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Global existence and blow-up of solution to a class of fourth-order equation with singular potential and logarithmic nonlinearity

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Abstract. In this paper, we consider the well-posedness and asymptotic behavior of Dirichlet initial boundary value problem for a fourth-order equation with strong damping and logarithmic nonlinearity. We establish the local solvability by the technique of cut-off combining with the method of Faedo–Galerkin approximation. By means of potential well method and Rellich inequality, we obtain the global existence and the decay estimate of global solutions under some appropriate conditions. Furthermore, we prove the finite time blow-up results of weak solutions, and establish the upper and lower bounds for blow-up time.

Keywords: fourth-order, singular potential, logarithmic nonlinearity, global existence, blow-up.

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1 Introduction

In this paper, we are concerned with the fourth-order parabolic problem

$$\begin{cases} \dfrac{u_t}{|x|^4} + \Delta^2 u - \Delta u_t = |u|^{p-2} u \ln |u|, & (x,t) \in \Omega \times (0,T); \\ u(x,t) = \Delta u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T); \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N\,(N > 4)$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < T \leq \infty$, $u_0 \in H_0^2(\Omega)$, $x = (x_1, x_2, \dots, x_N) \in R^N$ with $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$, and parameter p satisfies the following

$$2 < p < \bar{p} = \begin{cases} \dfrac{8}{N} + 2, & N \geq 8, \\ \dfrac{4}{N-4} + 2, & 4 < N < 8. \end{cases}$$

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Many scholars have been devoted to the topic on the global existence and blow-up phenomena of the second-order partial differential equations (or a system of partial differential equations) and there have been fruitful results (see [1–4,7,10]). However, there are fewer studies on higher order equations (see [6,15] and references therein). In particular, the fourth-order parabolic partial differential equations have some applications in the fields such as materials science, engineering, biological mathematics, image analysis, etc.

King et al. [11] investigated the fourth-order semilinear parabolic equation modeling epitaxial thin film growth

$$u_t + \Delta^2 u - \nabla \cdot (f(\nabla u)) = g, \qquad (x,t) \in \Omega \times (0,+\infty),$$

by using the technique of semi-discrete approximation, they obtained existence, uniqueness and regularity of the weak solutions under appropriate conditions and derived the long-time asymptotic behavior.

The authors of [5,16] considered the following p-biharmonic parabolic equation with logarithmic nonlinearity

$$\begin{cases} u_t - \Delta u_t + \Delta\left(|\Delta u|^{p-2}\Delta u\right) = |u|^{q-2}u\ln u, & x \in \Omega,\ t>0; \\ u(x,t) = \Delta u(x,t) = 0, & x \in \partial\Omega,\ t>0; \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Cömert and Pişkin [5] considered the case of $2 < p < q < p\left(1+\frac{4}{N}\right)$, they obtained the existence of the unique global weak solutions and decay polynomially of solutions by using the potential wells method and logarithmic Sobolev inequality. For $\max\left\{1, \frac{2N}{N+4}\right\} < p \le q < p\left(1+\frac{4}{N}\right)$, Liu and Fang [16] established the local and global solvability, infinite and finite time blow-up phenomena of weak solutions in different energy levels. Moreover, the life span in different energy cases and extinction phenomenon are discussed.

Do et al. [8] considered the following higher-order reaction-diffusion parabolic problem

$$\begin{cases} \dfrac{u_t}{|x|^4} + \Delta^2 u = k(t)|u|^{p-1}u, & x \in \Omega,\ t>0; \\ u(x,t) = \Delta u(x,t) = 0, & x \in \partial\Omega,\ t>0; \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

The main difficulty is that the methods of [9,14,19] are no longer valid due to the presentation of singular potential. To overcome this difficulty, the authors of [8] combined the technique of cut-off with Hardy–Sobolev inequality and Faedo–Galerkin approximation to establish the local well-posedness. They also obtained the existence and decay estimation of a global weak solution. What is more, they discussed the upper and lower bounds on the blow-up time of a weak solution in [18].

In view of the works mentioned above, we consider the problem (1.1) with strong damping and logarithmic nonlinearity. In fact, the third derivative term Δu_t can be regarded as a damping term, which has effect on the qualitative properties such as blow-up, decay and so on. Mathematically, the logarithmic nonlinearity does not satisfy monotonicity and may change signs, thus the problem with logarithmic nonlinearity is more difficult than the one with power source. To the best our knowledge, this is the first work in the literature that takes into account a singular fourth-order equation with strong damping and logarithmic nonlinearity.

The paper is organized as follows. In Section 2, we introduce some potential wells, basic definitions and important lemmas. In section 3, we prove the local existence and uniqueness theorem. In Section 4, we prove the global existence and discuss the asymptotic behavior of solutions. Finally, in Section 5, we discuss the finite time blow-up of weak solutions and give the upper and lower bounds for blow-up time.

2 Preliminaries

In this section, we introduce some notations, basic definitions and lemmas that will be used throughout the paper. For convenience, we denote the norms

$$\|u\|_p := \|u\|_{L^p(\Omega)}, \qquad \|u\|_2 := \|u\|_{L^2(\Omega)}, \qquad \|\Delta u\|_2 := \|u\|_{H_0^2(\Omega)}.$$

For $u \in H_0^2(\Omega)$, we define the potential energy functional and Nehari functional as

$$J(u) = \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p^2}\|u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln|u|\, dx, \tag{2.1}$$

$$I(u) = \|\Delta u\|_2^2 - \int_\Omega |u|^p \ln|u|\, dx. \tag{2.2}$$

Then it follows from (2.1) and (2.2) that

$$J(u) = \frac{1}{p}I(u) + \left(\frac{1}{2} - \frac{1}{p}\right)\|\Delta u\|_2^2 + \frac{1}{p^2}\|u\|_p^p. \tag{2.3}$$

