Fully nonlinear degenerate equations with applications to Grad equations

Priyank Oza

Indian Institute of Technology Gandhinagar, Palaj, Gandhinagar, Gujarat, India–382055

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Abstract. We consider a class of degenerate elliptic fully nonlinear equations with applications to Grad equations:

\[ |Du|^{\gamma} \mathcal{M}_{\lambda,\Lambda}^+(D^2u(x)) = f(|u \geq u(x)|) \quad \text{in } \Omega, \]

\[ u = g \quad \text{on } \partial \Omega, \]

where \( \gamma \geq 1 \) is a constant, \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with \( C^{1,1} \) boundary. We prove the existence of a \( W^{2,p} \)-viscosity solution to the above equation, which degenerates when the gradient of the solution vanishes.

Keywords: fully nonlinear degenerate elliptic equations, viscosity solution, Pucci’s extremal operator, Dirichlet boundary value problem.

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1 Introduction

We investigate the following degenerate problem:

\[ \left\{ \begin{array}{ll}
|Du|^{\gamma} \mathcal{M}_{\lambda,\Lambda}^+(D^2u(x)) = f(|u \geq u(x)|) & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega,
\end{array} \right. \]  

(1.1)

where \( \gamma \geq 1 \) is a constant, \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with \( C^{1,1} \) boundary, \( | \cdot | \) denotes the Lebesgue measure in \( \mathbb{R}^N \), \( f : [0,|\Omega|] \to \mathbb{R} \) is a non-decreasing, non-negative continuous function and \( u : \Omega \to \mathbb{R} \). Here, \( \mathcal{M}_{\lambda,\Lambda}^+ \) is the Pucci’s extremal operator. In our setting, by \( u \geq u(x) \), we mean,

\( \{ \omega \in \Omega : u(\omega) \geq u(x) \} \)

called the superlevel sets of \( u \). We establish the existence of a \( W^{2,p} \)-viscosity solution (also known as \( L^p \)-viscosity solution) to (1.1). It is worth mentioning that the notion of \( W^{2,p} \)-viscosity solution was defined by Caffarelli et al. [7]. In the case when \( \gamma = 0 \) in (1.1), the existence of a \( W^{2,p} \)-viscosity solution is proven by L. Caffarelli and I. Tomasetti [8].

Email: priyank.k@iitgn.ac.in
The pioneer contribution in this direction was due to H. Grad [12], who introduced such equations, which appear in plasma physics, called “Grad equations”. In their seminal work, Grad examined the following equation in three-dimension:

$$
\Delta \Psi = F(V, \Psi, \Psi', \Psi''),
$$

where the right hand side (R.H.S.) represents a second-order differential operator acting on $\Psi(V)$ for a surface defined by $\Psi = \text{constant}$. Here, $\Psi'(V)$ represents the derivative with respect to volume and $\Psi(V)$ stands for the inverse function to $V(\Psi)$, denoting the volume enclosed by $\Psi$. Furthermore, they pointed out the potential for simplifying plasma equations by introducing $u^*$ defined as:

$$
u^*(t):= \inf \left\{ s : |u < s| \geq t \right\}.
$$

These equations, also known as Queer Differential Equations in the literature, have a wide range of applications across various fields. One notable application is their appearance in plasma modeling, specifically in analyzing plasma confined within toroidal containers. We refer to [12] for the details. Moreover, these equations exhibit connections in financial mathematics, see [23]. R. Temam [22] pioneered the investigation of problems akin to (1.1) concerning the Laplacian, a direction extensively examined by several researchers. They notably established the existence of solutions to equations having the model structure:

$$
\Delta u = g(|u < u(x)|, u(x)) + f(x),
$$

by exploiting the properties of directional derivatives of $u^*$. For further insights into this topic, we refer the interested readers to the works of J. Mossino and R. Temam [17], as well as those by P. Laurence and E. Stredulinsky [15, 16], along with the related references therein.

