_{\max} \int_{\mathbb{R}^N} |v_n|^{p_1} \mathrm{d}x + o_n(1)\end{aligned}$$

for some constant $C_0 > 0$ independent of $\epsilon \in (0, \epsilon_0)$. Since $v_n \nrightarrow 0$ in X, we can assume that $\liminf_{n\to+\infty} \|v_n\|_X > C > 0$. Thus, there holds

$$\liminf_{n\to+\infty} |v_n|_{p_1}^{p_1} \ge C_3 \tag{3.2}$$

for some constant $C_3 > 0$. By (2.2), we can deduce

$$C_3 \le \liminf_{n\to+\infty} |v_n|_{p_1}^{p_1} \le C(\liminf_{n\to+\infty} |v_n|_2)^{(1-\nu_{p_1,q})p_1} K^{\nu_{p_1,q} p_1}, \tag{3.3}$$

where $K > 0$ is independent of $\epsilon \in (0, \epsilon_0)$ with $\|v_n\| \le K$ for all $n \in \mathbb{N}$. Then, combining with (3.2), and (3.3), we achieve the proof. □

Next, we consider $0<\rho<\min\{\frac{1}{2},\frac{\beta}{a^2}\}(m_\infty(a)-m_{\max}(a))$.

Lemma 3.4. *Assume that $0<\epsilon<\epsilon_0$ and $c<m_{\max}(a)+\rho$. Then, I_ϵ restricted to $S(a)$ satisfies the $(PS)_c$ condition.*

Proof. Firstly, we can get that (u_n) is bounded by Lemma 2.2, then let $(u_n)\subset S(a)$ be $(PS)_c$ sequence of I_ϵ with $u_n\rightharpoonup u_\epsilon$, where $u_\epsilon\neq 0$ by Lemma 3.2 and $c<m_{\max}(a)+\rho$. Set $v_n=u_n-u_\epsilon$. If $v_n\to 0$ in X, the proof is complete. If $v_n\nrightarrow 0$ in X and $|u_\epsilon|_2=b$, by Lemma 3.3, we have

$$\liminf_{n\to+\infty}|v_n|_2^2\geq\beta \tag{3.4}$$

for some $\beta>0$ which is independent of $\epsilon\in(0,\epsilon_0)$.

Let $|v_n|_2=d_n\to d\geq\beta^{\frac{1}{2}}$, we have $a^2=b^2+d^2$. From $d_n\in(0,a)$ for n large enough, we can deduce

$$c+o_n(1)=I_\epsilon(u_n)=I_\epsilon(v_n)+I_\epsilon(u_\epsilon)+o_n(1)\geq m_\infty(d_n)+m_{\max}(b)+o_n(1).$$

Applying Lemma 2.4(i) and inequality (2.5), letting $n\to+\infty$, we get

$$m_{\max}(a)+\rho>c\geq m_\infty(d)+m_{\max}(b)\geq\frac{d^2}{a^2}m_\infty(a)+\frac{b^2}{a^2}m_{\max}(a).$$

Then

$$\rho\geq\frac{d^2}{a^2}(m_\infty(a)-m_{\max}(a))\geq\frac{\beta}{a^2}(m_\infty(a)-m_{\max}(a)),$$

which is contradicted to the fact of $\rho<\frac{\beta}{a^2}(m_\infty(a)-m_{\max}(a))$. Then, it holds $v_n\to 0$ in X, that is, $u_n\to u_\epsilon$ in X, which implies that $u_\epsilon\in S(a)$ and

$$-\Delta u_\epsilon-\Delta_q u_\epsilon=\lambda_\epsilon u_\epsilon+h(\epsilon x)f(u_\epsilon),\quad x\in\mathbb{R}^N.$$

□

4 Multiplicity result

In the following, we do some technical stuff. Let $\rho_0,r_0>0$, e_j be defined in (h_3), satisfying:

- $\overline{B_{\rho_0}(e_i)}\cap\overline{B_{\rho_0}(e_j)}=\emptyset$ for $i\neq j$ and $i,j\in\{1,\dots,l\}$.
- $\bigcup_{i=1}^{l}B_{\rho_0}(e_i)\subset B_{r_0}(0)$.
- $K_{\frac{\rho_0}{2}}=\bigcup_{i=1}^{l}\overline{B_{\frac{\rho_0}{2}}(e_i)}$.