Furthermore, we introduce the following sets

$$\begin{aligned}
&W_1 = \left\{u \in H_0^2(\Omega) \mid J(u) < d\right\}, \quad W_2 = \left\{u \in H_0^2(\Omega) \mid J(u) = d\right\}, \quad W = W_1 \cup W_2,\\
&W_1^+ = \left\{u \in H_0^2(\Omega) \mid J(u) < d, I(u) > 0\right\}, \quad W_2^+ = \left\{u \in H_0^2(\Omega) \mid J(u) = d, I(u) > 0\right\},\\
&W_1^- = \left\{u \in H_0^2(\Omega) \mid J(u) < d, I(u) < 0\right\}, \quad W_2^- = \left\{u \in H_0^2(\Omega) \mid J(u) = d, I(u) < 0\right\},\\
&W^+ = W_1^+ \cup W_2^+, \quad W^- = W_1^- \cup W_2^-,
\end{aligned}$$

and the Nehari manifold

$$\mathcal{N} = \left\{u \in H_0^2(\Omega)\backslash\{0\},\ I(u) = 0\right\}.$$

The depth of potential well is defined as

$$d = \inf_{u \in \mathcal{N}} J(u).$$

Next, we give some definitions.

Definition 2.1 (Weak solution)**.** Let $T > 0$, the function $u \in L^\infty(0, T; H_0^2(\Omega))$ is a weak solution of problem (1.1) on $\Omega \times [0, T)$, if

$$u_t \in L^2(0, T; H_0^1(\Omega)), \qquad \frac{u_t}{|x|^2} \in L^2(0, T; L^2(\Omega)),$$

$u(x, 0) = u_0(x) \in H_0^2(\Omega)$ and $u(x, t)$ satisfies

$$\left\langle \frac{u_t}{|x|^4}, v\right\rangle + \langle \Delta u, \Delta v\rangle + \langle \nabla u_t, \nabla v\rangle = \left\langle |u|^{p-2}u \ln|u|, v\right\rangle,$$

for any $v \in H_0^2(\Omega)$ and $t \in [0, T)$.

Definition 2.2 (Maximal existence time [20])**.** Let $u(x,t)$ be a weak solution of problem (1.1), we define the maximal existence time T_{max} as follows

$$T_{max} = \sup\{T > 0;\ u(x,t) \text{ exists on } [0,T]\}.$$

(i) If $T_{max} = +\infty$, we say that the solution $u(t)$ is global;

(ii) If $T_{max} < +\infty$, we say that the solution $u(t)$ blows up in finite time and T_{max} is the blow-up time.

Definition 2.3 (Finite time blow-up)**.** Let $u(x,t)$ is a weak solution of problem (1.1). We say $u(x,t)$ blows up in finite time if the maximal existence time T_{max} is finite and

$$\lim_{t\to T_{max}^-} \int_0^t \left(\left\| \frac{u(\tau)}{|x|^2} \right\|_2^2 + \|\nabla u(\tau)\|_2^2 \right) d\tau = +\infty.$$

In order to deal with the singular potential, we introduce the cut-off function

$$\rho_n(x) = \min\left\{|x|^{-4}, n\right\}, \qquad n \in N^+,$$

and the following Lemma 2.4.

Lemma 2.4 (Rellich's inequality [8])**.** *Let $N > 4$ and $u \in H_0^2(\Omega)$. Then $\frac{u}{|x|^2} \in L^2(\Omega)$ and there exists a constant $R_N > 0$ such that*

$$\int_\Omega \frac{|u|^2}{|x|^4} dx \leq \frac{16}{N^2(N-4)^2} \int_\Omega |\Delta u|^2 dx =: R_N \int_\Omega |\Delta u|^2 dx.$$

Next, in Lemma 2.5, we describe some basic properties of the fiber mapping $J(\lambda u)$ that can be verified directly.

Lemma 2.5 ([5])**.** *Assume that $u \in H_0^2(\Omega)\backslash\{0\}$, then*

(i) $\lim_{\lambda\to 0^+} J(\lambda u) = 0$, $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$.

(ii) There exists a unique $\lambda^* = \lambda^*(u) > 0$ such that $\frac{d}{d\lambda} J(\lambda u)|_{\lambda=\lambda^*} = 0$.

(iii) $J(\lambda u)$ is increasing on $0 < \lambda < \lambda^*$, decreasing on $\lambda^* < \lambda < +\infty$, and attains the maximum at $\lambda = \lambda^*$.

(iv) $I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$, and $I(\lambda^* u) = 0$.

We introduce the following inequality to deal with the logarithmic nonlinearity.

Lemma 2.6. *Let μ be a positive number. Then we have the following inequalities:*

$$s^p \ln s \leq (e\mu)^{-1} s^{p+\mu}, \quad \text{for all } s \geq 1,$$

and

$$|s^p \ln s| \leq (ep)^{-1}, \quad \text{for all } 0 < s < 1.$$

The next result is the Gagliardo–Nirenberg inequality.

Lemma 2.7. *For any $u \in H_0^2(\Omega)$, it holds that:*

$$\|u\|_{p+\mu}^{p+\mu} \le C_G \|\Delta u\|_2^{(p+\mu)\theta} \|u\|_2^{(1-\theta)(p+\mu)},$$

where $\theta \in (0,1)$ is determined by $\theta = \frac{N(p+\mu-2)}{4(p+\mu)}$, $0 < \mu < \frac{8}{N} + 2 - p$, and the constant $C_G > 0$ depends on N, p.

In order to prove the decay estimation of weak solutions, we will introduce the following lemma.

Lemma 2.8 ([12]). *let $f : R^+ \to R^+$ be a nonincreasing function and σ be a positive constant such that:*

$$\int_t^{+\infty} f^{1+\sigma}(s)ds \le \frac{1}{\omega} f^\sigma(0) f(t), \ \ \forall\, t \ge 0.$$

Then we have

(i) $f(t) \le f(0)e^{1-\omega t}$, for all $t \ge 0$, whenever $\sigma = 0$.