In all the aforementioned research works, the problem was studied using variational methods. However, in a recent work, L. Caffarelli and I. Tomasetti [8] studied the equation similar to J. Mossino and R. Temam [17] for fully nonlinear uniformly elliptic operators using the viscosity approach. Specifically, they addressed the following problem:

$$
\begin{cases}
F(D^2u(x)) = f(|u \geq u(x)|) & \text{in } \Omega \\
u = g & \text{on } \partial \Omega,
\end{cases}
$$

where $F$ represents a convex, uniformly elliptic operator. They established the existence of a $W^{2,p}$-viscosity solution $u$ to this problem, satisfying the following estimate:

$$
\|u\|_{W^{2,p}(\Omega)} \leq C[\|u\|_{\infty, \Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f(|u \geq u(x)|)\|_{p, \Omega}].
$$

For further insights into the existence and qualitative questions pertaining to extremal Pucci’s equations, we refer to [10, 11, 18, 20, 21, 24–26].

Concurrently, equations involving gradient degenerate fully nonlinear elliptic operators have been widely investigated over the past decade. Pioneering works in this direction are attributed to I. Birindelli and F. Demengel. They proved several important results for these operators in a series of papers. These contributions involve comparison principle and Liouville-type results [3], regularity and uniqueness of eigenvalues and eigenfunctions [4, 5], $C^{1,\alpha}$ regularity in the radial case [6]. Furthermore, the equations of the form:

$$
|Du|^\gamma F(D^2u) = f \quad \text{in } B_1,
$$

(1.2)
Degenerate equations concerning to Grad equations when $\gamma \geq 0$ is a constant and $f \in L^\infty(B_1, \mathbb{R})$, were investigated by C. Imbert and L. Silvestre [13]. In particular, they established the interior $C^{1,\alpha}$ regularity of solutions for equations of the form (1.2). One may also see [19] for insights into variable exponent degenerate mixed fully nonlinear local and nonlocal equations.

Motivated by the above works and recently by the work of L. Caffarelli and I. Tomasetti [8], it is natural to ask the following question:

**Question:** Do we have the existence of a viscosity solution to (1.1)?

The aim of this paper is to answer this question affirmatively. The crucial difference to our problem from [8] is due to the fact that $|Du|^{\gamma}M^{+}_{\lambda,\Lambda}(D^2u)$ degenerates along the set of critical points,

$$C := \{ x : Du(x) = 0 \}.$$

The problem is challenging due to the following reasons:

(C1) The R.H.S. of (1.1) is a function of measure of superlevel sets. This makes the problem nonlocal.

(C2) The L.H.S. of (1.1) is degenerate. The fundamental theory of $L^p$-viscosity solutions does not work directly here since it requires the uniform ellipticity of the operator. Also, when $f \in C(\Omega)$, the problem can be discussed in the $C$-viscosity sense but in the case of discontinuous data, when $f \in L^p(\Omega)$, the problem needs to be treated in the $L^p$-viscosity sense. We point out that this situation occurs while approximating the R.H.S. of (1.1).

We use the $L^p$-viscosity solution approach for Monge–Ampère equation as in [1, 8] to (1.1). To handle the above mentioned challenges, we first consider the following approximate problem:

$$\begin{cases}
|Du|^{\gamma}M^{+}_{\lambda,\Lambda}(D^2u(x)) + \varepsilon \Delta u = f(|u \geq u(x)|) & \text{in } \Omega \\
u = g & \text{on } \partial \Omega,
\end{cases}$$

for $\varepsilon > 0$. Further, using the approximations in the R.H.S. of the equation and exploiting the results available for uniform elliptic operators, for instance, Theorem 2.5 and Theorem 2.7 (see next), we establish the existence of a viscosity solution to the approximate problem (1.3). This yields the existence of a viscosity solution to (1.1). More precisely, using the idea of Amadori et al. [1], we first get the existence of a $W^{2,p}$-viscosity solution to the approximate problem (1.3) by invoking Theorem 2.1 [8]. We recall that the estimate established in [8] is not adequate to claim the uniform bound on the $W^{2,p}$-viscosity solution of (1.3). To show the existence of a solution to the original problem (1.1), we seek the uniform bound on the solutions of (1.3), which is crucial in approaching $\varepsilon \to 0^+$. We invoke the Alexandroff–Bakelman–Pucci (ABP) estimates from Caffarelli et al. [7] to sort this issue. These estimates play a crucial role in obtaining uniform bounds on the $W^{2,p}$-viscosity solutions to (1.3).

Throughout the paper, we consider $\Omega$ to be a bounded $C^{1,1}$ domain in $\mathbb{R}^N$, $N \geq 2$.

