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Schrödinger–Hardy system without Ambrosetti–Rambinowitz condition on Carnot groups

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Abstract. In this paper, we study the following Schrödinger–Hardy system

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{r(\xi)^2} u = F_u(\xi, u, v) & \text{in } \Omega, \\ -\Delta_{\mathbb{G}} v - \nu \frac{\psi^2}{r(\xi)^2} v = F_v(\xi, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain on Carnot groups $\mathbb{G}$, whose homogeneous dimension is $Q \geq 3$, $\Delta_{\mathbb{G}}$ denotes the sub-Laplacian operator on $\mathbb{G}$, μ and ν are real parameters, $r(\xi)$ is the natural gauge associated with fundamental solution of $-\Delta_{\mathbb{G}}$ on $\mathbb{G}$, ψ is the geometrical function defined as $\psi = |\nabla_{\mathbb{G}} r|$, and $\nabla_{\mathbb{G}}$ is the horizontal gradient associated with $\Delta_{\mathbb{G}}$. The difficulty is not only the nonlinearities F_u and F_v without Ambrosetti–Rabinowitz condition, but also the hardy terms and the structure on Carnot groups. We obtain the existence of nonnegative solution for this system by mountain pass theorem in a new framework.

Keywords: Schrödinger–Hardy system, without Ambrosetti–Rabinowitz condition, Carnot groups, mountain pass theorem.

2020 Mathematics Subject Classification: 35H20, 35J47, 35J50.

1 Introduction and main results

In this paper, we consider the following Schrödinger–Hardy system

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{r(\xi)^2} u = F_u(\xi, u, v) & \text{in } \Omega, \\ -\Delta_{\mathbb{G}} v - \nu \frac{\psi^2}{r(\xi)^2} v = F_v(\xi, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain on Carnot groups $\mathbb{G}$, whose homogeneous dimension is $Q \geq 3$, $\Delta_{\mathbb{G}}$ denotes the sub-Laplacian operator on $\mathbb{G}$, μ and ν are real parameters, $r(\xi)$ is

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the natural gauge associated with fundamental solution of $-\Delta_{\mathbb{G}}$ on $\mathbb{G}$, ψ is the geometrical function defined as $\psi = |\nabla_{\mathbb{G}} r|$, and $\nabla_{\mathbb{G}}$ is the horizontal gradient associated with $\Delta_{\mathbb{G}}$. The difficulty in this paper is not only the nonlinearities F_u and F_v without Ambrosetti–Rabinowitz condition, but also the hardy terms and the structure on Carnot groups.

In the context of stratified groups, the problem has been intensively studied in last decades, starting with the pioneering papers [21,22]. In particular, a number of literatures are related to Heisenberg group, such as [4,15,16,23,35,36] and references therein. Only few results concern the general Carnot setting. For related topics, see [2,3,11,31,37] and references therein.

We mention that Ferrara et al. [17] obtained the existence of a weak solution for the following problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = \lambda f(\xi, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{G}$, $\lambda > 0$ is a real parameter, and f is a subcritical nonlinearity. For critical exponent subelliptic problem,

$$\begin{cases} -\Delta_{\mathbb{G}} u = |u|^{2^*-2} u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with $2^* = \frac{2Q}{Q-2}$. When $f = 0$, problem (1.2) does not admit any nonnegative non trivial solution on star-shaped domain, see [21, 22]. If Ω is a bounded domain of $\mathbb{G}$, Loiudice [27] established the existence of positive and sign changing solutions for $f = \lambda u$, extending the famous Brezis–Nirenberg results [8] to the subelliptic Carnot setting. Subsequently, by Nehari manifold and Ekeland variational principle, Loiudice [32] considered the general non-homogeneous problem (1.2) and proved the existence of at least two positive solutions, provided that non-homogeneous term f satisfies suitable assumptions.

Concerning the problem for sub-Laplacian operator involving critical Hardy–Sobolev nonlinearity

$$\begin{cases} -\Delta_{\mathbb{G}} u = \frac{\psi^{\alpha}}{r(\xi)^{\alpha}} |u|^{2^*(\alpha)-2} u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{G}$ is a bounded domain, $0 < \alpha < 2$, $2^*(\alpha) = \frac{2(Q-\alpha)}{Q-2}$ is the critical Sobolev–Hardy exponent, Loiudice [29] proved that if $\lambda = 0$, there is no nonnegative nontrivial solutions when Ω is a bounded star-shaped domain about the origin with respect to dilations of the group. Also, the existence of solution was established provided that $\lambda > 0$.

For more general nonlinearity with Hardy type potential, that is

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{r(\xi)^2} u = f(\xi, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is an open subset of $\mathbb{G}$, $0 \in \Omega$, $0 \le \mu < (\frac{Q-2}{2})^2$, the function f satisfies $f(\xi, u) \le C(|u| + |u|^{2^*-1})$, $\forall (\xi, u) \in \Omega \times \mathbb{R}$ and $C > 0$ is a constant. By L^p regularity of solutions and Moser's iteration, Loiudice [30] showed that any positive solution of (1.3) has a stronger singularity as $\mu \to (\frac{Q-2}{2})^2$. When f is a purely critical nonlinearity, that is $f(u) = |u|^{2^*-2}u$, the behavior of solutions at origin shows the decay of solution at infinity by Kelvin transform on $\mathbb{R}^n$ in Euclidean setting. However, this technique fails in Carnot group, because there does not exist a suitable inversion with good conformal properties. We point out this technique is true for a special subclass of stratified groups, that is the Iwasawa-type groups $\mathbb{H}$. Loiudice [28] showed that if $u \in \mathcal{S}_0^1(\Omega)$ is a solution to

$$-\Delta_{\mathbb{H}} u - \mu \frac{\psi^2}{r(\xi)^2} u = |u|^{2^*-2} u \quad \text{in } \Omega,$$

there is $C_1 > 0$ such that

$$|u(\xi)| \leq C_1 r(\xi)^{-\sqrt{\mu_{\mathbb{H}}} - \sqrt{\mu_{\mathbb{H}} - \mu}}, \quad \text{for } r(\xi) \text{ large}.$$

Moreover, if u is positive, there exists $C_2 > 0$ such that

$$|u(\xi)| \geq C_2 r(\xi)^{-\sqrt{\mu_{\mathbb{H}}} - \sqrt{\mu_{\mathbb{H}} - \mu}}, \quad \text{for } r(\xi) \text{ large},$$

where $\mathcal{S}_0^1(\Omega)$ is the Folland–Stein space, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{S}_0^1(\Omega)} = \Big(\int_\Omega |\nabla_{\mathbb{G}} u|^2 d\xi \Big)^{\frac{1}{2}},$$

and $\mu_{\mathbb{H}} = (\frac{Q-2}{2})^2$ is the best constant in Hardy inequality on Iwasawa-type groups,

$$\mu_{\mathbb{H}} \int_{\mathbb{H}} \psi^2 \frac{|u|^2}{r(\xi)^2} d\xi \leq \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^2 d\xi, \quad \forall u \in C_0^\infty(\mathbb{H}),$$

and it is never attained, some more details can be seen in [5, 10]. Moreover, this result was extended to the whole Carnot groups in [33] by using different methods, and Loiudice investigated the existence and nonexistence for subelliptic Brezis–Nirenberg type problem as follows

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{r(\xi)^2} u = u^{2^*-1} + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Concerning the results in the whole Carnot group, Zhang [39] considered the following equation

$$-\Delta_{\mathbb{G}} u = \lambda \frac{\psi^\alpha}{r(\xi)^\alpha} |u|^{2^*(\alpha)-2} u + \beta f(\xi) |u|^{p-2} u \quad \text{in } \mathbb{G},$$

where λ, $\beta > 0$ are parameters, $0 < \alpha \leq 2$, Zhang proved the existence and multiplicity of solutions by variational methods and the theory of genus. Concerning multiple Hardy nonlinearities, Zhang [38] proved the attainability of best Sobolev–Hardy constant of

$$S_{\mu,\alpha} = \inf_{u \in \mathcal{S}^1(\mathbb{G}) \setminus \{0\}} \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 d\xi - \mu \int_{\mathbb{G}} \frac{\psi(\xi)^2}{r(\xi)^2} |u|^2 d\xi}{\Big(\int_{\mathbb{G}} \frac{\psi(\xi)^\alpha}{r(\xi)^\alpha} |u|^{2^*(\alpha)} d\xi \Big)^{\frac{2}{2^*(\alpha)}}}.$$

Moreover, as an application, by variational methods and local compactness of Palais–Smale sequences, Zhang obtained the existence of nontrivial weak solution to the following singularity sub-elliptic equation and system

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi(\xi)^2}{r(\xi)^2} u = \frac{\psi(\xi)^\alpha}{r(\xi)^\alpha} |u|^{2^*(\alpha)-2} u + \frac{\psi(\xi)^\beta}{r(\xi)^\beta} |u|^{2^*(\beta)-2} u \quad \text{in } \mathbb{G},$$

and

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \frac{\psi(\xi)^2}{r(\xi)^2} u = \frac{\psi(\xi)^\alpha}{r(\xi)^\alpha} |u|^{2^*(\alpha)-2} u + \frac{\lambda \eta}{\eta+\theta} \frac{\psi(\xi)^\alpha}{r(\xi)^\alpha} |u|^{\eta-2} u |v|^\theta & \text{in } \mathbb{G}, \\ -\Delta_{\mathbb{G}} v - \mu \frac{\psi(\xi)^2}{r(\xi)^2} v = \frac{\psi(\xi)^\alpha}{r(\xi)^\alpha} |v|^{2^*(\alpha)-2} v + \frac{\lambda \eta}{\eta+\theta} \frac{\psi(\xi)^\alpha}{r(\xi)^\alpha} |u|^{\eta} |v|^{\theta-2} v & \text{in } \mathbb{G}, \end{cases}$$

where $0 \leq \alpha, \beta < 2$ and $\eta, \theta > 1$ with $\eta + \theta = 2^*(\alpha)$, $\lambda > 0$ is a parameter. Further, the problems with Hardy potential have been considered by [24] and [6,7,34,35] for Hardy nonlinearity in Heisenberg group. In particular, we mention that Bordoni and Pucci [6] first proved the existence of nontrivial nonnegative solutions of the Schrödinger system including multiple critical nonlinearities and Hardy potentials in Heisenberg groups.