(ii) $f(t) \le f(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}}$, for all $t \ge 0$, whenever $\sigma > 0$.

The following is the concavity lemma.

Lemma 2.9 ([13]). *Suppose that a positive, twice-differentiable function $\Psi(t)$ satisfies the inequality*

$$\Psi''(t)\Psi(t) - (1+\theta)(\Psi'(t))^2 \ge 0,$$

where $\theta > 0$. If $\Psi(0) > 0$ and $\Psi'(0) > 0$, then $\Psi(t) \to \infty$ as

$$t \to t_* \le t^* = \frac{\Psi(0)}{\theta \Psi'(0)}.$$

3 Local existence

In this section, we prove the local existence and uniqueness of weak solution to problem (1.1).

Lemma 3.1 ([16]). *Let $N > 4$, $2 < p < \bar{p}$. Then, for any $n \in N^+$ and any initial date $u_{n0} \in C_0^\infty(\Omega)$, there exists a unique weak solution $u_n \in L^\infty(0,T; H_0^2(\Omega))$ and $u_{nt} \in L^2(0,T; H_0^1(\Omega))$ satisfying the following equation*

$$\begin{cases} \rho_n(x)(u_n)_t + \Delta^2 u_n - \Delta(u_n)_t = |u_n|^{p-2} u_n \ln|u_n|, & x \in \Omega, t > 0; \\ u_n(x,t) = \Delta u_n(x,t) = 0, & x \in \partial\Omega, t > 0; \\ u_n(x,0) = u_{n0}, & x \in \Omega. \end{cases} \tag{3.1}$$

Theorem 3.2. *Let $u_0 \in H_0^2(\Omega) \setminus \{0\}$, $2 < p < \bar{p}$. Then there exist $T > 0$ and a unique weak solution $u(x,t) \in L^\infty(0,T; H_0^2(\Omega))$ of problem (1.1) with $u_t \in L^2(0,T; H_0^1(\Omega))$, $\frac{u_t}{|x|^2} \in L^2(0,T; L^2(\Omega))$ satisfying $u(x,0) = u_0(x)$. Moreover, $u(x,t)$ satisfies the energy equality*

$$\int_0^t \left(\left\| |x|^{-2} u_t \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + J(u(t)) = J(u_0), \qquad 0 \le t \le T.$$

Proof. We divide the proof of Theorem 3.2 into 3 steps.

Step 1. Local existence

We use Lemma 3.1 and approximation to prove the local existence of weak solutions to problem (1.1).

By Lemma 3.1, we know that $u_{n0} \in C_0^\infty(\Omega)$ such that

$$u_{n0} \to u_0(x) \quad \text{strongly in } H_0^2(\Omega), \tag{3.2}$$

and

$$\langle \rho_n(x)u_{nt}, \varphi\rangle + \langle \Delta u_n, \Delta\varphi\rangle + \langle \nabla u_{nt}, \nabla\varphi\rangle = \left\langle |u_n|^{p-2}u_n \ln|u_n|, \varphi\right\rangle. \tag{3.3}$$

Especially, taking $\varphi = u_n$ in (3.3), we get

$$\langle \rho_n(x)u_{nt}, u_n\rangle + \langle \Delta u_n, \Delta u_n\rangle + \langle \nabla u_{nt}, \nabla u_n\rangle = \left\langle |u_n|^{p-2}u_n \ln|u_n|, u_n\right\rangle. \tag{3.4}$$

Integrating the above equation over $[0,t]$, we have

$$\begin{aligned}&\frac{1}{2}\left\||\rho_n(x)|^{\frac{1}{2}}u_n(t)\right\|_2^2 + \int_0^t \|\Delta u_n(\tau)\|_2^2 d\tau + \frac{1}{2}\|\nabla u_n(t)\|_2^2\\ &= \frac{1}{2}\left\||\rho_n(x)|^{\frac{1}{2}}u_n(0)\right\|_2^2 + \frac{1}{2}\|\nabla u_n(0)\|_2^2 + \int_0^t\int_\Omega |u_n(\tau)|^p \ln|u_n(\tau)| dx d\tau.\end{aligned}$$

Let

$$S_n(t) = \frac{1}{2}\left\||\rho_n(x)|^{\frac{1}{2}}u_n(t)\right\|_2^2 + \frac{1}{2}\|\nabla u_n(t)\|_2^2 + \int_0^t \|\Delta u_n(\tau)\|_2^2 d\tau. \tag{3.5}$$

We observe that

$$S_n(t) = S_n(0) + \int_0^t\int_\Omega |u_n|^p \ln|u_n| dx d\tau. \tag{3.6}$$

From Lemma 2.6, we get

$$\begin{aligned}\int_\Omega |u_n|^p \ln|u_n| dx &= \int_{\Omega_1=\{x\in\Omega;|u_n(x)|\geq 1\}} |u_n|^p \ln|u_n| dx\\ &\quad + \int_{\Omega_2=\{x\in\Omega;|u_n(x)|<1\}} |u_n|^p \ln|u_n| dx\\ &\leq (e\mu)^{-1}\int_{\Omega_1=\{x\in\Omega;|u_n(x)|\geq 1\}} |u_n|^{p+\mu} dx\\ &\leq (e\mu)^{-1}\|u_n\|_{p+\mu}^{p+\mu}.\end{aligned} \tag{3.7}$$