The main result of this paper is the following:

**Theorem 1.1.** Let $\gamma \geq 1$ be a constant. Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain. Let $f \in C([0, |\Omega|], \mathbb{R})$ be a non-decreasing, non-negative function and $g \in W^{2,p}(\Omega) \cap C(\overline{\Omega})$, $p > N$. Consider the problem

$$\begin{cases}
|Du|^{\gamma}M^{+}_{\lambda,\Lambda}(D^2u(x)) = f(|u \geq u(x)|) & \text{in } \Omega \\
u = g & \text{on } \partial \Omega.
\end{cases}$$

(1.4)
Then, there exists a $W^{2,p}$-viscosity solution of (1.4). Moreover, the solution satisfies the following estimate:

$$\|u\|_{W^{2,p}(\Omega)} \leq C \left( \|u\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f(\text{and } u \geq u(x))\|_{p,\Omega} \right),$$

where $C > 0$ is a constant.

**Remark 1.2.** By Sobolev's embedding theorem we have that the solution is $C^{1,\alpha}(\Omega)$ regular for any $0 < \alpha < 1$.

The organization of the paper is as follows. In Section 2, we recall the basic definitions and several key results used in the ensuing sections of the paper. Section 3 is devoted to the proof of our main result. Here, we sketch the plan of our proof:

(i) Perturb the left-hand side (L.H.S.), i.e., the operator $|Du|^\gamma M_{\lambda,\lambda}^+(D^2u)$ by adding $\varepsilon \Delta u$, for $\varepsilon > 0$. (This makes the problem uniformly elliptic.)

(ii) Fix a Lipschitz function $v$ in the R.H.S. of (1.3).

(iii) Construct a sequence of $L^p$-functions converging to R.H.S. (for fixed Lipschitz function $v$) and obtain a sequence of solutions.

(iv) Obtain the existence of solution to equation pertaining $|Du|^\gamma M_{\lambda,\lambda}^+(D^2u) + \varepsilon \Delta u$ for fixed Lipschitz function $v$ in the R.H.S.

(v) Use Theorem 2.1 [8] (an application of Schaefer fixed point theorem) to show the existence of a solution to (1.3).

(vi) Establish the existence of a $W^{2,p}$-viscosity solution to (1.4).

## 2 Preliminaries

We recall that a continuous mapping $F : S^N \rightarrow \mathbb{R}$ is uniformly elliptic if:

For any $A \in S^N$, where $S^N$ is the set of all $N \times N$ real symmetric matrices, there exist two positive constants $\Lambda \geq \lambda > 0$ s.t.

$$\Lambda \|B\| \leq F(A + B) - F(A) \leq N\Lambda \|B\| \quad \text{for all positive semi-definite } B \in S^N,$$

and $\|B\|$ is the largest eigenvalue of $B$. Here, we have the usual partial ordering: $A \preceq B$ in $S^N$ means that $\langle A\xi, \xi \rangle \leq \langle B\xi, \xi \rangle$ for any $\xi \in \mathbb{R}^N$. In other words, $B - A$ is positive semidefinite.

Let $S \in S^N$, then for the given two parameters $\Lambda \geq \lambda > 0$, Pucci’s maximal operator is defined as follows:

$$M_{\lambda,\lambda}^+(S) := \sum_{e_i \geq 0} \lambda e_i + \lambda \sum_{e_i < 0} e_i,$$

where $\{e_i\}_{i=1}^N$ are the eigenvalues of $S$. This operator is uniformly elliptic and subadditive, that is

$$M_{\lambda,\lambda}^+(A + B) \leq M_{\lambda,\lambda}^+(A) + M_{\lambda,\lambda}^+(B),$$

for $A, B \in S^N$. Clearly, for $\lambda = \Lambda = 1$, $M_{1,1}^+ \equiv \Delta$, the classical Laplace operator.

Next, we recall the notion of a viscosity solution. M. G. Crandall and P.-L. Lions [9] were the first to introduce the concept of a viscosity solution. Now, we recall the definition of continuous viscosity solution to the following equation:

$$|Du|^\gamma F(D^2u(x)) = f \quad \text{in } \Omega, \quad (2.1)$$
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for \( f \in C(\Omega) \).

**Definition 2.1 ([3])**. Let \( u : \overline{\Omega} \rightarrow \mathbb{R} \) be an upper semicontinuous (USC) function in \( \Omega \). Then, \( u \) is called a **viscosity subsolution** of (2.1) if

\[
|D\varphi(x)|^{\gamma} F(D^2\varphi(x)) \geq f(x),
\]

whenever \( \varphi \in C^2(\Omega) \) and \( x \in \Omega \) is a local maximizer of \( u - \varphi \) with \( D\varphi \neq 0 \in \mathbb{R}^N \).