In order to deal with (1.1), we introduce the Sobolev-type inequality: there exists a positive constant $C > 0$ such that

$$\int_\Omega |u|^{2^*} d\xi \leq C \Big(\int_\Omega |\nabla_{\mathbb{G}} u|^2 d\xi \Big)^{\frac{2^*}{2}}, \quad \forall u \in C_0^\infty(\Omega), \tag{1.4}$$

where 2^* is the critical exponent for $\Delta_{\mathbb{G}}$, the embedding $\mathcal{S}_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $1 \leq q < 2^*$ but only continuous for $q = 2^*$, and the Hardy-type inequality is: for every $u \in C_0^\infty(\Omega)$, there holds

$$\Big(\frac{Q-2}{2} \Big)^2 \Big(\int_\Omega \frac{\psi^2}{r(\xi)^2} |u|^2 d\xi \Big) \leq \int_\Omega |\nabla_{\mathbb{G}} u|^2 d\xi, \tag{1.5}$$

where $\big(\frac{Q-2}{2}\big)^2$ is the optimal constant but never attained (see [12,20]). (1.5) is first proved by Garofalo and Lanconelli [20] in Heisenberg group, then, D'Ambrosio [12] extended this result to all Carnot groups. Moreover, the best Hardy constant $\mathcal{K} > 0$ of (1.5) is given by

$$\mathcal{K} = \inf_{u \in \mathcal{S}_0^1(\Omega), u \neq 0} \frac{\|u\|^2_{\mathcal{S}_0^1(\Omega)}}{\|u\|^2_\psi} \quad \text{with} \quad \|u\|^2_\psi = \int_\Omega \frac{\psi^2}{r(\xi)^2} |u|^2 d\xi. \tag{1.6}$$

Now, let us define a suitable solution space $W = \mathcal{S}_0^1(\Omega) \times \mathcal{S}_0^1(\Omega)$, which is a separable, reflexive Banach space and endowed with the norm

$$\|(u,v)\| = \Big(\|u\|^2_{\mathcal{S}_0^1(\Omega)} + \|v\|^2_{\mathcal{S}_0^1(\Omega)} \Big)^{\frac{1}{2}}, \tag{1.7}$$

we denote

$$\|(u,v)\|_p = \Big(\int_\Omega |(u,v)|^p d\xi \Big)^{\frac{1}{p}} = \Big(\int_\Omega |(u^2+v^2)^{\frac{1}{2}}|^p d\xi \Big)^{\frac{1}{p}},$$

for $1 \leq p < \infty$, and let

$$\lambda^* = \inf_{(u,v) \in W \setminus \{(0,0)\}} \frac{\|(u,v)\|^2}{\|(u,v)\|_2^2} > 0.$$

Throughout the paper, we assume that $F(\xi, u, v) : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $F(\xi, 0, 0) = 0$ in Ω, and it satisfies the following assumptions.

(f_1) The partial derivatives F_u, $F_v \in C(\Omega \times \mathbb{R}^2)$, $F(\xi, u, v) \geq 0$ in $\Omega \times \mathbb{R}^2$. Moreover, for each $\xi \in \Omega$,

$$F_u(\xi, u, v) = 0 \begin{cases} \text{if } u \leq 0 \text{ and } v \in \mathbb{R}, \\ \text{if } v \leq 0 \text{ and } u \in \mathbb{R}. \end{cases}$$

(f_2) There exists $s \in (2, 2^*)$ and $\lambda \in [0, \lambda^*)$, then for each $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that
$$|F_w(\xi, w)| \leq (\lambda + \epsilon)|w| + C_\epsilon |w|^{s-1},$$
for every $(\xi, w) \in \Omega \times \mathbb{R}^2$, $w = (u, v)$, $|w| = \sqrt{u^2 + v^2}$, where $F_w = (F_u, F_v)$.

(f_3) $\lim\limits_{|w|\to\infty} \frac{2F(\xi,w)}{|w|^2} = \infty$, uniformly in Ω.

(f_4) For any $w = (u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $0 < \tau < 1$, there exists a nonnegative function $g \in L^1(\Omega)$ and a constant $C_F \geq 1$ such that $H(\xi, \tau w) \leq C_F H(\xi, w) + g(\xi)$, where $H(\xi, w) = F_w(\xi, w)w - 2F(\xi, w)$.

The main result can be stated as follows.

Theorem 1.1. *Assume that F satisfies (f_1)–(f_4). Then (1.1) has at least a nonnegative solution $(u, v) \in W$ for any μ, $\nu \in (-\infty, \mathcal{K})$ such that*
$$\Theta - \frac{2\lambda}{\lambda^*} > 0, \tag{1.8}$$
where $\lambda \in [0, \lambda^*)$, $\Theta = \min\left\{1 - \frac{\mu^+}{\mathcal{K}}, 1 - \frac{\nu^+}{\mathcal{K}}\right\}$, $\mu^+ = \max\{0, \mu\}$ and $\nu^+ = \max\{0, \nu\}$.

In this paper, the main difficulty is that the energy functional does not satisfy Palais–Smale condition since the nonlinearities F_u and F_v loss the Ambrosetti–Rabinowitz condition, see also [14, 25, 26]. It should be mentioned that the (f_4) plays an important role in proving the boundless of Palais–Smale sequence.

The rest of the paper is organized as follows. In Section 2, we recall the main notations and definitions related to the Carnot groups, and present some preparatory results. In Section 3, we prove that the energy functional satisfies the mountain pass geometry structures. In Section 4, we obtain the compactness theorem and prove the main result. Finally, we show two lemmas in Section 5.

2 The functional setting of Carnot groups

We briefly recall the definitions and notations related to the Carnot groups functional setting. For a complete treatment, we refer to [5, 18, 19].

2.1 The Carnot groups

A Carnot group is a homogeneous group, denoted as $\mathbb{G} = (\mathbb{R}^n, \circ, \mathfrak{F})$, whose Lie algebra $\mathfrak{g}$ is stratified, that is, $\mathfrak{g} = \bigoplus_{i=1}^r V_i$, where $r > 0$ is a integer number and called the *step* of $\mathbb{G}$, $\mathfrak{g}$ is the Lie algebra of left invariant vector fields on $\mathbb{G}$, V_i is a linear subspace of $\mathfrak{g}$, $i = 1, \ldots, r$, and satisfies
$$\begin{aligned} &\dim V_i = n_i, \text{ for } i = 1, \ldots, r, \\ &[V_1, V_i] = V_{i+1}, \text{ for } 1 \leq i \leq r-1, \text{ and } [V_1, V_r] = \{0\}. \end{aligned}$$

From these, we can see that $[V_1, V_i]$ stands for the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$, $Y \in V_i$.

In fact, $(\mathbb{R}^n, \circ)$ is a Lie group equipped with a family of group automorphisms (namely *dilatations*) $\mathfrak{F} := \{\delta_\eta\}_{\eta>0}$ such that, for every $\eta > 0$, the map

$$\delta_\eta : \prod_{i=1}^{r} \mathbb{R}^{n_i} \to \prod_{i=1}^{r} \mathbb{R}^{n_i},$$

shows that $\delta_\eta(\xi^{(1)}, \ldots, \xi^{(r)}) = (\eta\xi^{(1)}, \eta^2\xi^{(2)}, \ldots, \eta^r\xi^{(r)})$, where $\xi^{(i)} \in \mathbb{R}^{n_i}$, $i = 1, \ldots, r$, and $\sum_{i=1}^{r} n_i = n$. The structure $\mathbb{G} = (\mathbb{R}^n, \circ, \mathfrak{F})$ is called a *homogeneous group*, and $Q = \dim_h \mathbb{G} := \sum_{k=1}^{r} kn_k$ is called the *homogeneous dimension* of $\mathbb{G}$. In this paper, we pay attention to $\dim_h \mathbb{G} \geq 3$. In particular, $\mathbb{G}$ is the Euclidean space provided that $\dim_h \mathbb{G} \leq 3$, i.e. $\mathbb{G} = (\mathbb{R}^{\dim_h \mathbb{G}}, +)$.

Let $\{X_j\}_{j=1}^{n_1}$ be a basis of V_1, then the associated subelliptic operator $\Delta_{\mathbb{G}}$ is given by

$$\Delta_{\mathbb{G}} := \sum_{j=1}^{n_1} X_j^2,$$

which is the second order differential operator on $\mathbb{G}$. Here, n_1 is the dimension of the first step, moreover, the subelliptic gradient is $\nabla_{\mathbb{G}} := (X_1, X_2, \ldots, X_{n_1})$. As proved in [18], there exists a suitable homogeneous norm $r(\xi)$, called *gauge norm*, such that $\Gamma(\xi) = \frac{C}{r(\xi)^{Q-2}}$ is the fundamental solution of $-\Delta_{\mathbb{G}}$, where $C > 0$ is a constant. By definition, a homogeneous norm is any continuous function from $\mathbb{G}$ to $[0, +\infty)$ such that for $\eta > 0$, $\xi \in \mathbb{G}$, $r(\delta_\eta(\xi)) = \eta r(\xi)$, $r(\xi^{-1}) = r(\xi)$, $r(\xi) = 0$ if and only if $\xi = 0$.