Then, by Lemma 2.7, Young's inequality and (3.7), we obtain

$$\begin{aligned}\int_\Omega |u_n|^p \ln|u_n| dx &\leq (e\mu)^{-1}\|u_n\|_{p+\mu}^{p+\mu}\\ &\leq (e\mu)^{-1}C_G\|\Delta u_n\|_2^{\theta(p+\mu)}\|u_n\|_2^{(1-\theta)(p+\mu)}\\ &\leq (e\mu)^{-1}C_G\varepsilon\|\Delta u_n\|_2^2 + (e\mu)^{-1}C_G C(\varepsilon)\|u_n\|_2^{\frac{2(1-\theta)(p+\mu)}{2-\theta(p+\mu)}}\\ &\leq (e\mu)^{-1}C_G\varepsilon\|\Delta u_n\|_2^2 + (e\mu)^{-1}C_G C(\varepsilon)B_1\|\nabla u_n\|_2^{\frac{2(1-\theta)(p+\mu)}{2-\theta(p+\mu)}},\end{aligned} \tag{3.8}$$

where B_1 is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, $\varepsilon \in (0,1)$, and we choose $\mu > 0$ with $2 < p + \mu < \frac{8}{N} + 2$. Substituting (3.8) into (3.6), we get

$$S_n(t) \leq C_1 + C_2\int_0^t [S_n(\tau)]^\alpha d\tau, \tag{3.9}$$

where $C_1 = \frac{S_n(0)}{1-(e\mu)^{-1}C_G\varepsilon}$, $C_2 = \frac{(e\mu)^{-1}C_G C(\varepsilon)2^\alpha B_1}{1-(e\mu)^{-1}C_G\varepsilon}$, $\alpha = \frac{4p+4\mu-Np-N\mu+2N}{8-N(p+\mu-2)} = 1 + \frac{4(p+\mu)-8}{2(4+N)-N(p+\mu)} > 1$. By direct calculation, we obtain

$$S_n(t) \le C_T, \tag{3.10}$$

where C_T is a positive constant dependent on T.

Now, multiplying the first equation of problem (1.1) by u_{nt} and integrating on $\Omega \times (0,t)$, we obtain

$$\int_0^t \left(\left\| |\rho_n(x)|^{\frac{1}{2}} u_{nt} \right\|_2^2 + \|\nabla u_{nt}\|_2^2 \right) d\tau + J(u_n(t)) = J(u_{n0}), \qquad 0 \le t \le T. \tag{3.11}$$

By the continuity of the functional $J(u)$ in $H_0^2(\Omega)$ and (3.2), there exists a constant $C > 0$ satisfying

$$J(u_{n0}) \le C, \quad \text{for any positive integer n.} \tag{3.12}$$

Applying (2.1), (3.5), (3.8), (3.11) and (3.12), we obtain

$$\begin{aligned} C \ge J(u_{n0}) \ge J(u_n(t)) &= \frac{1}{2}\|\Delta u_n\|_2^2 + \frac{1}{p^2}\|u_n\|_p^p - \frac{1}{p}\int_\Omega |u_n|^p \ln|u_n|\,dx \\ &\ge \frac{1}{2}\|\Delta u_n\|_2^2 + \frac{1}{p^2}\|u_n\|_p^p - \frac{C_G\varepsilon}{pe\mu}\|\Delta u_n\|_2^2 - \frac{C_G C(\varepsilon) B_1}{pe\mu}\|\nabla u_n\|_2^{2\alpha} \\ &\ge \left(\frac{1}{2} - \frac{C_G\varepsilon}{pe\mu}\right)\|\Delta u_n\|_2^2 + \frac{1}{p^2}\|u_n\|_p^p - \frac{C_G C(\varepsilon) B_1 2^\alpha}{pe\mu}(C_T)^\alpha, \end{aligned}$$

namely

$$\|\Delta u_n\|_2^2 + \|u_n\|_p^p \le C. \tag{3.13}$$

From (3.11)–(3.13), it follows that

$$\|u_n(t)\|_{L^\infty(0,T;H_0^2(\Omega))} \le C, \quad \text{for any positive integer } n, \tag{3.14}$$

$$\|u_n(t)\|_{L^\infty(0,T;L^p(\Omega))} \le C, \quad \text{for any positive integer } n, \tag{3.15}$$

$$\|u_{nt}(t)\|_{L^2(0,T;H_0^1(\Omega))} \le C, \quad \text{for any positive integer } n, \tag{3.16}$$

$$\left\| |\rho_n(x)|^{\frac{1}{2}} u_{nt} \right\|_{L^2(0,T;L^2(\Omega))} \le C, \quad \text{for any positive integer } n. \tag{3.17}$$

By (3.14), (3.16) and the Aubin–Lions–Simon lemma (see [17], Corollary 4), we get

$$u_n \to u \quad \text{in } C(0,T;L^2(\Omega)). \tag{3.18}$$

Therefore, $u_n(x,0) \to u(x,0) = u_0(x)$ in $L^2(\Omega)$. By (3.18), we have $u_n \to u$ a.e. $(x,t) \in \Omega \times (0,T)$, this implies

$$|u_n|^{p-2} u_n \ln|u_n| \to |u|^{p-2} u \ln|u| \quad \text{a.e. } (x,t) \in \Omega \times (0,T).$$

On the other hand, by a direct calculation and the Sobolev inequality, we have

$$\begin{aligned} \int_\Omega \left| |u_n|^{p-2} u_n \ln|u_n| \right|^2 dx &= \int_{\Omega_1=\{x\in\Omega; |u_n(x)|\ge 1\}} \left| |u_n|^{p-2} u_n \ln|u_n| \right|^2 dx \\ &\quad + \int_{\Omega_2=\{x\in\Omega; |u_n(x)|<1\}} \left| |u_n|^{p-2} u_n \ln|u_n| \right|^2 dx \\ &\le (e\mu)^{-2}\|u_n\|_{2(p-1+\mu)}^{2(p-1+\mu)} + [e(p-1)]^{-2}|\Omega| \\ &\le (e\mu)^{-2} B_2 \|\Delta u_n\|_2^{2(p-1+\mu)} + [e(p-1)]^{-2}|\Omega| < C, \end{aligned}$$

where B_2 is the best constant of the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^{2(p-1+\mu)}(\Omega)$. Here we choose $0 < \mu \leq \frac{4}{N-4} + 2 - p$, $p < \frac{4}{N-4} + 2$, we know that

$$\left\| |u_n|^{p-2} u_n \ln |u_n| \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad \text{for any positive integer } n. \tag{3.19}$$