**Definition 2.2 ([3])**. Let \( u : \overline{\Omega} \rightarrow \mathbb{R} \) be a lower semicontinuous (LSC) function in \( \Omega \). Then, \( u \) is called a **viscosity supersolution** of (2.1) if

\[
|D\psi(x)|^{\gamma} F(D^2\psi(x)) \leq f(x),
\]

whenever \( \psi \in C^2(\Omega) \) and \( x \in \Omega \) is a local minimizer of \( u - \psi \) with \( D\psi \neq 0 \in \mathbb{R}^N \).

**Definition 2.3 ([3])**. A continuous function \( u \) is said to be a **viscosity solution** to (2.1) if it is a supersolution as well as subsolution to (2.1).

Let \( h \in L^p(\Omega) \), \( g \in W^{2,p}(\Omega) \cap C(\overline{\Omega}) \) for \( p > N \). Let us consider the problem

\[
\begin{align*}
\left| Du \right|^{\gamma} M_{\lambda,\Lambda}^{+}(D^2 u) &= h \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial \Omega.
\end{align*}
\] (2.2)

We mention that the classical definition of \( W^{2,p} \)-viscosity solution can not be applied for (2.2), due to the lack of uniform ellipticity. Consider the problem:

\[
\begin{align*}
\left| Du \right|^{\gamma} M_{\lambda,\Lambda}^{+}(D^2 u) + \varepsilon \Delta u &= h \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial \Omega,
\end{align*}
\] (2.3)

for \( p \in \mathbb{R}^N \). Motivated by Caffarelli et al. [7] and Ishii et al. [14], we define the \( L^p \)-viscosity subsolution (supersolution) to (2.3) as follows.

**Definition 2.4.** Let \( u \) be an USC (respectively, LSC) function on \( \overline{\Omega} \). We say that \( u \) is an \( L^p \)-viscosity subsolution (respectively, supersolution) to (2.3) if

\[
\text{ess lim inf}_{x \to y} \left( |D\phi(x)|^{\gamma} M_{\lambda,\Lambda}^{+}(D^2\phi(x)) + \varepsilon \Delta\phi(x) - h(x) \right) \geq 0
\]

\[
\left( \text{resp., ess lim sup}_{x \to y} \left( |D\phi(x)|^{\gamma} M_{\lambda,\Lambda}^{+}(D^2\phi(x)) + \varepsilon \Delta\phi(x) - h(x) \right) \leq 0 \right),
\]

for \( y \in \Omega \), the point of local maxima (respectively, minima) to \( u - \phi \).

We say that any continuous function \( u \) is an \( L^p \)-viscosity solution to (2.3) if it is both \( L^p \)-viscosity subsolution and supersolution to (2.3). Now, we state a result concerning the existence and uniqueness of \( W^{2,p} \)-viscosity solution to the operator \( F \) under certain hypotheses. The following result is due to N. Winter [27].
**Theorem 2.5** ([27, Theorem 4.6]). Let $\Omega$ be a bounded $C^{1,1}$ domain in $\mathbb{R}^N$. Let $F(p,M)$ be a uniformly elliptic operator and convex in $M$-variable. Also, let $F(0,0) \equiv 0$ in $\Omega$, $f \in L^p(\Omega)$ and $g \in W^{2,p}(\Omega)$ for $p > N$. Then, there exists a unique $W^{2,p}$-viscosity solution to

$$
\begin{cases}
F(Du,D^2u) = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega.
\end{cases}
$$

Moreover, $u \in W^{2,p}(\Omega)$ and

$$
\|u\|_{W^{2,p}(\Omega)} \leq C(\|u\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f\|_{p,\Omega}),
$$

for some positive constant $C$.

**Theorem 2.6** ([2, Theorem 1.1]). Let $\Omega$ be a bounded domain with $C^2$-boundary. Let $\gamma \geq 0$ and $F$ be a uniformly elliptic operator and $f \in C(\overline{\Omega})$, $g \in C^{1,\beta}(\partial \Omega)$ for some $\beta \in (0,1)$. Then, any viscosity solution $u$ of

$$
\begin{cases}
|Du|^\gamma F(D^2u) = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}
$$

is in $C^{1,\alpha}$ for some $\alpha = \alpha(\lambda, \Lambda, \|f\|_{\infty,\Omega}, N, \Omega, \gamma, \beta)$. Moreover, $u$ satisfies the following estimate

$$
\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\left(\|g\|_{C^{1,\beta}(\partial \Omega)} + \|u\|_{\infty,\Omega} + \|f\|_{p,\Omega}^\frac{1}{\gamma}\right),
$$

for some positive constant $C = C(\alpha)$.