2.2 Functional setting and preliminary results

In this subsection, we present some useful results and comments, (1.1) has a variational structure and the Euler–Lagrange functional $I_{\mu,\nu} : W \to \mathbb{R}$ is given by

$$I_{\mu,\nu}(u, v) = \frac{1}{2}\|u\|^2_{\mathcal{S}_0^1(\Omega)} + \frac{1}{2}\|v\|^2_{\mathcal{S}_0^1(\Omega)} - \frac{\mu}{2}\|u\|^2_\psi - \frac{\nu}{2}\|v\|^2_\psi - \int_\Omega F(\xi, u, v)d\xi,$$

for all $u, v \in W$. Indeed, $I_{\mu,\nu}$ is well defined and be of class $C^1(W)$ under the assumptions (f_1) and (f_2). A function $(u, v) \in W$ is a weak solution of (1.1) if holds

$$\langle u, \Phi\rangle + \langle v, \Psi\rangle - \mu\langle u, \Phi\rangle_\psi - \nu\langle v, \Psi\rangle_\psi = \int_\Omega \Big(F_u(\xi, u, v)\Phi + F_v(\xi, u, v)\Psi\Big)d\xi,$$

for every $(\Phi, \Psi) \in W$, where

$$\langle u, \Phi\rangle = \int_\Omega (\nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \Phi)d\xi, \qquad \langle v, \Psi\rangle = \int_\Omega (\nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} \Psi)d\xi,$$
$$\langle u, \Phi\rangle_\psi = \int_\Omega \frac{\psi^2}{r(\xi)^2} u\Phi d\xi, \qquad \langle v, \Psi\rangle_\psi = \int_\Omega \frac{\psi^2}{r(\xi)^2} v\Psi d\xi.$$

Moreover, for all $(u, v) \in W$, there holds

$$\begin{aligned}\langle I'_{\mu,\nu}(u, v), (\Phi, \Psi)\rangle &= \langle u, \Phi\rangle + \langle v, \Psi\rangle - \mu\langle u, \Phi\rangle_\psi - \nu\langle v, \Psi\rangle_\psi \\ &\quad - \int_\Omega \Big(F_u(\xi, u, v)\Phi + F_v(\xi, u, v)\Psi\Big)d\xi, \quad \text{for every } (\Phi, \Psi) \in W.\end{aligned}$$

Therefore, the weak solutions of (1.1) are exactly the critical points of $I_{\mu,\nu}$.

Lemma 2.1. *The embedding $W \hookrightarrow L^q(\Omega) \times L^q(\Omega)$ is continuous for $1 \le q \le 2^*$ and $\|(u,v)\|_q \le C_q' \|(u,v)\|$ for all $(u,v) \in W$ and $C_q' > 0$ is a constant.*

Proof. From [19], we know that $\mathcal{S}_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le q \le 2^*$, thus, there is $C_q > 0$ such that

$$\|u\|_q \le C_q \|u\|_{\mathcal{S}_0^1(\Omega)} \quad \text{and} \quad \|v\|_q \le C_q \|v\|_{\mathcal{S}_0^1(\Omega)}.$$

Moreover, by (f_2), (1.7) and a fact $a + b \le \sqrt{2(a^2+b^2)}$ for each $a, b \in \mathbb{R}$, there holds

$$\begin{aligned} \|(u,v)\|_q &= \|\sqrt{u^2+v^2}\|_q \le \|\sqrt{(u+v)^2}\|_q \le \|u\|_q + \|v\|_q \\ &\le C_q(\|u\|_{\mathcal{S}_0^1(\Omega)} + \|v\|_{\mathcal{S}_0^1(\Omega)}) \le C_q \sqrt{2(\|u\|^2_{\mathcal{S}_0^1(\Omega)} + \|v\|^2_{\mathcal{S}_0^1(\Omega)})} = C_q' \|(u,v)\|. \end{aligned}$$

This proof is finished. □

Lemma 2.2 ([1])**.** *Let $\{(u_k, v_k)\} \subset W$ be such that $(u_k, v_k) \rightharpoonup (u,v)$ weakly in W as $k \to \infty$, then up to a subsequence, $(u_k, v_k) \to (u,v)$ a.e. in Ω as $k \to \infty$.*

Lemma 2.3. *Let $\Omega \subset \mathbb{G}$ be a smooth bounded domain, then, the embedding $W \hookrightarrow L^q(\Omega) \times L^q(\Omega)$ is compact when $1 \le q < 2^*$.*

Proof. From [19], it holds that $S_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $1 \le q < 2^*$, that is, if $\{u_k\}$ and $\{v_k\}$ are bounded sequences in $S_0^1(\Omega)$, then there exist $u, v \in W$ such that,

$$u_k \to u \quad \text{and} \quad v_k \to v \quad \text{in} \quad L^q(\Omega).$$

Hence, if $\{(u_k, v_k)\} \subset W$ be a bounded sequence, we have

$$\|(u_k, v_k) - (u,v)\|_q \le \|u_k - u\|_q + \|v_k - v\|_q \to 0.$$

It follows that $\{(u_k, v_k)\}$ strongly in $L^q(\Omega) \times L^q(\Omega)$. □

In the following, we recall the definition of Cerami sequence and Cerami condition.

Definition 2.4. Let $X = (X, \|\cdot\|)$ be a Banach space, X' denotes its dual space, the functional $I : X \to \mathbb{R}$ be of $C^1(X)$.

(i) Cerami sequence: A sequence $u_k \in X$ is called a Cerami sequence if for every $u_k \in X$, $I(u_k)$ is bounded and $(1 + \|u_k\|)\|I'(u_k)\|_{X'} \to 0$ as $k \to \infty$. In particular, $\|I'(u_k)\|_{X'} \to 0$ as $k \to \infty$.

(ii) Cerami condition: A functional I satisfies the Cerami condition if any Cerami sequence associated with I has a strongly convergent subsequence in X.

3 Mountain pass structure

In this section, the results concern the existence of Palais–Smale sequence for $I_{\mu,\nu}$.

Lemma 3.1 ([9])**.** *Let E be a real Banach space, $\mathcal{I} \in C^1(E)$ with $\mathcal{I}(0) = 0$. There are constants $\rho, \tau > 0$ and $e \in E$ with $\|e\|_E > \rho$ such that*

$$\inf_{\|u\|_E = \rho} \mathcal{I}(u) \ge \tau \quad \text{and} \quad \mathcal{I}(e) < 0.$$

Then there is a Cerami sequence $\{u_k\} \subset E$ *such that*

$$\mathcal{I}(u_k) \to c, \quad (1 + \|u_k\|_E)\|I'(u_k)\|_E \to 0,$$

where

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}(\gamma(t)) \geq \tau,$$

and

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

The number c *is called mountain pass level. If the functional* $\mathcal{I}$ *satisfies the Cerami condition at the minimax level* c*, then* c *is a critical value of* $\mathcal{I}$ *in* E*.*

We first show that the energy functional $\mathcal{I}_{\mu,\nu}$ satisfies the geometric structure required by Lemma 3.1.

Lemma 3.2. *Assume that* (f_2) *holds, then there exist* ζ, $\rho > 0$ *such that*

$$I_{\mu,\nu}(u,v) \geq \zeta, \quad \textit{if } \|(u,v)\| = \rho.$$

Proof. Let us set $\chi = \min\{(1 - \frac{\mu^+}{\mathcal{K}}), (1 - \frac{\nu^+}{\mathcal{K}})\}$ and from (f_2), we have

$$\begin{aligned}
I_{\mu,\nu}(u,v) &= \frac{1}{2}\|u\|^2_{\mathcal{S}^1_0(\Omega)} + \frac{1}{2}\|v\|^2_{\mathcal{S}^1_0(\Omega)} - \frac{\mu}{2}\|u\|^2_\psi - \frac{\nu}{2}\|v\|^2_\psi - \int_\Omega F(\xi,u,v)d\xi \\
&\geq \frac{1}{2}\|u\|^2_{\mathcal{S}^1_0(\Omega)}\Big(1 - \frac{\mu^+}{\mathcal{K}}\Big) + \frac{1}{2}\|v\|^2_{\mathcal{S}^1_0(\Omega)}\Big(1 - \frac{\nu^+}{\mathcal{K}}\Big) \\
&\quad - \int_\Omega \Big(\frac{1}{2}(\lambda+\epsilon)|(u,v)|^2 + \frac{1}{s}C_\epsilon|(u,v)|^s\Big)d\xi \\
&\geq \frac{\chi}{2}(\|u\|^2_{\mathcal{S}^1_0(\Omega)} + \|v\|^2_{\mathcal{S}^1_0(\Omega)}) - \frac{1}{2}(\lambda+\epsilon)\|(u,v)\|^2_2 - \frac{1}{s}C_\epsilon\|(u,v)\|^s_s \\
&\geq \frac{\chi}{2}\|(u,v)\|^2 - \frac{1}{2\lambda^*}(\lambda+\epsilon)\|(u,v)\|^2 - \frac{1}{s}C_\epsilon C_s\|(u,v)\|^s \\
&= \frac{1}{2}\Big(\chi - \frac{\lambda+\epsilon}{\lambda^*} - \frac{2}{s}C_\epsilon C_s\|(u,v)\|^{s-2}\Big)\|(u,v)\|^2,
\end{aligned}$$

where $C_s > 0$ is a constant, $\mathcal{K}$ is given in (1.6) and $s \in (2, 2^*)$. Thus, if ρ is small enough such that

$$\chi - \frac{\lambda+\epsilon}{\lambda^*} - \frac{2}{s}C_\epsilon C_s\rho^{s-2} > 0,$$

it holds $I_{\mu,\nu}(u,v) \geq \frac{1}{2}(\chi - \frac{\lambda+\epsilon}{\lambda^*} - \frac{2}{s}C_\epsilon C_s\rho^{s-2})\rho^2 = \zeta > 0$ for all $(u,v) \in W$ with $\|(u,v)\| = \rho$. We obtain this lemma. □

Lemma 3.3. *Suppose that* (f_3) *holds, then there exists* $(\tilde{u},\tilde{v}) \in W$ *with* $\|(\tilde{u},\tilde{v})\| > \rho$ *such that* $I_{\mu,\nu}(\tilde{u},\tilde{v}) < 0$*.*

Proof. It suffices to prove that for a fixed $(u_0,v_0) \in W$, $I_{\mu,\nu}(tu_0,tv_0) \to -\infty$ as $t \to +\infty$. We assume that $(u,v) \in W$ with compact support $\mathbf{D_c}$. From (f_3), there are constants c_1, c_2, $\delta > 0$, such that for $|u|$, $|v| > \delta$, one has

$$F(\xi,u,v) \geq c_1|(u,v)|^2 \geq c_1|(u,v)|^2 - c_2, \quad \text{for } (u,v) \in W.$$