By (3.14)–(3.19), there exist functions u and a subsequence of $\{u_n\}_{n=1}^\infty$ which we still denote it by $\{u_n\}_{n=1}^\infty$ such that

$$u_n \to u \quad \text{weakly star in } L^\infty(0,T;H_0^2(\Omega)), \tag{3.20}$$

$$u_{nt} \to u_t \quad \text{weakly star in } L^2(0,T;H_0^1(\Omega)), \tag{3.21}$$

$$|\rho_n(x)|^{\frac{1}{2}} u_{nt} \to |x|^{-2} u_t \quad \text{weakly star in } L^2(0,T;L^2(\Omega)), \tag{3.22}$$

$$|u_n|^{p-2} u_n \ln |u_n| \to |u|^{p-2} u \ln |u| \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)). \tag{3.23}$$

By (3.20)–(3.23), passing to the limit in (3.3), as $n \to +\infty$, it follows that u satisfies the initial condition $u(0) = u_0$,

$$\left\langle |x|^{-4} u_t, \varphi \right\rangle + \langle \Delta u, \Delta \varphi \rangle + \langle \nabla u_t, \nabla \varphi \rangle = \left\langle |u|^{p-2} u \ln |u|, \varphi \right\rangle,$$

for all $\varphi \in H_0^2(\Omega)$, and for a.e. $t \in [0,T]$.

Step 2. Uniqueness

Suppose there are two solutions u_1 and u_2 to the problem (1.1) with the same initial condition $u_1(x,0) = u_2(x,0) = u_0(x) \in H_0^2(\Omega)$, we have

$$\left\langle |x|^{-4} u_{1t}, v \right\rangle + \langle \Delta u_1, \Delta v \rangle + \langle \nabla u_{1t}, \nabla v \rangle = \left\langle |u_1|^{p-2} u_1 \ln |u_1|, v \right\rangle, \tag{3.24}$$

and

$$\left\langle |x|^{-4} u_{2t}, v \right\rangle + \langle \Delta u_2, \Delta v \rangle + \langle \nabla u_{2t}, \nabla v \rangle = \left\langle |u_2|^{p-2} u_2 \ln |u_2|, v \right\rangle. \tag{3.25}$$

Let $w = u_1 - u_2$ and $w(0) = 0$, then by subtracting the (3.24) and (3.25), we can derive

$$\int_\Omega |x|^{-4} w_t v dx + \int_\Omega \Delta w \Delta v dx + \int_\Omega \nabla w_t \nabla v dx = \int_\Omega \left(|u_1|^{p-2} u_1 \ln |u_1| - |u_2|^{p-2} u_2 \ln |u_2| \right) v dx,$$

Let $v = w$ and integrating it on $[0,t]$, we obtain

$$\frac{1}{2} \left\| |x|^{-2} w \right\|_2^2 + \int_\Omega \|\Delta w\|_2^2 dx + \frac{1}{2} \|\nabla w\|_2^2 = \int_0^t \int_\Omega \frac{|u_1|^{p-2} u_1 \ln |u_1| - |u_2|^{p-2} u_2 \ln |u_2|}{w} w^2 dx d\tau,$$

then

$$\|\nabla w\|_2^2 \leq 2 \int_0^t \int_\Omega \frac{f(u_1) - f(u_2)}{w} w^2 dx d\tau,$$

where $f(s) = |s|^{p-2} s \ln |s|$. By the Lipschitz continuity of $f : R^+ \to R^+$, we have

$$\|\nabla w\|_2^2 \leq 2C_U \int_0^t \|\nabla w\|_2^2 d\tau.$$

Employing the Gronwall's inequality, the above inequality yields that $\|\nabla w\|_2^2 = 0$. Thus, we have $w = 0$ a.e. in $\Omega \times (0,T)$. Therefore, the uniqueness of problem (1.1) can be deduced.

Step 3. Energy equality

Multiplying (1.1) with u_t and integrating over $\Omega \times (0,t)$ to obtain the energy equality

$$\int_0^t \left(\left\| |x|^{-2} u_t \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + J(u(t)) = J(u_0), \qquad 0 \leq t \leq T. \tag{3.26}$$

The proof of Theorem 3.2 is completed □

4 Global existence and decay rate

In this section, we are concerned with the existence of global weak solutions to problem (1.1) and show that the norm $\|u(t)\|_{H_0^2(\Omega)}$ decays exponentially.

Theorem 4.1. *Assume that $u_0 \in W^+$, then problem (1.1) admits a global weak solution $u \in L^\infty(0,\infty; H_0^2(\Omega))$, $u_t \in L^2(0,\infty; H_0^1(\Omega))$ with $\frac{u_t}{|x|^2} \in L^2(0,\infty; L^2(\Omega))$, and $u(t) \in W^+$ for $0 \le t \le \infty$. Moreover, if $u_0 \in {W_1}^+$, then*

$$\|\Delta u\|_2^2 \le \|\Delta u_0\|_2^2 e^{1-\frac{C_3}{C_4}t}, \quad t \ge 0,$$

where $C_3 = 1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2}{p}-1}$, $C_4 = \frac{R_N + B_1}{2}$, B_1 is the best embedding constant.

Step 1. Global existence

Proof. In order to prove the existence of global weak solutions, we consider two following cases.

Case 1. The initial data $u_0 \in W_1^+$.