The following result plays an important role in Step 5 of the proof of our main result.

**Theorem 2.7** ([7, Theorem 3.8]). Let $F_i$, $F$ be uniformly elliptic and $p > N$. Let $f, f_i \in L^p(\Omega)$. Let $u_i \in C(\Omega)$ be $W^{2,p}$-viscosity subsolutions (supersolutions) to

$$
F_i(D^2u_i) = f_i \quad \text{in } \Omega,
$$

for $i = 1, 2, \ldots$. Assume that $u_i \to u$ locally uniformly in $\Omega$. Also, assume that if for each $B(x_0, r) \subset \Omega$ and $g \in W^{2,p}(B(x_0, r))$, we have

$$
\| (F_i(D^2u_i) - f_i(x) - F(D^2u)) + f(x) \|^+_p \to 0, \quad \| (F_i(D^2u_i) - f_i(x) - F(D^2u)) - f(x) \|^-_p \to 0.
$$

Then, $u$ is a $W^{2,p}$-viscosity subsolution (supersolution) to

$$
F(D^2u) = f \quad \text{in } \Omega.
$$

### 3 Proof of our main result

**Proof of Theorem 1.1.** The original problem is

$$
\begin{cases}
|Du|^\gamma \mathcal{M}^{+}_{\gamma,L}(D^2u(x)) = f(|u \geq u(x)|) & \text{in } \Omega \\
u = g & \text{on } \partial \Omega.
\end{cases}
$$

(3.1)
Step 1: Perturbing the L.H.S. by adding $\varepsilon \Delta u$. Consider the approximate problem:

$$\begin{cases}
|Du|^{r} \mathcal{M}_{\lambda,\Lambda}^{+} (D^{2}u) + \varepsilon \Delta u = f(|u| \geq u(x)) & \text{in } \Omega \\
u = g & \text{on } \partial \Omega,
\end{cases}$$

(3.2)

for $\varepsilon > 0$. Since $Gu := |Du|^{r} \mathcal{M}_{\lambda,\Lambda}^{+} (D^{2}u) + \varepsilon \Delta u$ is uniformly elliptic, so by Theorem 2.1 [8], we immediately have the existence of a $W^{2,p}$-viscosity solution (say $u_{\varepsilon}$) to (3.2) satisfying the following estimate:

$$\|u_{\varepsilon}\|_{W^{2,p}(\Omega)} \leq C \left( \|u_{\varepsilon}\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f(|u_{\varepsilon}| \geq u_{\varepsilon}(x))\|_{p,\Omega} \right).$$

By the above estimate, one can not directly claim the uniform bound on $u_{\varepsilon}$, which is crucial in order to pass the limit $\varepsilon \to 0$ to establish the existence of $W^{2,p}$-viscosity solution to (3.1). To overcome this difficulty, we further approximate problem (3.2).

Step 2: Freeze a Lipschitz function $v$ for the R.H.S.. Next, following the arguments similar to [8], we fix a Lipschitz function $v$ in $\Omega$, and consider $h_{v}(x) := f(|v \geq v(x)|)$ and reduce to the following problem:

$$\begin{cases}
|Du|^{r} \mathcal{M}_{\lambda,\Lambda}^{+} (D^{2}u) + \varepsilon \Delta u = h_{v} & \text{in } \Omega \\
u = g & \text{on } \partial \Omega.
\end{cases}$$

(3.3)

We approximate the function $h_{v}(x) (= f(|v \geq v(x)|))$ in the R.H.S. of (3.3) by the sequence of functions $\{h_{v}^{i}\}_{i=1}^{\infty}$. Hence, we have the following approximate problem:

$$\begin{cases}
|Du|^{r} \mathcal{M}_{\lambda,\Lambda}^{+} (D^{2}u) + \varepsilon \Delta u = h_{v}^{i} & \text{in } \Omega \\
u = g & \text{on } \partial \Omega,
\end{cases}$$

(3.4)

for $i \geq 1$. Since $\{h_{v}^{i}\} \in L^{p}(\Omega)$. For each $i$, by Theorem 2.5, we have the existence of a unique $W^{2,p}$-viscosity solution to (3.4).