Now, choosing arbitrarily $(u_0, v_0) \in W$ with $u_0, v_0 > 0$, and $\|(u_0, v_0)\| = 1$, hence, for all $t > 0$, we set $\mu^- = \min\{0, \mu\}$ and $\nu^- = \min\{0, \nu\}$, then

$$\begin{aligned} I_{\mu,\nu}(tu_0, tv_0) &\leq \frac{t^2}{2}\Big(\|u_0\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_0\|^2_{\mathcal{S}^1_0(\Omega)} - \mu\|u_0\|^2_\psi - \nu\|v_0\|^2_\psi\Big) - \int_\Omega \Big(c_1|(tu_0, tv_0)|^2 - c_2\Big) d\xi \\ &\leq \frac{t^2}{2}\Big(\|u_0\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_0\|^2_{\mathcal{S}^1_0(\Omega)} + |\mu^-|\|u_0\|^2_\psi + |\nu^-|\|v_0\|^2_\psi - 2c_1\|(u_0, v_0)\|^2_2\Big) + c_2|\mathbf{D_c}|. \end{aligned}$$

If c_1 is large enough, there holds

$$0 < \|u_0\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_0\|^2_{\mathcal{S}^1_0(\Omega)} + |\mu^-|\|u_0\|^2_\psi + |\nu^-|\|v_0\|^2_\psi < 2c_1\|(u_0, v_0)\|^2_2.$$

Therefore, we have $I_{\mu,\nu}(tu_0, tu_0) \to -\infty$ as $t \to \infty$. Setting $(\tilde{u}, \tilde{v}) = (t_0 u_0, t_0 u_0) \in W$, such that $\|(\tilde{u}, \tilde{v})\| > \rho$ and $I_{\mu,\nu}(\tilde{u}, \tilde{v}) < 0$. We obtain this lemma. □

4 Cerami sequence and existence of solutions

4.1 Cerami sequence

In this section, we give an analysis of Cerami sequence and prove that $I_{\mu,\nu}$ satisfies Cerami condition.

Lemma 4.1. *Assume that (f_1)–(f_4) hold, then for each $\mu, \nu \in (-\infty, \mathcal{K})$, any Cerami sequence of $I_{\mu,\nu}$ is bounded in W.*

Proof. Let $\{(u_k, v_k)\} \subset W$ be a Cerami sequence of $I_{\mu,\nu}$, then, there exists $\mathcal{L} > 0$ independent of k such that

$$|I_{\mu,\nu}(u_k, v_k)| \leq \mathcal{L} \quad \text{for all } k, \quad (1 + \|(u_k, v_k)\|) I'_{\mu,\nu}(u_k, v_k) \to 0 \quad \text{as } k \to \infty. \tag{4.1}$$

Thus, there is $\tau_k > 0$ and $\tau_k \to 0$ as $k \to \infty$, such that

$$|\langle I'_{\mu,\nu}(u_k, v_k), (\Phi, \Psi)\rangle| \leq \frac{\tau_k \|(\Phi, \Psi)\|}{1 + \|(u_k, v_k)\|}, \quad \forall (\Phi, \Psi) \in W. \tag{4.2}$$

Let us set $(\Phi, \Psi) = (u_k, v_k)$, then

$$\begin{aligned} &\Big|\langle u_k, u_k\rangle + \langle v_k, v_k\rangle - \mu\langle u_k, u_k\rangle_\psi - \nu\langle v_k, v_k\rangle_\psi - \int_\Omega \Big(F_u(\xi, u_k, v_k)u_k + F_v(\xi, u_k, v_k)v_k\Big) d\xi\Big| \\ &\quad = |\langle I'_{\mu,\nu}(u_k, v_k), (u_k, v_k)\rangle| \leq \frac{\tau_k \|(u_k, v_k)\|}{1 + \|(u_k, v_k)\|} \leq \tau_k \leq C, \end{aligned}$$

that is

$$\begin{aligned} &-\|u_k\|^2_{\mathcal{S}^1_0(\Omega)} - \|v_k\|^2_{\mathcal{S}^1_0(\Omega)} + \mu\|u_k\|^2_\psi + \nu\|v_k\|^2_\psi \\ &\qquad + \int_\Omega \Big(F_u(\xi, u_k, v_k)u_k + F_v(\xi, u_k, v_k)v_k\Big) d\xi \leq C. \end{aligned} \tag{4.3}$$

Now, we prove that (u_k, v_k) is bounded in W. Suppose, by contradiction, $\|(u_k, v_k)\| \to \infty$ as $k \to \infty$. We define a sequence as $(w_k, z_k) = \frac{(u_k, v_k)}{\|(u_k, v_k)\|}$, then, $\|(w_k, z_k)\| = 1$. By Lemmas 2.2 and 2.3, there exists $(w, z) \in W$ such that

$$\begin{aligned} &(w_k, z_k) \rightharpoonup (w, z) \quad \text{weakly in } W, \\ &(w_k, z_k) \to (w, z) \quad \text{strongly in } L^q(\Omega) \times L^q(\Omega) \quad \text{for } q \in [1, 2^*), \\ &(w_k, z_k) \to (w, z) \quad \text{a.e. in } \Omega. \end{aligned} \tag{4.4}$$

We divide the argument into several steps.

Step 1: We prove $w \geq 0$ and $z \geq 0$ *a.e.* in Ω. Let us set $w_k^- = \min\{0, w_k\}$ and $z_k^- = \min\{0, z_k\}$, then (w_k^-, z_k^-) is bounded because (w_k, z_k) in W is bounded. We choose $(\Phi, \Psi) = (w_k^-, z_k^-)$ in (4.2), it follows that

$$o(1) = \frac{|\langle I'_{\mu,\nu}(u_k, v_k), (w_k^-, z_k^-)\rangle|}{\|(u_k, v_k)\|} \quad \text{for } \|(u_k, v_k)\| \to \infty.$$

Therefore, from (f_1), the elementary inequality $|a^- - b^-|^2 \leq (a-b)(a^- - b^-)$, $(a, b \in \mathbb{R})$, (1.5) and a fact that μ, $\nu < \mathcal{K}$, one has

$$\begin{aligned}
o(1) &= \frac{1}{\|(u_k, v_k)\|}\bigg(\langle u_k, w_k^-\rangle + \langle v_k, z_k^-\rangle - \mu\langle u_k, w_k^-\rangle_\psi - \nu\langle v_k, z_k^-\rangle_\psi \\
&\quad - \int_\Omega \Big(F_u(\xi, u_k, v_k)w_k^- + F_v(\xi, u_k, v_k)z_k^-\Big)d\xi\bigg) \\
&= \frac{1}{\|(u_k, v_k)\|^2}\Big(\langle u_k, u_k^-\rangle + \langle v_k, v_k^-\rangle - \mu\langle u_k, u_k^-\rangle_\psi - \nu\langle v_k, v_k^-\rangle_\psi\Big) \\
&\quad - \int_\Omega \frac{\Big(F_u(\xi, u_k, v_k)u_k^- + F_v(\xi, u_k, v_k)v_k^-\Big)}{\|(u_k, v_k)\|^2}d\xi \\
&= \frac{1}{\|(u_k, v_k)\|^2}\Big(\langle u_k, u_k^-\rangle + \langle v_k, v_k^-\rangle - \mu\langle u_k, u_k^-\rangle_\psi - \nu\langle v_k, v_k^-\rangle_\psi\Big) \\
&\geq \frac{1}{\|(u_k, v_k)\|^2}\Big(\|u_k^-\|^2_{\mathcal{S}_0^1(\Omega)} + \|v_k^-\|^2_{\mathcal{S}_0^1(\Omega)} - \mu\|u_k^-\|^2_\psi - \nu\|v_k^-\|^2_\psi\Big) \\
&\geq \left(1 - \frac{\mu^+}{\mathcal{K}}\right)\|w_k^-\|^2_{\mathcal{S}_0^1(\Omega)} + \left(1 - \frac{\nu^+}{\mathcal{K}}\right)\|z_k^-\|^2_{\mathcal{S}_0^1(\Omega)}.
\end{aligned}$$

It follows that

$$\|w_k^-\|_{\mathcal{S}_0^1(\Omega)} \to 0 \quad \text{and} \quad \|z_k^-\|_{\mathcal{S}_0^1(\Omega)} \to 0.$$

Hence, $(w_k^-, z_k^-) \to (0, 0)$ in W as $k \to \infty$, $(w_k^-, z_k^-) = (0, 0)$ *a.e* in Ω, by the definition of w_k^- and z_k^-, we get that $w \geq 0$ and $z \geq 0$ *a.e.* in Ω.

Step 2: We prove $(w, z) = (0, 0)$ *a.e* in Ω. Let us set $\mathbf{D}_+ = \{\xi \in \Omega : w > 0 \text{ or } z > 0\}$ and $\mathbf{D_0} = \{\xi \in \Omega : (w, z) = (0, 0)\}$. Assume that the Haar measure of $\mathbf{D}_+$ is positive. From the assumption that $\|(u_k, v_k)\| \to \infty$, we have

$$|(u_k, v_k)| = \|(u_k, v_k)\||(w_k, z_k)| \to \infty \quad a.e. \text{ in } \mathbf{D}_+.$$

Then, from (f_3), we get

$$\lim_{k\to\infty} \frac{F(\xi, u_k, v_k)}{\|(u_k, v_k)\|^2} = \lim_{k\to\infty} \frac{F(\xi, u_k, v_k)|(w_k, z_k)|^2}{|(u_k, v_k)|^2} = \infty \quad a.e. \text{ in } \mathbf{D}_+. \tag{4.5}$$

Moreover, by Fatou's lemma and (4.5), there holds

$$\begin{aligned}
\liminf_{k\to\infty} \int_\Omega \frac{F(\xi, u_k, v_k)}{\|(u_k, v_k)\|^2}d\xi &\geq \int_\Omega \liminf_{k\to\infty} \frac{F(\xi, u_k, v_k)}{\|(u_k, v_k)\|^2}d\xi \\
&= \int_\Omega \liminf_{k\to\infty} \frac{F(\xi, u_k, v_k)|(w_k, z_k)|^2}{|(u_k, v_k)|^2}d\xi = \infty \quad a.e. \text{ in } \mathbf{D}_+.
\end{aligned} \tag{4.6}$$