Combining $J(u_0) < d$ with (3.26), then we get

$$\int_0^t \left(\left\| \frac{u_t}{|x|^2} \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + J(u(t)) = J(u_0) < d, \qquad 0 \le t \le T_{max}, \tag{4.1}$$

where T_{max} is the maximal existence time of solution $u(t)$, we shall prove that $T_{max} = +\infty$. Next, we will show that

$$u(x,t) \in W_1^+ \quad \text{for all } 0 \le t \le T_{max}. \tag{4.2}$$

In fact, assume that (4.2) does not hold and let t_* be the smallest time for which $u(t_*) \notin W_1^+$. Then, by the continuity of $u(t)$, we have $u(t_*) \in \partial W_1^+$. Hence, it follows that

$$J(u(t_*)) = d, \tag{4.3}$$

or

$$I(u(t_*)) = 0. \tag{4.4}$$

Nevertheless, it is clear that (4.3) is invalid by (4.1). On the other hand, if (4.4) holds, by the definition of d, we have

$$J(u(t_*)) \ge \inf_{u \in \mathcal{N}} J(u) = d,$$

which also contradicts with (4.1). Hence, we have $u(x,t) \in W_1^+$ such that $I(u(t)) > 0$. Consequently, it follows from this fact and (2.3) that

$$\int_0^t \left(\left\| \frac{u_t}{|x|^2} \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + \frac{1}{p} I(u) + \left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u\|_2^2 + \frac{1}{p^2} \|u\|_p^p < d, \tag{4.5}$$

namely

$$\int_0^t \left(\left\| \frac{u_t}{|x|^2} \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + \left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u\|_2^2 + \frac{1}{p^2} \|u\|_p^p < d. \tag{4.6}$$

This estimation allows us to take $T_{max} = +\infty$. Hence, we can conclude that there is a unique global weak solution $u(t) \in W_1^+$ of the problem (1.1).

Case 2. The initial data $u_0 \in W_2^+$.

Firstly, we choose a sequence $\{\theta_m\}_{m=1}^{\infty} \subset (0,1)$ such that $\lim_{m\to\infty} \theta_m = 1$. Then we consider the following problem

$$\begin{cases} \frac{u_t}{|x|^4} + \Delta^2 u - \Delta u_t = |u|^{p-2} u \ln |u|, & (x,t) \in \Omega \times (0,T); \\ u(x,t) = \Delta u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T); \\ u\,(x,0) = u_{0m} = \theta_m u_0(x), & x \in \Omega. \end{cases} \tag{4.7}$$

Due to $I\left(\lambda\left(u_0\right)\right) = I\left(u_0\right) > 0$, we have $\lambda = 1$. From lemma 2.5, it follows that $\lambda^* > \lambda = 1$. Hence, from $\theta_m < 1 < \lambda^*$ we can deduce that $I(u_{0m}) = I(\theta_m u_0) > 0$ and $J(u_{0m}) = J(\theta_m u_0) < J(u_0) = d$, which means $u_{0m} \in W_1^+$. Using the similar arguments as the Case 1. We find that problem (4.7) admits a global weak solution u.

Step 2. Decay estimate

From $u_0 \in W_1^+$ and the conclusions of the global weak solutions, we know that $u(t) \in W_1^+$. Hence, by (2.3), we have

$$\left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u\|_2^2 + \frac{1}{p^2} \|u\|_p^p \le J(u(t)) \le J(u_0) < d. \tag{4.8}$$

Through a direct calculation, we arrive that

$$\lambda_0 \left[\left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u\|_2^2 + \frac{1}{p^2} \|u\|_p^p\right] \ge J\left(\lambda^* u\left(t\right)\right) \ge d, \tag{4.9}$$

where $\lambda_0 = \max\{(\lambda^*)^2, (\lambda^*)^p\}$. Combining with (4.8), we get

$$\lambda_0 \ge \frac{d}{J\left(u_0\right)} > 1, \tag{4.10}$$

so we can infer that $\lambda_* > 1$, it implies

$$\lambda^* \ge \left(\frac{d}{J\left(u_0\right)}\right)^{\frac{1}{p}} > 1. \tag{4.11}$$

From (2.2), we have

$$\begin{aligned} 0 = I(\lambda^* u) &= (\lambda^*)^2 \|\Delta u\|_2^2 - (\lambda^*)^p \int_\Omega |u|^p \ln |u|\, dx - (\lambda^*)^p \ln(\lambda^*) \|u\|_p^p \\ &= (\lambda^*)^p I(u) - \left[(\lambda^*)^p - (\lambda^*)^2\right] \|\Delta u\|_2^2 - (\lambda^*)^p \ln(\lambda^*) \|u\|_p^p. \end{aligned} \tag{4.12}$$

In view of (4.11) and (4.12) we have

$$I\left(u\right) = \|u\|_p^p \ln\left(\lambda^*\right) + \left[1 - (\lambda_*)^{2-p}\right] \|\Delta u\|_2^2 \ge C_3 \|\Delta u\|_2^2, \tag{4.13}$$

where $C_3 = 1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2}{p}-1}$.

According to equation (2.2) and lemma 2.4, we obtain

$$\begin{aligned}
\int_t^T I(u)\,ds &= \int_t^T \left(\|\Delta u\|_2^2 - \int_\Omega |u|^p \ln|u|\,dx \right) ds \\
&= -\frac{1}{2}\int_t^T \left(\frac{d}{dt}\left\| \frac{u}{|x|^2} \right\|_2^2 + \frac{d}{dt}\|\nabla u\|_2^2 \right) ds \\
&= \frac{1}{2}\left(\left\| \frac{u(t)}{|x|^2} \right\|_2^2 + \|\nabla u(t)\|_2^2 \right) - \frac{1}{2}\left(\left\| \frac{u(T)}{|x|^2} \right\|_2^2 + \|\nabla u(T)\|_2^2 \right) \\
&\le \frac{1}{2}\left(\left\| \frac{u(t)}{|x|^2} \right\|_2^2 + \|\nabla u(t)\|_2^2 \right) \\
&\le \left(\frac{R_N + B}{2} \right) \|\Delta u(t)\|_2^2 = C_4 \|\Delta u(t)\|_2^2,
\end{aligned} \tag{4.14}$$

where $C_4 = \frac{R_N + B_1}{2}$, B_1 is the best embedding constant.