Set $\kappa:\mathbb{R}^N\to\mathbb{R}^N$ with

$$\kappa(x):=\begin{cases}x, & \text{if } |x|\leq r_0,\\ r_0\frac{x}{|x|}, & \text{if } |x|>r_0.\end{cases}$$

Now we consider the function $G_\epsilon: X\backslash\{0\}\to\mathbb{R}^N$ with

$$G_\epsilon(u):=\frac{\int_{\mathbb{R}^N}\kappa(\epsilon x)|u|^2\mathrm{d}x}{\int_{\mathbb{R}^N}|u|^2\mathrm{d}x},$$

Then, we will get the existence of (PS) sequences of I_ϵ, which is restricted to $S(a)$ by the next two lemmas.

Lemma 4.1. *Decreasing ϵ_0 if necessary, there exists a positive constant $\delta_0 < \rho$ such that*

$$G_\epsilon(u) \in K_{\frac{\rho_0}{2}}, \quad \forall \epsilon \in (0, \epsilon_0),$$

where $u \in S(a)$ and $I_\epsilon(u) \leq m_{\max}(a) + \delta_0$.

Proof. We assume that the conclusion is false, so there exist $\delta_n \to 0$, $u_n \in S(a)$ and $\epsilon_n \to 0$ such that

$$I_{\epsilon_n}(u_n) \leq m_{\max}(a) + \delta_n$$

and

$$G_{\epsilon_n}(u_n) \notin K_{\frac{\rho_0}{2}}.$$

Firstly, we know

$$m_{\max}(a) \leq I_{\max}(u_n) \leq I_{\epsilon_n}(u_n) \leq m_{\max}(a) + \delta_n,$$

then,

$$I_{\max}(u_n) \to m_{\max}(a), \quad \text{as } n \to \infty.$$

We will analyze the following two cases by Proposition 2.5.

(i) $u_n \to u$ in X, where $u \in S(a)$. According to the Lebesgue dominated convergence theorem, we can deduce that

$$G_{\epsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \kappa(\epsilon_n x)|u_n|^2 dx}{\int_{\mathbb{R}^N} |u_n|^2 dx} \to \frac{\int_{\mathbb{R}^N} \kappa(0)|u|^2 dx}{\int_{\mathbb{R}^N} |u|^2 dx} = 0 \in K_{\frac{\rho_0}{2}},$$

which contradicts to $G_{\epsilon_n}(u_n) \notin K_{\frac{\rho_0}{2}}$ for n large.

(ii) There exists a sequence $v_n(\cdot) = u(\cdot + y_n)$ with $|y_n| \to +\infty$ and $(y_n) \subset \mathbb{R}^N$, which is convergent in X for some $v \in S(a)$. Then, we can also study the following two cases:

When $|\epsilon_n y_n| \to +\infty$, we can deduce that

$$I_{\epsilon_n}(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla v_n|^q dx - \int_{\mathbb{R}^N} h(\epsilon_n x + \epsilon_n y_n)F(v_n) dx \to I_\infty(v).$$

Since $I_{\epsilon_n}(u_n) \leq m_{\max}(a) + \delta_n$, there holds

$$m_{\max}(a) \geq I_\infty(v) \geq m_\infty(a),$$

which contradicts to (3.1).

When $\epsilon_n y_n \to y$ for some $y \in \mathbb{R}^N$, we get

$$I_{\epsilon_n}(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla v_n|^q dx - \int_{\mathbb{R}^N} h(\epsilon_n x + \epsilon_n y_n)F(v_n) dx \to I_{h(y)}(v),$$

then we obtain

$$m_{h(y)}(a) \leq m_{\max}(a). \tag{4.1}$$

If $h(y) < h_{\max}$, Corollary 2.7 implies that $m_{h(y)}(a) > m_{\max}(a)$, which contradicts to (4.1). Thus, it holds $h(y) = h_{\max}$, which means $y = e_i$ for some $i = 1, \ldots, l$. Then we have

$$G_{\epsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \kappa(\epsilon_n x)|u_n|^2 dx}{\int_{\mathbb{R}^N} |u_n|^2 dx} = \frac{\int_{\mathbb{R}^N} \kappa(\epsilon_n x + \epsilon_n y_n)|v_n|^2 dx}{\int_{\mathbb{R}^N} |v_n|^2 dx} \to \frac{\int_{\mathbb{R}^N} \kappa(y)|v|^2 dx}{\int_{\mathbb{R}^N} |v|^2 dx} = e_i \in K_{\frac{\rho_0}{2}},$$

which contradicts to $G_{\epsilon_n}(u_n) \notin K_{\frac{\rho_0}{2}}$ for n large. □

Next, we introduce some notations:

- $\theta_\epsilon^i := \{u \in S(a); |G_\epsilon(u) - e_i| \leq \rho_0\}$,
- $\partial\theta_\epsilon^i := \{u \in S(a); |G_\epsilon(u) - e_i| = \rho_0\}$,
- $\eta_\epsilon^i := \inf_{u\in\theta_\epsilon^i} I_\epsilon(u)$,
- $\tilde{\eta}_\epsilon^i := \inf_{u\in\partial\theta_\epsilon^i} I_\epsilon(u)$.