On the other hand, from (4.1), a fact that $\|u_k\|^2_{\mathcal{S}^1_0(\Omega)} \le \|(u_k, v_k)\|^2$, $\|v_k\|^2_{\mathcal{S}^1_0(\Omega)} \le \|(u_k, v_k)\|^2$ and (1.5), we get

$$\begin{aligned}\int_\Omega F(\xi, u_k, v_k)d\xi &\le \frac{1}{2}\|u_k\|^2_{\mathcal{S}^1_0(\Omega)} + \frac{1}{2}\|v_k\|^2_{\mathcal{S}^1_0(\Omega)} - \frac{\mu}{2}\|u_k\|^2_\psi - \frac{\nu}{2}\|v_k\|^2_\psi + \mathcal{L}\\ &\le \|(u_k, v_k)\|^2 + \frac{|\mu^-|}{2\mathcal{K}}\|(u_k, v_k)\|^2 + \frac{|\nu^-|}{2\mathcal{K}}\|(u_k, v_k)\|^2 + \mathcal{L} \quad \text{for } \ k \in \mathbb{N},\end{aligned}$$

where we have used a fact that $\|(u_k, v_k)\| \ge 1$ because the hypothesis $(u_k, v_k) \to \infty$. Hence

$$\limsup_{k\to\infty} \int_\Omega \frac{F(\xi, u_k, v_k)}{\|(u_k, v_k)\|^2} d\xi \le 1 + \frac{|\mu^-|}{2\mathcal{K}} + \frac{|\nu^-|}{2\mathcal{K}} + \frac{Z}{\|(u_k, v_k)\|^2},$$

it contradicts with (4.6). Hence, the measure of $\mathbf{D}_+$ is zero, that is $(w, z) = (0, 0)$ *a.e* in Ω.

Step 3: We prove that $\{(u_k, v_k)\} \subset W$ is bounded. Choosing τ_k is the smallest value of $\tau \in [0, 1]$ such that $I_{\mu,\nu}(\tau_k u_k, \tau_k v_k) = \max_{0\le\tau\le1} I_{\mu,\nu}(\tau u_k, \tau v_k)$. For $\Lambda > 0$, we set $(W_k, Z_k) = \sqrt{2\Lambda}(w_k, z_k) = \sqrt{2\Lambda}\frac{(u_k, v_k)}{\|(u_k, v_k)\|}$, then, by (4.4) and Step 2, we obtain

$$\lim_{k\to\infty}(W_k, Z_k) = \lim_{k\to\infty}\sqrt{2\Lambda}(w_k, z_k) = \sqrt{2\Lambda}(w, z) = \sqrt{2\Lambda}(0, 0), \tag{4.7}$$

in $L^q(\Omega) \times L^q(\Omega)$ for $q \in [1, 2^*)$. By (f_1), (f_2), (4.7) and let $\epsilon = 1$, it holds

$$\begin{aligned}0 \le \int_\Omega F(\xi, W_k, Z_k)d\xi &\le \int_\Omega \Big((\lambda + 1)|(W_k, Z_k)| + C_1|(W_k, Z_k)|^s\Big)d\xi\\ &\le (\lambda + 1)\|(W_k, Z_k)\|_1 + C_1\|(W_k, Z_k)\|^s_s \to 0, \quad \text{as } k \to \infty,\end{aligned}$$

for $s \in (2, 2^*)$, that is

$$\lim_{k\to\infty}\int_\Omega F(\xi, W_k, Z_k)d\xi = 0. \tag{4.8}$$

From $\|(u_k, v_k)\| \to \infty$, we assume that there is $k_0 \ge k$, such that $\frac{\sqrt{2\Lambda}}{\|(u_k, v_k)\|} \in (0, 1)$, then

$$\begin{aligned}I_{\mu,\nu}(\tau_k u_k, \tau_k v_k) &\ge I_{\mu,\nu}\left(\sqrt{2\Lambda}\frac{u_k}{\|(u_k, v_k)\|}, \sqrt{2\Lambda}\frac{v_k}{\|(u_k, v_k)\|}\right)\\ &\ge \Lambda\|w_k\|^2(1 - \frac{\mu^+}{\mathcal{K}}) + \|z_k\|^2(1 - \frac{\nu^+}{\mathcal{K}}) - \int_\Omega F(\xi, W_k, Z_k)d\xi\\ &\ge \Lambda\chi(\|w_k\|^2 + \|z_k\|^2) - \int_\Omega F(\xi, W_k, Z_k)d\xi\\ &\ge \frac{1}{2}\Lambda\chi - \int_\Omega F(\xi, W_k, Z_k)d\xi,\end{aligned}$$

where χ is defined in Lemma 3.2, $\|w_k\|^2_{\mathcal{S}^1_0(\Omega)} + \|z_k\|^2_{\mathcal{S}^1_0(\Omega)} = 1$ because $\|(w_k, z_k)\| = 1$. By (4.8), there is $k_1 \ge k_0$ such that $\int_\Omega F(\xi, W_k, Z_k)d\xi \le \frac{1}{2}\Lambda\chi$ for $k \ge k_1$. It follows that

$$\lim_{k\to\infty} I_{\mu,\nu}(\tau_k u_k, \tau_k v_k) = \infty. \tag{4.9}$$

Since $0 < \tau_k < 1$, by (f_4), one has

$$\int_\Omega H(\xi, \tau_k u_k, \tau_k v_k)d\xi \le C_F\int_\Omega H(\xi, u_k, v_k)d\xi + \int_\Omega g(\xi)d\xi. \tag{4.10}$$

From the facts that $I_{\mu,\nu}(0,0)=0$, $I_{\mu,\nu}(u_k,v_k)\to c\in\mathbb{R}$, (4.9), and $\tau_k\in(0,1)$, there holds

$$\begin{aligned}0=\tau_k\frac{d}{d\tau}I_{\mu,\nu}(\tau u_k,\tau v_k)\Big|_{\tau=\tau_k}&=\langle I'_{\mu,\nu}(\tau_k u_k,\tau_k v_k),(\tau_k u_k,\tau_k v_k)\rangle\\&=\|\tau_k u_k\|^2_{\mathcal{S}^1_0(\Omega)}+\|\tau_k v_k\|^2_{\mathcal{S}^1_0(\Omega)}-\mu\|\tau_k u_k\|^2_\psi-\nu\|\tau_k v_k\|^2_\psi\\&\quad-\int_\Omega\Big(F_u(\xi,\tau_k u_k,\tau_k v_k)\tau_k u_k+F_v(\xi,\tau_k u_k,\tau_k v_k)\tau_k v_k\Big)d\xi.\end{aligned}$$

By (f_4) and (4.10), it follows that

$$\begin{aligned}&\|\tau_k u_k\|^2_{\mathcal{S}^1_0(\Omega)}+\|\tau_k v_k\|^2_{\mathcal{S}^1_0(\Omega)}-\mu\|\tau_k u_k\|^2_\psi-\nu\|\tau_k v_k\|^2_\psi\\&\quad=\int_\Omega\Big(F_u(\xi,\tau_k u_k,\tau_k v_k)\tau_k u_k+F_v(\xi,\tau_k u_k,\tau_k v_k)\tau_k v_k\Big)d\xi\\&\quad=2\int_\Omega F(\xi,\tau_k u_k,\tau_k v_k)d\xi+\int_\Omega H(\xi,\tau_k u_k,\tau_k v_k)d\xi\\&\quad\le 2\int_\Omega F(\xi,\tau_k u_k,\tau_k v_k)d\xi+C_F\int_\Omega H(\xi,u_k,v_k)d\xi+\int_\Omega g(\xi)d\xi.\end{aligned}\tag{4.11}$$

From (4.9) and (4.11), one has

$$\begin{aligned}2I_{\mu,\nu}(\tau_k u_k,\tau_k v_k)&=\|\tau_k u_k\|^2_{\mathcal{S}^1_0(\Omega)}+\|\tau_k v_k\|^2_{\mathcal{S}^1_0(\Omega)}-\mu\|\tau_k u_k\|^2_\psi-\nu\|\tau_k v_k\|^2_\psi\\&\quad-2\int_\Omega F(\xi,\tau_k u_k,\tau_k v_k)d\xi\\&\le C_F\int_\Omega H(\xi,u_k,v_k)d\xi+\int_\Omega g(\xi)d\xi\to\infty\quad\text{as }k\to\infty.\end{aligned}$$

Hence, we deduce that

$$\frac{1}{C_F}\Big(-C+\int_\Omega H(\xi,u_k,v_k)d\xi\Big)\to\infty\quad\text{as }k\to\infty.\tag{4.12}$$

On the other hand, by (4.1), (f_4) and (4.3), we have

$$\begin{aligned}\tilde{\mathcal{L}}&\ge 2I_{\mu,\nu}(u_k,v_k)\\&=\|u_k\|^2_{\mathcal{S}^1_0(\Omega)}+\|v_k\|^2_{\mathcal{S}^1_0(\Omega)}-\mu\|u_k\|^2_\psi-\nu\|v_k\|^2_\psi-2\int_\Omega F(\xi,u_k,v_k)d\xi\\&=\|u_k\|^2_{\mathcal{S}^1_0(\Omega)}+\|v_k\|^2_{\mathcal{S}^1_0(\Omega)}-\mu\|u_k\|^2_\psi-\nu\|v_k\|^2_\psi\\&\quad-\int_\Omega\Big(F_u(\xi,u_k,v_k)u_k+F_v(\xi,u_k,v_k)v_k\Big)d\xi+\int_\Omega H(\xi,u_k,v_k)d\xi\\&\ge -C+\int_\Omega H(\xi,u_k,v_k)d\xi,\end{aligned}\tag{4.13}$$

where $\tilde{\mathcal{L}}$ is a positive constant. Since $C_F\ge 1$ in (f_4) and by (4.13), we obtain

$$\frac{1}{C_F}\Big(-C+\int_\Omega H(\xi,u_k,v_k)d\xi\Big)\le -C+\int_\Omega H(\xi,u_k,v_k)d\xi\le\tilde{\mathcal{L}}.$$

This contradicts with (4.12), it follows that $\{(u_k,v_k)\}\subset W$ is a bounded Cerami sequence. We finish the proof of this lemma. □

In the following, we verify that $I_{\mu,\nu}$ satisfies the Cerami condition at level c.