By (4.13) and (4.14), we get

$$\int_t^T \|\Delta u(s)\|_2^2 ds \le \frac{C_4}{C_3}\|\Delta u(t)\|_2^2, \quad \text{for all } t \in [0, T], \tag{4.15}$$

let $T \to +\infty$ in (4.15), by virtue of lemma 2.8, it follows that

$$\|\Delta u(t)\|_2^2 \le \|\Delta u_0\|_2^2 e^{1 - \frac{C_3}{C_4}t}.$$

The proof of Theorem 4.1 is completed. □

5 Blow-up phenomena of weak solutions

In this section, we consider the finite time blow-up results of weak solutions with $u_0 \in W^-$, and give the upper and lower bounds for blow up time to problem (1.1). For simplicity, we shall write

$$L(t) = \frac{1}{2}\left(\left\| \frac{u(t)}{|x|^2} \right\|_2^2 + \|\nabla u(t)\|_2^2 \right).$$

5.1 Upper bound for blow-up time

Theorem 5.1. *Assume that $u_0 \in W^-$, $2 < p < \bar{p}$. Then the weak solution $u(t)$ of problem (1.1) blows up in finite time, the upper bound for blow-up time T_{max} is given by*

$$T_{max} \le \frac{\beta b^2}{(p-2)\beta b - \left(\left\| \frac{u_0}{|x|^2} \right\|_2^2 + \|\nabla u_0\|_2^2 \right)},$$

where

$$\beta \in \left(0, \frac{p(d - J(u_0))}{p-1} \right], \qquad b > \max\left\{ 0, \frac{\left\| \frac{u_0}{|x|^2} \right\|_2^2 + \|\nabla u_0\|_2^2}{(p-2)\beta} \right\}.$$

Proof. We will divide the proof into two cases.

Case 1: $u_0 \in W_1^-$.

We claim that $u(t) \in W_1^-$ for $t \in [0, T_{max}]$ provided that $u_0 \in W_1^-$. Indeed, by contradiction, there exists a $t_0 \in (0, T_{max})$ such that $I(u(t)) > 0$ for $t \in [0, t_0)$ and $I(u(t_0)) = 0$. Recalling the definition of d, it is clear that $J(u(t_0)) \geq d$ which contradicts with $J(u(t)) \leq J(u_0) < d$. Hence, we get $u(t) \in W_1^-$ for $t \in [0, T_{max}]$.

From lemma 2.5, as $I(u(t)) < 0$, there is a $\lambda^* < 1$ such that $I(\lambda^* u) = 0$. Then

$$\begin{aligned} d \leq J(\lambda^* u) &= \frac{1}{p} I(\lambda^* u) + (\lambda^*)^2 \left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u\|_2^2 + \frac{(\lambda^*)^p}{p^2} \|u\|_p^p \\ &< \left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u\|_2^2 + \frac{1}{p^2} \|u\|_p^p. \end{aligned} \tag{5.1}$$

We show that $T_{max} < +\infty$. For any $T \in [0, T_{max})$, define the positive function

$$F(t) = \int_0^t L(t)\, d\tau + (T - t) L(0) + \frac{\beta}{2}(t + b)^2, \tag{5.2}$$

where $\beta > 0, b > 0$. We compute the first-order differential and second-order differential of $F(t)$, respectively, as follows:

$$\begin{aligned} F'(t) &= L(t) - L(0) + \beta(t + b) \\ &= \int_0^t \frac{d}{dt} L(t)\, d\tau + \beta(t + b) \\ &= \int_0^t \left(\int_\Omega \frac{u \cdot u_t}{|x|^4} dx + \int_\Omega \nabla u \cdot \nabla u_t dx\right) d\tau + \beta(t + b), \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} F''(t) &= L'(t) + \beta = -I(u) + \beta \\ &= -pJ(u) + \left(\frac{p}{2} - 1\right) \|\Delta u\|_2^2 + \frac{1}{p} \|u\|_p^p + \beta. \end{aligned} \tag{5.4}$$

From (5.2)–(5.4), through a direct calculation, we have

$$\begin{aligned} &F(t)F''(t) - (1+\theta)\left[F'(t)\right]^2 = F(t)F''(t) \\ &+ (1+\theta)\left\{H(t) - \left[2F(t) - 2(T-t)L(0)\right]\left[\int_0^t \left(\left\|\frac{u_t}{|x|^2}\right\|_2^2 + \|\nabla u_t\|_2^2\right) d\tau + \beta\right]\right\}, \end{aligned} \tag{5.5}$$

the definition of $H(t)$ is following

$$\begin{aligned} H(t) = &\left[\int_0^t \left(\left\|\frac{u}{|x|^2}\right\|_2^2 + \|\nabla u\|_2^2\right) d\tau + \beta(t+b)^2\right] \cdot \left[\int_0^t \left(\left\|\frac{u_t}{|x|^2}\right\|_2^2 + \|\nabla u_t\|_2^2\right) d\tau + \beta\right] \\ &- \left[\int_0^t \int_\Omega \left(\frac{u u_t}{|x|^4} + \nabla u \cdot \nabla u_t\right) dx d\tau + \beta(t+b)\right]^2. \end{aligned}$$