Lemma 4.2. *Let $0 < \delta_0 < \rho < \min\{\frac{1}{2}, \frac{\beta}{a^2}\}(m_\infty(a) - m_{\max}(a))$. Then, there holds*

$$\eta_\epsilon^i < m_{\max}(a) + \rho \quad \text{and} \quad \eta_\epsilon^i < \tilde{\eta}_\epsilon^i, \quad \forall \epsilon \in (0, \epsilon_0).$$

Proof. By Proposition (2.5), we set that

$$m_{\max}(a) = I_{\max}(u),\ I'_{\max}(u) = 0,$$

where $u \in S(a)$. Let $u_\epsilon^i : \mathbb{R}^N \to \mathbb{R}$ be $u_\epsilon^i(x) = u(x - e_i/\epsilon)$ for $1 \leq i \leq l$. By direct calculation, we get

$$I_\epsilon(u_\epsilon^i(x)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x + e_i) F(u) \mathrm{d}x,$$

which implies that

$$\limsup_{\epsilon\to 0} I_\epsilon(u_\epsilon^i(x)) \leq I_{\max}(u) = m_{\max}(a). \tag{4.2}$$

If $\epsilon \to 0^+$, there holds

$$G_\epsilon(u_\epsilon^i) = \frac{\int_{\mathbb{R}^N} \kappa(\epsilon x)|u_\epsilon^i|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u_\epsilon^i|^2 \mathrm{d}x} = \frac{\int_{\mathbb{R}^N} \kappa(\epsilon x + e_i)|u|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x} \to e_i.$$

Then we can infer that $u_\epsilon^i \in \theta_\epsilon^i$ when ϵ is small enough. Moreover, by (4.2),

$$I_\epsilon(u_\epsilon^i(x)) \leq m_{\max}(a) + \frac{\delta_0}{4}, \quad \forall \epsilon \in (0, \epsilon_0).$$

From this, decreasing ϵ_0 if necessary,

$$\eta_\epsilon^i \leq m_{\max}(a) + \frac{\delta_0}{4}, \quad \forall \epsilon \in (0, \epsilon_0).$$

Then,

$$\eta_\epsilon^i \leq m_{\max}(a) + \rho, \quad \forall \epsilon \in (0, \epsilon_0),$$

showing the first inequality.

If there holds $u \in \partial\theta_\epsilon^i$, i.e.,

$$u \in S(a) \quad \text{and} \quad |G_\epsilon(u) - e_i| = \rho_0 > \frac{\rho_0}{2},$$

which implies $G_\epsilon(u) \notin K_{\frac{\rho_0}{2}}$. Then, combining with Lemma 4.1, we have

$$I_\epsilon(u) > m_{\max}(a) + \frac{\delta_0}{2}, \quad \forall u \in \partial\theta_\epsilon^i, \quad \forall \epsilon \in (0, \epsilon_0),$$

and so,

$$\tilde{\eta}_\epsilon^i \geq m_{\max}(a) + \frac{\delta_0}{2}, \quad \forall \epsilon \in (0, \epsilon_0),$$

from which it follows that

$$\eta_\epsilon^i < \tilde{\eta}_\epsilon^i, \quad \forall \epsilon \in (0, \epsilon_0).$$

□

4.1 Proof of Theorem 1.1

By Ekeland's variational principle, we can get that there exists a sequence $(u_n^i) \subset S(a)$ such that

$$I_\epsilon(u_n^i) \to \eta_\epsilon^i$$

and

$$I_\epsilon(v) - I_\epsilon(u_n^i) \geq -\frac{1}{n}\|v - u_n^i\|, \quad \forall v \in \theta_\epsilon^i \quad \text{with} \quad v \neq u_n^i$$

for each $i \in \{1, \ldots, l\}$. Then, we get $u_n^i \in \theta_\epsilon^i \setminus \partial\theta_\epsilon^i$ for n large enough by Lemma 4.2.