Lemma 3.1. There exists a unique $W^{2,p}$-viscosity solution to (3.4). Moreover, it satisfies the following estimate:

$$\|u_{\varepsilon}^{i}\|_{W^{2,p}(\Omega)} \leq C \left( \max_{\partial \Omega} g + \|g\|_{W^{2,p}(\Omega)} + f(|\Omega|)|\Omega|^{\frac{1}{p}} \right).$$

Proof. By Theorem 2.5, we have the existence of a unique $W^{2,p}$-viscosity solution $u_{\varepsilon}^{i}$ to (3.4) satisfying the following estimate:

$$\|u_{\varepsilon}^{i}\|_{W^{2,p}(\Omega)} \leq C \left( \|u_{\varepsilon}^{i}\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|h_{v}^{i}\|_{p,\Omega} \right).$$

(3.5)

Also, it is easy to observe that

$$\|h_{v}^{i}\|_{\infty,\Omega} \leq f(|\Omega|),$$

(3.5)
and
\[
\|h_i^\varepsilon\|_{p,\Omega} = \left( \int_\Omega |h_i^\varepsilon(x)|^p\,dx \right)^{\frac{1}{p}} 
\leq \|h_i^\varepsilon\|_{\infty,\Omega}|\Omega|^{\frac{1}{p}} 
\leq f(|\Omega|)|\Omega|^{\frac{1}{p}},
\]  
(3.6)

for each \(i \geq 1\). Thus the sequence of functions \(h_i^\varepsilon\) is uniformly bounded. Now, by ABP estimates established in [7], we have
\[
\sup_{\Omega} u_i^\varepsilon \leq \sup_{\partial \Omega} u_i^\varepsilon + C\|h_i^\varepsilon\|_{p,\Omega},
\]
and similarly for the \(\inf_{\Omega} u_i^\varepsilon\). For more details, see Proposition 3.3 [7]. Using this along with the estimates (3.5) and (3.6), we have the following:
\[
\|u_i^\varepsilon\|_{W^{2,p}(\Omega)} \leq \tilde{C},
\]
where \(\tilde{C}\) is a positive constant independent of \(i\) and \(\varepsilon\).

**Step 4: Establish the existence of solution to (3.3).** It further gives that \(\{u_i^\varepsilon\}\) is uniformly bounded in \(W^{2,p}(\Omega)\) (with respect to \(i\)). Now, by reflexivity of \(W^{2,p}(\Omega)\), \(u_i^\varepsilon\) converges weakly in \(W^{2,p}(\Omega)\). Moreover, since \(p > N/2\). Using the similar arguments as above, we have the existence of a subsequence such that \(u_i^\varepsilon \rightharpoonup u_{\varepsilon,p}\) in the Lipschitz norm. As a consequence of Theorem 2.7, \(u_{\varepsilon,p}\) is a \(W^{2,p}\)-viscosity solution to (3.3). Moreover, \(u_{\varepsilon,p}\) satisfies the following estimate:
\[
\|u_{\varepsilon,p}\|_{W^{2,p}(\Omega)} \leq C\left( \max_{\partial \Omega} |g| + \|g\|_{W^{2,p}(\Omega)} + \|f(\{u \geq v(x)\})\|_{p,\Omega} \right).
\]

**Step 5: Establish the existence of solution to (3.2).** Further, using Theorem 2.1 [8] (an application of Schaefer fixed point theorem), we have the existence of a \(W^{2,p}\)-viscosity solution to (3.2) for each \(0 < \varepsilon < 1\), say \(u_\varepsilon\) (a Lipschitz fixed point). Moreover, \(u_\varepsilon\) satisfies the following estimate:
\[
\|u_\varepsilon\|_{W^{2,p}(\Omega)} \leq C\left( \max_{\partial \Omega} |g| + \|g\|_{W^{2,p}(\Omega)} + \|f(\{u \geq u(x)\})\|_{p,\Omega} \right).
\]
(3.7)