Lemma 4.2. *Assume that (f_2) with $\epsilon = 1$ holds. Then, for all μ, $\nu \in (-\infty, \mathcal{K})$, $I_{\mu,\nu}$ satisfies the Cerami condition in W.*

Proof. Assume that $\{(u_k, v_k)\} \subset W$ is a Cerami sequence of $I_{\mu,\nu}$. Then, by Lemma 4.1, we know that $\{(u_k, v_k)\}$ is bounded. Then, up to subsequence, from (1.5), Lemmas 2.2 and 2.3, for $1 \le q < 2^*$, there exists $(u, v) \in W$ such that

$$\begin{aligned} (u_k, v_k) \rightharpoonup (u, v) \ \text{ in } W, \quad \|u_k - u\|_{\mathcal{S}_0^1(\Omega)} \to \bar{a}, \ \ \|v_k - v\|_{\mathcal{S}_0^1(\Omega)} \to \check{a}, \\ u_k \rightharpoonup u \ \text{ in } L^2(\Omega, \psi^2 r^{-2}), \ \ \|u_k - u\|_\psi \to \acute{a}, \\ v_k \rightharpoonup v \ \text{ in } L^2(\Omega, \psi^2 r^{-2}), \ \ \|v_k - v\|_\psi \to \grave{a}, \\ (u_k, v_k) \to (u, v) \ \text{ in } L^q(\Omega) \times L^q(\Omega), \quad (u_k, v_k) \to (u, v) \ \text{ a.e in } \Omega, \\ \nabla_{\mathbb{G}} u_k \rightharpoonup \nabla_{\mathbb{G}} u \ \text{ in } L^2(\Omega, \mathbb{R}^{2n}), \quad \nabla_{\mathbb{G}} v_k \rightharpoonup \nabla_{\mathbb{G}} v \ \text{ in } L^2(\Omega, \mathbb{R}^{2n}), \\ \nabla_{\mathbb{G}} u_k \rightharpoonup \vartheta \ \text{ in } L^2(\Omega, \mathbb{R}^{2n}), \quad \nabla_{\mathbb{G}} v_k \rightharpoonup \varsigma \ \text{ in } L^2(\Omega, \mathbb{R}^{2n}), \end{aligned} \tag{4.14}$$

where ϑ, $\varsigma \in L^2(\Omega, \mathbb{R}^{2n})$ are two vector field functions in Ω, and $\bar{a}$, $\acute{a}$, $\check{a}$, $\grave{a}$ are four nonnegative numbers.

From (4.14), we conclude that

$$\int_\Omega \frac{\psi^2}{r(\xi)^2} u_k \Phi d\xi \to \int_\Omega \frac{\psi^2}{r(\xi)^2} u \Phi d\xi \quad \text{and} \int_\Omega \frac{\psi^2}{r(\xi)^2} v_k \Psi d\xi \to \int_\Omega \frac{\psi^2}{r(\xi)^2} v \Psi d\xi, \tag{4.15}$$

for $(\Phi, \Psi) \in W$. We choose $\epsilon = 1$ in (f_2), and by Hölder inequality, then

$$\begin{aligned} &\int_\Omega \Big| \Big(F_u(\xi, u_k, v_k) - F_u(\xi, u, v) \Big) (u_k - u) \\ &\qquad + \Big(F_v(\xi, u_k, v_k) - F_v(\xi, u, v) \Big) (v_k - v) \Big| d\xi \\ &= \int_\Omega \Big| F_w(\xi, w_k)(w_k - w) - F_w(\xi, w)(w_k - w) \Big| d\xi \\ &\le \int_\Omega \Big((\lambda + 1)(|w_k| + |w|)|w_k - w| + C_1(|w_k|^{s-1} + |w|^{s-1})|w_k - w| \Big) d\xi \\ &\le C_\lambda (\|w_k - w\|_2 + \|w_k - w\|_s) \to 0 \quad \text{as } k \to \infty, \end{aligned} \tag{4.16}$$

where $C_\lambda > 0$ is a suitable constant. From (4.1), it holds that $I'_{\mu,\nu}(u_k, v_k) \to 0$ in W' as $k \to \infty$, then for every $(\Phi, \Psi) \in W$, we have

$$\begin{aligned} 0 \leftarrow &\langle I'_{\mu,\nu}(u_k, v_k), (\Phi, \Psi) \rangle \\ = &\int_\Omega (\nabla_{\mathbb{G}} u_k, \nabla_{\mathbb{G}} \Phi) d\xi + \int_\Omega (\nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}} \Psi) d\xi - \mu \int_\Omega \frac{\psi^2}{r(\xi)^2} u_k \Phi d\xi - \nu \int_\Omega \frac{\psi^2}{r(\xi)^2} v_k \Psi d\xi \\ &- \int_\Omega \Big(F_u(\xi, u_k, v_k) \Phi + F_v(\xi, u_k, v_k) \Psi \Big) d\xi. \end{aligned} \tag{4.17}$$

Subsequently, we prove that the (PS) sequence satisfies compactness condition by means of the Brézis–Lieb lemma.

From (4.17) and Lemma A.1 (it shows that (u_k, v_k) satisfies the Brézis–Lieb lemma's condition, see in the Appendix), one has

$$\nabla_{\mathbb{G}} u_k \to \nabla_{\mathbb{G}} u \quad \text{and } \nabla_{\mathbb{G}} v_k \to \nabla_{\mathbb{G}} v \quad a.e. \text{ in } \Omega, \tag{4.18}$$

and by (4.14), there holds $\nabla_G u_k \rightharpoonup \vartheta$ and $\nabla_G v_k \rightharpoonup \varsigma$ in $L^2(\Omega, \mathbb{R}^{2n})$. Hence, from Proposition A.7 in [1], we obtain $\nabla_G u = \vartheta$ and $\nabla_G v = \varsigma$ a.e. in Ω. It yields that $\nabla_G u_k \rightharpoonup \nabla_G u$ and $\nabla_G v_k \rightharpoonup \nabla_G v$ in $L^2(\Omega, \mathbb{R}^{2n})$, therefore, for any $(\Phi, \Psi) \in W$, one has

$$\int_\Omega (\nabla_G u_k, \nabla_G \Phi) d\xi \to \int_\Omega (\nabla_G u, \nabla_G \Phi) d\xi \quad \text{and} \quad \int_\Omega (\nabla_G v_k, \nabla_G \Phi) d\xi \to \int_\Omega (\nabla_G v, \nabla_G \Phi) d\xi.$$

It follows $\langle u_k, u\rangle \to \|u\|^2_{\mathcal{S}^1_0(\Omega)}$, $\langle u, u_k\rangle \to \|u\|^2_{\mathcal{S}^1_0(\Omega)}$ and $\langle v_k, v\rangle \to \|v\|^2_{\mathcal{S}^1_0(\Omega)}$, $\langle v, v_k\rangle \to \|v\|^2_{\mathcal{S}^1_0(\Omega)}$. Moreover, by (4.15) and (4.16), the weak limit $w = (u, v)$ is a critical point of $I_{\mu,\nu}$ in W. From (4.14) and (4.18), the Brézis–Lieb lemma holds that

$$\|u_k\|^2_{\mathcal{S}^1_0(\Omega)} = \|u_k - u\|^2_{\mathcal{S}^1_0(\Omega)} + \|u\|^2_{\mathcal{S}^1_0(\Omega)} + o(1), \quad \|v_k\|^2_{\mathcal{S}^1_0(\Omega)} = \|v_k - v\|^2_{\mathcal{S}^1_0(\Omega)} + \|v\|^2_{\mathcal{S}^1_0(\Omega)} + o(1),$$
$$\|u_k\|^2_\psi = \|u_k - u\|^2_\psi + \|u\|^2_\psi + o(1), \quad \|v_k\|^2_\psi = \|v_k - v\|^2_\psi + \|v\|^2_\psi + o(1).$$

Consequently, one has

$$\begin{aligned}
o(1) &= \langle I'_{\mu,\nu}(w_k) - I'_{\mu,\nu}(w), w_k - w\rangle \\
&= \|u_k\|^2_{\mathcal{S}^1_0(\Omega)} + \|u\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_k\|^2_{\mathcal{S}^1_0(\Omega)} + \|v\|^2_{\mathcal{S}^1_0(\Omega)} - \langle u_k, u\rangle - \langle u, u_k\rangle - \langle v_k, v\rangle - \langle u, u_k\rangle \\
&\quad - \mu\Big(\|u_k\|^2_\psi + \|u\|^2_\psi - \langle u_k, u\rangle_\psi - \langle u, u_k\rangle_\psi\Big) \\
&\quad - \nu\Big(\|v_k\|^2_\psi + \|v\|^2_\psi - \langle v_k, v\rangle_\psi - \langle v v_k\rangle_\psi\Big) + o(1) \\
&= \|u_k\|^2_{\mathcal{S}^1_0(\Omega)} - \|u\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_k\|^2_{\mathcal{S}^1_0(\Omega)} - \|v\|^2_{\mathcal{S}^1_0(\Omega)} - \mu(\|u_k\|^2_\psi - \|u\|^2_\psi) \\
&\quad - \nu(\|v_k\|^2_\psi - \|v\|^2_\psi) + o(1) \\
&= \|u_k - u\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_k - v\|^2_{\mathcal{S}^1_0(\Omega)} - \mu\|u_k - u\|^2_\psi - \nu\|v_k - v\|^2_\psi + o(1).
\end{aligned}$$

From (4.14) and above equality, it follows that

$$\begin{aligned}
\bar{a}^2 + \check{a}^2 &= \lim_{k\to\infty} \|u_k - u\|^2_{\mathcal{S}^1_0(\Omega)} + \lim_{k\to\infty} \|v_k - v\|^2_{\mathcal{S}^1_0(\Omega)} \\
&= \mu \lim_{k\to\infty} \|u_k - u\|^2_\psi + \nu \lim_{k\to\infty} \|v_k - v\|^2_\psi \\
&= \mu \acute{a}^2 + \nu \grave{a}^2.
\end{aligned} \tag{4.19}$$

Thus, when either $\mu^+ + \nu^+ = 0$ or $\acute{a} + \grave{a} = 0$, we get $(u_k, v_k) \to (u, v)$ in W as $k \to \infty$ and finish the proof about compactness condition for (PS) sequence of I_ϵ. In order to achieve this aim, we assume by contradiction, that is $\mu^+ + \nu^+ > 0$ and $\acute{a} + \grave{a} > 0$.