Given $v \in T_{u_n^i} S(a) = \{w \in X : \int_{\mathbb{R}^N} u_n^i w \mathrm{d}x = 0\}$, we can define the path $\sigma : (-\xi, \xi) \to S(a)$ with

$$\sigma(t) = a\frac{(u_n^i + tv)}{|u_n^i + tv|_2},$$

where $\xi > 0$. It is obvious to know that $\sigma \in C^1((-\xi, \xi), S(a))$ and we have

$$\sigma(t) \in \theta_\epsilon^i \backslash \partial\theta_\epsilon^i, \ \forall t \in (-\xi, \xi), \ \sigma(0) = u_n^i \text{ and } \sigma'(0) = v.$$

Then we get

$$I_\epsilon(\sigma(t)) - I_\epsilon(u_n^i) \geq -\frac{1}{n}\|\sigma(t) - u_n^i\|$$

for $t \in (-\xi, \xi)$, which implies that

$$\begin{aligned}
\frac{I_\epsilon(\sigma(t)) - I_\epsilon(\sigma(0)))}{t} &= \frac{I_\epsilon(\sigma(t)) - I_\epsilon(u_n^i)}{t} \\
&\geq -\frac{1}{n}\left\|\frac{\sigma(t) - u_n^i}{t}\right\| \\
&= -\frac{1}{n}\left\|\frac{\sigma(t) - \sigma(0)}{t}\right\|, \quad \forall t \in (0, \xi).
\end{aligned}$$

Taking the limit of $t \to 0^+$, we have

$$\langle I_\epsilon'(u_n^i), v\rangle \geq -\frac{1}{n}\|v\|.$$

Then, we can replace v by $-v$ to deduce

$$\sup\{|\langle I_\epsilon'(u_n^i), v\rangle| : \|v\| \leq 1\} \leq \frac{1}{n},$$

which implies that

$$I_\epsilon(u_n^i) \to \eta_\epsilon^i \quad \text{and} \quad \|I_\epsilon|'_{S(a)}(u_n^i)\| \to 0 \quad \text{as } n \to +\infty,$$

which means $(u_n^i) \subset S(a)$ is a $(PS)_{\eta_\epsilon^i}$ sequence of I_ϵ. Combining with Lemma 3.4 and $\eta_\epsilon^i < m_{\max}(a) + \rho$, we can infer that there is u^i such that $u_n^i \to u^i$ in X. So, we have

$$u^i \in \theta_\epsilon^i, \ I_\epsilon(u^i) = \eta_\epsilon^i \quad \text{and} \quad I_\epsilon|'_{S(a)}(u^i) = 0.$$

According to our assumptions, we have

$$G_\epsilon(u^i) \in \overline{B_{\rho_0}(e_i)}, \quad G_\epsilon(u^j) \in \overline{B_{\rho_0}(e_j)}$$

and

$$\overline{B_{\rho_0}(e_i)} \cap \overline{B_{\rho_0}(e_j)} = \emptyset \text{ for } i \neq j,$$

which means $u^i \neq u^j$ for $i \neq j$ while $1 \leq i, j \leq l$. Thus, for any $\epsilon \in (0, \epsilon_0)$, I_ϵ has at least l nontrivial critical points, i.e.,

$$-\Delta u^i - \Delta_q u^i = \lambda_i u^i + h(\epsilon x) f(u^i), \quad \forall i \in \{1, 2, \dots, l\},$$

which ensures

$$\int_{\mathbb{R}^N} |\nabla u^i|^2 \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u^i|^q \mathrm{d}x - \int_{\mathbb{R}^N} \lambda_i |u^i|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x) f(u^i) u^i \mathrm{d}x = 0.$$

Combining with $I_\epsilon(u^i) < 0$, we have

$$\begin{aligned}
0 &> I_\epsilon(u^i) - \frac{1}{q}\Big(\int_{\mathbb{R}^N} |\nabla u^i|^2 \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u^i|^q \mathrm{d}x - \int_{\mathbb{R}^N} \lambda_i |u^i|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x) f(u^i) u^i \mathrm{d}x \Big) \\
&= \left(\frac{1}{2} - \frac{1}{q}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} \lambda_i |u^i|^2 \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} h(\epsilon x) f(u^i) u^i \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x) F(u^i) \mathrm{d}x \\
&\geq \frac{1}{q} \int_{\mathbb{R}^N} \lambda_i |u^i|^2 \mathrm{d}x,
\end{aligned}$$

which implies $\lambda_i < 0$. This proves the desired result.

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