**Step 6: Establish the existence of solution to (3.1) on \(\varepsilon \rightarrow 0\).** Since \(u_\varepsilon\) is uniformly bounded in \(W^{2,p}(\Omega)\) (with respect to \(\varepsilon\)) so we have that along some subsequence, \(u_\varepsilon\) converges weakly in \(W^{2,p}(\Omega)\). Moreover, by the Rellich–Kondrasov theorem, along some subsequence \(u_\varepsilon \rightharpoonup u\) in \(C(\overline{\Omega})\) (since \(p > N\)) to a Lipschitz function \(u\). We further claim that \(u\) is an \(L^p\)-viscosity solution to (3.1). We use the idea of [1]. We just check the subsolution part. Further, one can check for the subsolution part using the similar arguments. Let, if possible, assume that \(u\) is not an \(L^p\)-viscosity supersolution to (3.1). Then by definition, there exists a point \(x_0 \in \Omega\) and a function \(\phi \in W^{2,p}(\Omega)\) with \(D\phi \neq 0\) such that \(u - \phi\) has local minimum at \(x_0\) and
\[
|D\phi|^\gamma M_{\lambda,\Lambda}^+(D^2\phi) - f(|u \geq u(x_0)|) \geq \alpha \text{ a.e. in some ball } B(x_0, r),
\]
(3.8)
for some constant \(\alpha > 0\). In other words, \(u - \phi\) restricted to \(\overline{B(x_0, r)}\) has a global strict minimum at \(x_0\). Next, using the above information, we get a contradiction by constructing a function \(\phi_\varepsilon = \phi - \psi_\varepsilon\) corresponding to \(u_\varepsilon\) such that
\[
|D\phi_\varepsilon|^\gamma M_{\lambda,\Lambda}^+(D^2\phi_\varepsilon) + \varepsilon \Delta \phi_\varepsilon - f(|u \geq u(x_0)|) \geq \alpha \text{ a.e. in } B(x_0, r)
\]
(3.9)
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for small enough \( \varepsilon > 0 \) and

\[
\phi_\varepsilon \rightarrow \phi \quad \text{uniformly.}
\]

Now, since \( u_\varepsilon \) is an \( L^p \)-viscosity solution to (3.2) so (3.9) implies that \( u_\varepsilon - \phi_\varepsilon \) can not attain minimum in the ball \( B(x_0, r) \). However, since \( u_\varepsilon - \phi_\varepsilon \) is continuous and \( B(x_0, r) \) is compact. Therefore, \( u_\varepsilon - \phi_\varepsilon \) attains minimum in \( B(x_0, r) \). Let it be \( x_\varepsilon \). It gives that \( x_\varepsilon \rightarrow x_0 \) along some subsequence. It further implies that \( x_\varepsilon \in B(x_0, r) \) for small enough \( \varepsilon \), which is a contradiction.

Thus such a function \( \phi \) constructed in (3.8) does not exist, which proves our claim that \( u \) is an \( L^p \)-supersolution to (3.1). Similarly, one can check the subsolution part.

Next, we show that \( u \) is the limit function of the sequence of functions \( u_\varepsilon \) as \( \varepsilon \rightarrow 0 \). Let if possible, \( \varepsilon_i \) and \( \varepsilon_{i+1} \) be two sequences approaching 0 with \( u \) and \( \tilde{u} \) being the corresponding limit functions to the sequences, respectively. Up to subsequences, we may assume that

\[
\cdots \leq \tilde{\varepsilon}_{i+1} \leq \varepsilon_i \leq \tilde{\varepsilon}_i \leq \varepsilon_{i-1} \leq \cdots .
\]

Our aim is to show that \( w = u_{\varepsilon_i} - u_{\varepsilon_{i+1}} \leq 0 \). If we show that

\[
|Dw|^\gamma \mathcal{M}_{\lambda, \Lambda}^+(D^2w) + \varepsilon \Delta w \geq 0 \quad \text{in } \Omega \quad \text{(in C-viscosity sense)},
\]

we are done. As by comparison principle, we would immediately get \( w \leq 0 \). Therefore, \( u_{\varepsilon_i} \leq u_{\varepsilon_{i+1}} \). Thus, in order to show that

\[
w = u_{\varepsilon_i} - u_{\varepsilon_{i+1}} \leq 0,
\]

we only need to show that

\[
|Dw|^\gamma \mathcal{M}_{\lambda, \Lambda}^+(D^2w) + \varepsilon_i \Delta w \geq 0.
\]