(1) If either $\mu^+ + \grave{a} = 0$ or $\nu^+ + \acute{a} = 0$, then either $\acute{a} > 0$ and $\bar{a} = 0$, or $\grave{a} > 0$ and $\check{a} = 0$. However, all of cases are impossible because the nonnegative of norm in (4.14).

(2) If either $\mu^+ + \acute{a} = 0$ or $\nu^+ + \grave{a} = 0$, then either $\grave{a} > 0$, $\nu^+ > 0$ and $\check{a}^2 \le \nu^+ \grave{a}^2 < \mathcal{K}\grave{a}^2 \le \check{a}^2$, or $\acute{a} > 0$, $\mu^+ > 0$ and $\bar{a}^2 \le \mu^+ \acute{a}^2 < \mathcal{K}\acute{a}^2 \le \bar{a}^2$, it appears a contradiction.

(3) $\mu^+ > 0$, $\nu^+ > 0$, $\acute{a} > 0$ and $\grave{a} > 0$, from (4.19) and (1.5), we get

$$\bar{a}^2 + \check{a}^2 = \mu \acute{a}^2 + \nu \grave{a}^2 < \mathcal{K}\acute{a}^2 + \mathcal{K}\grave{a}^2 \le \bar{a}^2 + \check{a}^2,$$

and a contradiction arises. From above discussions, we get $\acute{a} + \grave{a} = 0$, that is $(u_k, v_k) \to (u, v)$ in W as $k \to \infty$, from (4.19), the proof of this lemma is finished. □

4.2 The existence of solution

In this part, we study the existence of nonnegative solution for (1.1).

Proof of Theorem 1.1. From Lemmas 3.2 and 3.3, we know that $I_{\mu,\nu}$ satisfies the mountain pass geometry structures. Moreover, the Cerami condition holds by Lemma 4.2. Therefore, for every $(\Phi, \Psi) \in W$, there exists $(u, v) \in W$, $(u, v) \neq (0, 0)$ such that

$$\langle u, \Phi\rangle + \langle v, \Psi\rangle - \mu\langle u, \Phi\rangle_\psi - \nu\langle v, \Psi\rangle_\psi = \int_\Omega \Big(F_u(\xi, u, v)\Phi + F_v(\xi, u, v)\Psi\Big) d\xi.$$

Now, we prove that (u, v) is nonnegative. Let us set $\Phi = u^- = \min\{0, u\}$ and $\Psi = v^- = \min\{0, v\}$, then, from (f_1), (1.6) and (1.8), one has

$$\begin{aligned} 0 &= \int_\Omega \Big(F_u(\xi, u, v)u^- + F_v(\xi, u, v)v^-\Big) d\xi \\ &= \langle u, u^-\rangle + \langle v, v^-\rangle - \mu\langle u, u^-\rangle_\psi - \nu\langle v, v^-\rangle_\psi \\ &\geq (1 - \frac{\mu^+}{\mathcal{K}})\|u^-\|^2_{\mathcal{S}^1_0(\Omega)} + (1 - \frac{\nu^+}{\mathcal{K}})\|v^-\|^2_{\mathcal{S}^1_0(\Omega)} \geq 0. \end{aligned}$$

Thus, $u^- = 0$ and $v^- = 0$ a.e. in Ω, that is $u \geq 0$ and $v \geq 0$ a.e. in Ω, it shows that any solution of (1.1) is nonnegative. We finish the proof. □

A Appendix

In this section, we give a proof for the following lemma.

Lemma A.1. *Let (u_k, v_k) and (u, v) belongs to W and satisfying*

(i) $(u_k, v_k) \rightharpoonup (u, v)$ in W,

(ii) $(u_k, v_k) \to (u, v)$ a.e. in Ω,

(iii) $I'_{\mu,\nu}(u_k, v_k) \to 0$ strongly in W',

(iv) ϑ, $\varsigma \in \Omega$ are two vector field functions with ϑ, $\varsigma \in L^2(\Omega, \mathbb{R}^{2n})$ such that $\nabla_{\mathbb{G}} u_k \rightharpoonup \vartheta$ and $\nabla_{\mathbb{G}} v_k \rightharpoonup \varsigma$ in $L^2(\Omega, \mathbb{R}^{2n})$, then, it holds

$$\nabla_{\mathbb{G}} u_k \to \nabla_{\mathbb{G}} u \quad \textit{and} \quad \nabla_{\mathbb{G}} v_k \to \nabla_{\mathbb{G}} v \quad \textit{a.e. in} \;\; \Omega. \tag{A.1}$$

Proof. Let function $\beta_R \in C_0^\infty(\Omega)$ with $R > 0$, such that $0 \leq \beta_R \leq 1$ in Ω and $\beta_R \equiv 1$ in B_R. For every $z \in \mathbb{R}$, we define

$$\varrho_\epsilon(z) = \begin{cases} z, & \text{if } |z| < \epsilon, \\ \epsilon\dfrac{z}{|z|}, & \text{if } |z| \geq \epsilon. \end{cases}$$

We set $\phi_k = \beta_R \varrho_\epsilon \circ (u_k - u)$ and $\varphi_k = \beta_R \varrho_\epsilon \circ (v_k - v)$, thus, by Lemma 2.1, there holds ϕ_k,

$\varphi_k \in W^{1,2}(\Omega)$. Let $\Phi = \phi_k$ and $\Psi = \varphi_k$ in (4.17), then

$$\begin{aligned}
&\int_\Omega \beta_R \Big((\nabla_{\mathbb{G}} u_k - \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \Big(\varrho_\epsilon \circ (u_k - u) \Big) \Big) d\xi \\
&\qquad + \int_\Omega \beta_R \Big(\nabla_{\mathbb{G}} v_k - \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} \Big(\varrho_\epsilon \circ (v_k - v) \Big) \Big) d\xi \\
&\quad = - \int_\Omega \varrho_\epsilon \circ (u_k - u)(\nabla_{\mathbb{G}} u_k, \nabla_{\mathbb{G}} \beta_R) d\xi - \int_\Omega \beta_R \Big(\nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \Big(\varrho_\epsilon \circ (u_k - u) \Big) \Big) d\xi \\
&\qquad - \int_\Omega \varrho_\epsilon \circ (v_k - v)(\nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}} \beta_R) d\xi - \int_\Omega \beta_R \Big(\nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} \Big(\varrho_\epsilon \circ (v_k - v) \Big) \Big) d\xi \\
&\qquad + \langle I'_{\mu,\nu}(u_k, v_k), (\phi_k, \varphi_k) \rangle + \mu \int_\Omega \frac{\psi^2}{r(\xi)^2} u_k \phi_k d\xi + \nu \int_\Omega \frac{\psi^2}{r(\xi)^2} v_k \varphi_k d\xi \\
&\qquad + \int_\Omega \Big(F_u(\xi, u_k, v_k)\phi_k + F_v(\xi, u_k, v_k)\varphi_k \Big) d\xi.
\end{aligned} \tag{A.2}$$

Now, we prove the each term in (A.2).

(1) We choose that $\tilde{\beta}_R$ be the support of β_R and contained in a suitable ball of Ω, since $|\varrho_\epsilon \circ (u_k - u) \nabla_{\mathbb{G}} \beta_R| \to 0$ in $L^2(\tilde{\beta}_R)$ and $|\varrho_\epsilon \circ (v_k - v) \nabla_{\mathbb{G}} \beta_R| \to 0$ in $L^2(\tilde{\beta}_R)$, and by (4.14), $\nabla_{\mathbb{G}} u_k \rightharpoonup \vartheta$ in $L^2(\Omega, \mathbb{R}^{2n})$, $\nabla_{\mathbb{G}} v_k \rightharpoonup \varsigma$ in $L^2(\Omega, \mathbb{R}^{2n})$, then

$$\int_\Omega \varrho_\epsilon \circ (u_k - u)(\nabla_{\mathbb{G}} u_k, \nabla_{\mathbb{G}} \beta_R) d\xi \to 0 \quad \text{and} \quad \int_\Omega \varrho_\epsilon \circ (v_k - v)(\nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}} \beta_R) d\xi \to 0.$$

(2) Since $\nabla_{\mathbb{G}} \Big(\varrho_\epsilon \circ (u_k - u) \Big)$ in $L^2(\Omega, \mathbb{R}^{2n})$, $\nabla_{\mathbb{G}} \Big(\varrho_\epsilon \circ (v_k - v) \Big)$ in $L^2(\Omega, \mathbb{R}^{2n})$, $\nabla_{\mathbb{G}} u_k \in L^2(\Omega, \mathbb{R}^{2n})$, $\nabla_{\mathbb{G}} v_k \in L^2(\Omega, \mathbb{R}^{2n})$. From Lemma 2.1, $u_k \rightharpoonup u$ and $v_k \rightharpoonup v$ in W, one has

$$\int_\Omega \beta_R \Big(\nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} (\varrho_\epsilon \circ (u_k - u)) \Big) d\xi \to 0 \quad \text{and} \quad \int_\Omega \beta_R \Big(\nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} (\varrho_\epsilon \circ (v_k - v)) \Big) d\xi \to 0.$$

(3) From $I'_{\mu,\nu}(u_k, v_k) \to 0$ in W' and $(\varphi_k, \phi_k) \rightharpoonup 0$ in W as $k \to \infty$, we have

$$\langle I'_{\mu,\nu}(u_k, v_k), (\varphi_k, \phi_k) \rangle \rightharpoonup 0.$$