Applying the Cauchy–Schwarz inequality, Young's inequality and Hölder's inequality, it is easy to verify that $H(t) \geq 0$ for any $t \in (0, T)$. Therefore, choosing $\theta = \frac{p-2}{2} > 0$, there are

$$\begin{aligned}
&F(t) F''(t) - \frac{p}{2}\left[F'(t)\right]^2 \\
&\geq F(t) F''(t) - \frac{p}{2}\left[2F(t) - 2(T-t) L(0)\right]\left[\int_0^t \left(\left\|\frac{u_t}{|x|^2}\right\|_2^2 + \|\nabla u_t\|_2^2\right) d\tau + \beta\right] \\
&\geq F(t)\left[F''(t) - p\int_0^t \left(\left\|\frac{u_t}{|x|^2}\right\|_2^2 + \|\nabla u_t\|_2^2\right) d\tau - p\beta\right] \\
&= F(t)\left[-pJ(u) + \left(\frac{p}{2} - 1\right)\|\Delta u\|_2^2 + \frac{1}{p}\|u\|_p^p\right. \\
&\quad \left. -p\int_0^t \left(\left\|\frac{u_t}{|x|^2}\right\|_2^2 + \|\nabla u_t\|_2^2\right) d\tau + (1-p)\beta\right] \\
&= F(t)\varsigma(t),
\end{aligned}$$

we denote $\varsigma(t)$ as follows

$$\varsigma(t) = -pJ(u) + \left(\frac{p}{2} - 1\right)\|\Delta u\|_2^2 + \frac{1}{p}\|u\|_p^p - p\int_0^t \left(\left\|\frac{u_t}{|x|^2}\right\|_2^2 + \|\nabla u_t\|_2^2\right) d\tau + (1-p)\beta.$$

From (5.1) and $u(t) \in W_1^-$, when we chose $\beta \in \left(0, \frac{p(d-J(u_0))}{p-1}\right]$, we have

$$\begin{aligned}
\varsigma(t) =& -pJ(u_0) + \left(\frac{p}{2} - 1\right)\|\Delta u\|_2^2 + \frac{1}{p}\|u\|_p^p + (1-p)\beta \\
&\geq p(d - J(u_0)) + (1-p)\beta \geq 0.
\end{aligned}$$

Hence, by the above discussions, (5.5) becomes that

$$F(t) F''(t) - (1+\theta)\left[F'(t)\right]^2 \geq 0.$$

Therefore, Lemma 2.9 guarantees that $F(0) > 0$, $F'(0) = \beta b > 0$, then there is a T_1 satisfies $0 < T_1 < \frac{2F(0)}{(p-2)F'(0)}$ such that $F(t) \to \infty$, $t \to T_1$, we can obtain that

$$T_{max} \leq \frac{\beta b^2}{(p-2)\beta b - \left(\left\|\frac{u_0}{|x|^2}\right\|_2^2 + \|\nabla u_0\|_2^2\right)},$$

where

$$b > \max\left\{0, \frac{\left\|\frac{u_0}{|x|^2}\right\|_2^2 + \|\nabla u_0\|_2^2}{(p-2)\beta}\right\}.$$

Case 2: $u_0 \in W_2^-$

By the similar arguments as those in the proof of Case 1. When $u_0 \in W_2^-$, by continuity we see that there exists a $t_1 > 0$ such that $I(u(t_1)) < 0$, $\|\frac{u_t}{|x|^2}\|_2^2 > 0$ and $\|\nabla u_t\|_2^2 > 0$ for all $t \in [0, t_1)$. From energy equality we get

$$J(u(t_1)) \leq J(u_0) - \int_0^{t_1} \left(\left\| \frac{u_t}{|x|^2} \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau < J(u_0) = d.$$

The remainder of the proof is the same as Case 1. □

5.2 Lower bound for blow-up time

In this subsection, we shall derive a lower bound for the blow-up time T_{max}.

Theorem 5.2. *Assume that $u_0 \in W^-$, $2 < p < \bar{p}$. Then the weak solution $u(t)$ of problem (1.1) blows up in finite time, the lower bound for blow-up time T_{max} is given by*

$$T_{max} \geq \frac{L^{1-\alpha}(0)}{C_L(\alpha - 1)},$$

where $C_L = 2^\alpha (e\mu)^{-1} C_G C(\varepsilon) B_2$, $\alpha = \frac{4p+4\mu-Np-N\mu+2N}{8-N(p+\mu-2)} > 1$.

Proof. According to the proof proof of Theorem 5.1, we can get $u(t) \in W^-$. From problem (1.1) and equation (2.2), we obtain

$$\begin{aligned} L'(t) &= \int_\Omega \frac{u \cdot u_t}{|x|^4} dx + \int_\Omega \nabla u \cdot \nabla u_t dx \\ &= -\|\Delta u\|_2^2 + \int_\Omega |u|^p \ln |u| \, dx \\ &= -I(u) > 0. \end{aligned} \tag{5.6}$$

Recalling the inequality (3.8) and combining (3.8) and (5.6), it follows that

$$L'(t) \leq \left[(e\mu)^{-1} C_G \varepsilon - 1 \right] \|\Delta u\|_p^p + (e\mu)^{-1} C_G C(\varepsilon) B_2 \|\nabla u\|_2^{2\alpha}. \tag{5.7}$$

In view of $(e\mu)^{-1} C_G \varepsilon - 1 < 0$, $\alpha = \frac{4p+4\mu-Np-N\mu+2N}{8-N(p+\mu-2)} > 1$ and the definition of $L(t)$, we get

$$L'(t) \leq (e\mu)^{-1} C_G C(\varepsilon) B_2 \|\nabla u\|_2^{2\alpha} \leq C_L L^\alpha(t), \tag{5.8}$$

where $C_L = 2^\alpha (e\mu)^{-1} C_G C(\varepsilon) B_2$. Integrating (5.8) over $[0, t)$, we get

$$\frac{1}{1-\alpha} \left[L^{1-\alpha}(t) - L^{1-\alpha}(0) \right] \leq C_L t.$$

Since $\alpha > 1$, letting $t \to T_{max}$ in the above inequality and recalling that $\lim_{t \to T_{max}} L(t) = +\infty$, we obtain

$$T_{max} \geq \frac{L^{1-\alpha}(0)}{C_L(\alpha - 1)}.$$

The proof of Theorem 5.1 and Theorem 5.2 are finished. □

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