As shown above, it immediately gives \( w \leq 0 \). Let us assume the contrary, i.e., there exists some point \( x_0 \in \Omega \) such that for some \( \varphi \in C^2(\Omega) \), \( w - \varphi \) attains local maxima at \( x_0 \), i.e., there exists a ball \( B(x_0, r) \) such that

\[
|D\varphi|^\gamma \mathcal{M}_{\lambda, \Lambda}^+(D^2\varphi) + \varepsilon_i \Delta \varphi \leq -\alpha \quad \text{in } B(x_0, r),
\]

for some \( \alpha > 0 \) and \( w - \varphi = (u_{\varepsilon_i} - u_{\varepsilon_{i+1}}) - \varphi = u_{\varepsilon_i} - (u_{\varepsilon_{i+1}} + \varphi) \) has a global strict maximum at \( x_0 \) in \( B(x_0, r) \). Now, consider a function

\[
\Psi := \varphi + u_{\varepsilon_{i+1}}.
\]

Clearly, \( \Psi \in W^{2,p}(\Omega) \) and touches \( u_{\varepsilon_i} \) from above at \( x_0 \). Also, consider a test function, \( \Phi \) for
$u_{\xi_{i+1}}$ touching from below with $|D\Phi(x_0)|$ sufficiently larger than $|D\varphi(x_0)|$. We have

$$
|D\Psi(x_0)\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2\Psi(x_0)) + \varepsilon_i \Delta \Psi(x_0) - f(|u \geq u(x_0)|) + \alpha
\leq |D\Psi(x_0)\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi(x_0)) + \mathcal{M}^+_{\lambda,\Lambda}(D^2\Phi(x_0)) + \varepsilon_i \Delta \varphi(x_0)
+ \varepsilon_i \mathcal{D}(u) - f(|u \geq u(x_0)|) + \alpha
= |D\Psi(x_0)\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi(x_0)) + \varepsilon_i \Delta \varphi(x_0) + |D\Psi(x_0)\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2\Phi(x_0))
+ \varepsilon_i \Delta \Phi(x_0) - f(|u \geq u(x_0)|) + \alpha
\leq \frac{|D\varphi(x_0) + D\Phi(x_0)|}{|D\Phi(x_0)|} \left(-\alpha - \varepsilon_i \Delta \varphi(x_0)\right) + \varepsilon_i \Delta \varphi(x_0) + \alpha
+ \frac{2^{i-1}|D\varphi(x_0)| + |D\Phi(x_0)|}{|D\Phi(x_0)|} \left(f(|u \geq u(x_0)|) - \varepsilon_i \Delta \Phi(x_0)\right) + \varepsilon_i \Delta \Phi(x_0)
\left(-\alpha - \varepsilon_i \Delta \varphi(x_0)\right) \left( \frac{|D\varphi(x_0) + D\Phi(x_0)|}{|D\Phi(x_0)|} - 1 \right)
+ \left(f(|u \geq u(x_0)|) - \varepsilon_i \Delta \Phi(x_0)\right) \left(2^{i-1} \frac{|D\varphi(x_0)| + |D\Phi(x_0)|}{|D\Phi(x_0)|} - 1 \right),
$$

(3.10)

for all large enough $i \in \mathbb{N}$. Note that in the second last step we used the fact that for any positive real numbers $a, b$ and $r \geq 1$, we have

$$
|a + b|^r \leq 2^{i-1}(|a|^r + |b|^r).
$$

Further, by the choice of test function $\Phi$ made before (3.10), we have

$$
\mathcal{M}^+_{\lambda,\Lambda}(D^2\Psi) + \varepsilon_i \Delta \Psi - f(|u \geq u(x_0)|) \leq -\alpha < 0,
$$

which contradicts the fact that $u_{\xi_i}$ is an $L^p$-viscosity solution to (3.2). Thus we have that $u_{\xi_i} \leq u_{\xi_{i+1}}$. Letting $i \to \infty$, we get $u \leq \bar{u}$. Also, following the similar arguments, one can show that $u_{\xi_{i+1}} \leq u_{\xi_i}$. Thus, we have $\bar{u} \leq u$ and hence $u = \bar{u}$.

Therefore, we have the existence of a $W^{2,p}$-viscosity solution, $u$ to (3.1). Moreover, by (3.7), we have the following estimate:

$$
\|u\|_{W^{2,p}(\Omega)} \leq C \left(\|u\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f(|u \geq u(x)|)\|_{p,\Omega}\right).
$$

\[ \square \]

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