(4) For simplicity, we denote

$$M_k = \mu \frac{\psi^2}{r(\xi)^2} u_k + F_u(\xi, u_k, v_k), \quad N_k = \nu \frac{\psi^2}{r(\xi)^2} v_k + F_v(\xi, u_k, v_k), \tag{A.3}$$

by $0 \le \beta_R \le 1$ in Ω, the definition of ϕ_k, φ_k and $\varrho_\epsilon(z)$, Lemma A.2, there holds

$$\begin{aligned}
\int_\Omega (M_k \varphi_k + N_k \phi_k) d\xi &\le \int_{\tilde{\beta}_R} \Big(|M_k| \cdot |\varrho_\epsilon \circ (u_k - u)| + |N_k| \cdot |\varrho_\epsilon \circ (v_k - v)| \Big) d\xi \\
&\le \epsilon \int_{\tilde{\beta}_R} (|M_k| + |N_k|) d\xi \le \epsilon C_R,
\end{aligned}$$

where $C_R > 0$ is a constant. Moreover

$$\begin{aligned}
&\beta_R \Big(\nabla_{\mathbb{G}} u_k - \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} (\varrho_\epsilon \circ (u_k - u)) \Big) \ge 0, \\
&\beta_R \Big(\nabla_{\mathbb{G}} v_k - \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} (\varrho_\epsilon \circ (v_k - v)) \Big) \ge 0 \quad \textit{a.e.} \text{ in } \Omega.
\end{aligned} \tag{A.4}$$

Furthermore

$$\begin{aligned}
&\int_{B_R} \beta_R\Big(\nabla_{\mathbb{G}} u_k - \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (u_k - u)\Big)\Big) d\xi \\
&\qquad + \int_{B_R} \beta_R\Big(\nabla_{\mathbb{G}} v_k - \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (v_k - v)\Big)\Big) d\xi \\
&\quad \le \int_{\Omega} \beta_R\Big(\nabla_{\mathbb{G}} u_k - \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (u_k - u)\Big)\Big) d\xi \\
&\qquad + \int_{\Omega} \beta_R\Big(\nabla_{\mathbb{G}} v_k - \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (v_k - v)\Big)\Big) d\xi.
\end{aligned}$$

From (1)–(4) and a fact that $\beta_R \equiv 1$ in B_R, then (A.2) becomes

$$\begin{aligned}
&\limsup_{k\to\infty} \Big[\int_{B_R} \Big(\nabla_{\mathbb{G}} u_k - \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (u_k - u)\Big)\Big) d\xi \\
&\qquad + \int_{B_R} \Big(\nabla_{\mathbb{G}} v_k - \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (v_k - v)\Big)\Big) d\xi\Big] \\
&\quad \le \limsup_{k\to\infty} \Big[\int_{\Omega} \Big(\nabla_{\mathbb{G}} u_k - \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (u_k - u)\Big)\Big) d\xi \\
&\qquad + \int_{\Omega} \Big(\nabla_{\mathbb{G}} v_k - \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (v_k - v)\Big)\Big) d\xi\Big] \le \epsilon C_R. \qquad \text{(A.5)}
\end{aligned}$$

Subsequently, let $g_k = g_{u,k} + g_{v,k}$ with $g_{u,k} = \big(\nabla_{\mathbb{G}} u_k - \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}}(u_k - u)\big)$ and $g_{v,k} = \big(\nabla_{\mathbb{G}} v_k - \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}}(v_k - v)\big)$. We will show that g_k is nonnegative and bounded in $L^1(\Omega)$.

Firstly, if we assume that g_k is negative, it appears a contradiction with (A.4), thus, g_k is nonnegative. Secondly, since $\nabla_{\mathbb{G}} u_k$ is bounded in $L^2(\Omega, \mathbb{R}^{2n})$, and by (4.14), we know that $\nabla_{\mathbb{G}} v_k$ is bounded in $L^2(\Omega, \mathbb{R}^{2n})$. Therefore

$$0 \le \int_{\Omega} g_k(\xi) d\xi \le \|\nabla_{\mathbb{G}} u_k - \nabla_{\mathbb{G}} u\|_2^2 + \|\nabla_{\mathbb{G}} v_k - \nabla_{\mathbb{G}} v\|_2^2 \le C_0, \qquad \text{(A.6)}$$

where C_0 is a suitable constant and independent of k.

We select $t \in (0,1)$ and divide the ball B_R into four parts,

$$\begin{aligned}
B^\epsilon_{u,k}(R) &= \{\xi \in B_R : |u_k(\xi) - u(\xi)| \le \epsilon\}, \qquad \tilde{B}^\epsilon_{u,k}(R) = B_R \setminus B^\epsilon_{u,k}(R), \\
B^\epsilon_{v,k}(R) &= \{\xi \in B_R : |v_k(\xi) - v(\xi)| \le \epsilon\}, \qquad \tilde{B}^\epsilon_{v,k}(R) = B_R \setminus B^\epsilon_{v,k}(R).
\end{aligned}$$

Since $\nabla_{\mathbb{G}}\big(\varrho_\epsilon \circ (u_k - u)\big) = \nabla_{\mathbb{G}}(u_k - u)$ in $B^\epsilon_{u,k}(R)$ and $\nabla_{\mathbb{G}}\big(\varrho_\epsilon \circ (v_k - v)\big) = \nabla_{\mathbb{G}}(v_k - v)$ in $B^\epsilon_{v,k}(R)$, and from (A.6), we get

$$\begin{aligned}
\int_{B_R} g_k^t d\xi &\le \int_{B_R} g_{u,k}^t d\xi + \int_{B_R} g_{v,k}^t d\xi \\
&= \int_{B^\epsilon_{u,k}(R)} g_{u,k}^t d\xi + \int_{\tilde{B}^\epsilon_{u,k}(R)} g_{u,k}^t d\xi + \int_{B^\epsilon_{v,k}(R)} g_{v,k}^t d\xi + \int_{\tilde{B}^\epsilon_{v,k}(R)} g_{v,k}^t d\xi \\
&\le \Big(\int_{B^\epsilon_{u,k}(R)} g_{u,k} d\xi\Big)^t |B^\epsilon_{u,k}(R)|^{1-t} + \Big(\int_{\tilde{B}^\epsilon_{u,k}(R)} g_k d\xi\Big)^t |\tilde{B}^\epsilon_{u,k}(R)|^{1-t} \\
&\quad + \Big(\int_{B^\epsilon_{v,k}(R)} g_{v,k} d\xi\Big)^t |B^\epsilon_{v,k}(R)|^{1-t} + \Big(\int_{\tilde{B}^\epsilon_{v,k}(R)} g_k d\xi\Big)^t |\tilde{B}^\epsilon_{v,k}(R)|^{1-t} \\
&\le (\epsilon C_R)^t \Big(|B^\epsilon_{u,k}(R)|^{1-t} + |\tilde{B}^\epsilon_{v,k}(R)|^{1-t}\Big) + C_0^t \Big(|B^\epsilon_{u,k}(R)|^{1-t} + |\tilde{B}^\epsilon_{v,k}(R)|^{1-t}\Big).
\end{aligned}$$

Moreover, the definition of $B^{\epsilon}_{u,k}(R)$ and $B^{\epsilon}_{v,k}(R)$ follows that, $|\tilde{B}^{\epsilon}_{u,k}(R)|$ and $|\tilde{B}^{\epsilon}_{v,k}(R)|$ tends to 0 as k goes to ∞. Thus, $0 \le \limsup_{k\to\infty}\int_{B_R} g_k^t d\xi \le (\epsilon C_R)^t |B_R|^{1-t}$, which means that $g_k^t \to 0$ as $\epsilon \to 0$ in $L^1(B_R)$. Hence, $g_k \to 0$ a.e. in Ω for R is arbitrary, then, (A.1) is valid from Lemma 3 in [13]. □

Finally, the following result shows that the hardy term is bounded in W.

Lemma A.2. *Let $\{(u_k, v_k)\} \subset W$ be a bounded sequence and Ω_0 represent a compact set of Ω, M_k and N_k are given in* (A.3)*. Then there is a constant $C(\Omega_0) > 0$ such that*

$$\sup_k \int_{\Omega_0} (|M_k| + |N_k|) d\xi \le C(\Omega_0).$$

Proof. Since $\psi = |\psi| \le 1$ and the Jacobian determinant is r^4, thus, $\psi^2 r^{-2}$ be of class $L^1_{loc}(\Omega)$, by (1.5), one has

$$\int_{\Omega_0} \left(\left(\frac{\psi}{r}\right)^2 |u_k| + \left(\frac{\psi}{r}\right)^2 |v_k| \right) d\xi \le \left\| \frac{\psi}{r} \right\|_2 \sup_k \|u_k\|_\psi + \left\| \frac{\psi}{r} \right\|_2 \sup_k \|v_k\|_\psi = C_2(\Omega_0),$$

where $C_2(\Omega_0)$ is a positive constant depending on Ω_0. Moreover, from (f_2), it holds

$$\begin{aligned}
&\int_{\Omega_0} \Big| F_u(\xi, u_k, v_k) + F_v(\xi, u_k, v_k) \Big| d\xi \\
&\le \sqrt{2} \int_{\Omega_0} \left| \frac{\sqrt{H_u^2(\xi, u_k, v_k) + H_v^2(\xi, u_k, v_k)}}{2} \right| d\xi \\
&\le \sqrt{2} \int_{\Omega_0} \Big| (\lambda + 1)|(u_k, v_k)| + C_\epsilon |(u_k, v_k)|^{s-1} \Big| d\xi \\
&\le \sqrt{2} \Big((\lambda + 1) \sup_k \|(u_k, v_k)\|_{2^*} |\Omega_0|^{\frac{1}{t}} + C_\epsilon |\Omega_0|^{2^*-2+1} \sup_k \|(u_k, v_k)\|_{2^*}^{2-1} \Big) = C_3(\Omega_0),
\end{aligned}$$

where $t > 1$ and $t = \frac{2^*}{2^*-1}$ is the Lebesgue exponent for $s \in (2, 2^*)$. From above argument, we get $\sup_k \int_{\Omega_0} (|M_k| + |N_k|) d\xi \le C(\Omega_0)$, the proof of this lemma is completed. □

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