Structural stability
for scalar reaction-diffusion equations

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Abstract. In this paper, we prove the structural stability for a family of scalar reaction-diffusion equations. Our arguments consist of using invariant manifold theorem to reduce the problem to a finite dimension and then, we use the structural stability of Morse–Smale flows in a finite dimension to obtain the corresponding result in infinite dimension. As a consequence, we obtain the optimal rate of convergence of the attractors and estimate the Gromov–Hausdorff distance of the attractors using continuous $\epsilon$-isometries.

Keywords: Morse–Smale semiflows, rate of convergence of attractors, structural stability, invariant manifolds, Gromov–Hausdorff distance.

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1 Introduction and statement of the results

The continuity of attractors is an important feature to study the stability of the semilinear evolution equations. For a family of attractors $\{A_{\epsilon}\}_{\epsilon \in [0,1]}$ the continuity at $\epsilon = 0$ means that the symmetric Hausdorff distance $d_H(A_{\epsilon}, A_0) \to 0$ as $\epsilon \to 0$. The work [8] obtained positive results in the class of gradient systems, assuming structural conditions on the unperturbed attractor, together with information on the continuity of unstable manifolds of equilibria. In particular, if $\{u_{\epsilon}^*\}_{\epsilon \in [0,1]}$ is the family of equilibrium points then $d(u_{\epsilon}^*, u_0^*) \to 0$ as $\epsilon \to 0$ for the phase space metric $d$.

There is a natural question, as follows.

Question 1. Is the order in which $d_H(A_{\epsilon}, A_0)$ goes to zero the same as $d(u_{\epsilon}^*, u_0^*)$?

There are many works concerning the rate of convergence of attractors to different situations, as we can see in [1, 3, 6] and [7]. The case of reaction-diffusion equation in a smooth domain, [1] has been shown that

$$d(u_{\epsilon}^*, u_0^*) \sim \epsilon \quad \text{and} \quad d_H(A_{\epsilon}, A_0) \sim \epsilon^\beta, \quad 0 < \beta < 1.$$ \hfill (1.1)
In [3], the authors have analyzed the reaction-diffusion equation in a thin domain under perturbations, where they have obtained

\[ d(u^\varepsilon, u^0) \sim \varepsilon \quad \text{and} \quad d_H(A_\varepsilon, A_0) \sim \varepsilon \ln(\varepsilon). \]  

Notice that both above problems do not provide an answer to Question 1 because the rate of convergence of attractors is worse than equilibria.

The work [6] was able to answer Question 1 considering the reaction-diffusion equation where the diffusion coefficient becomes large in all domains when \( \varepsilon \to 0 \). The optimal rate state

\[ d(u^\varepsilon, u^0) \sim \varepsilon \quad \text{and} \quad d_H(A_\varepsilon, A_0) \sim \varepsilon. \]  

The figure below shows (1.2) is better than (1.1) and (1.3) improves (1.2) as the parameter \( \varepsilon \) goes to zero.

The main argument to obtain (1.2) and (1.3) is the existence of a finite-dimensional invariant manifold that allows us to reduce the problem to finite dimension and, then, we can use properties of Morse–Smale dynamical systems in finite-dimensional closed manifolds. For instance, [3] have used that in a neighborhood of the attractor, a Morse–Smale flow has the Lipschitz Shadowing property to estimate \( d_H(A_\varepsilon, A_0) \) by the continuity of the solution \( T_\varepsilon(\cdot) \to T_0(\cdot) \) in a neighborhood of the \( \cup_t A_\varepsilon \).

The purpose of this paper is to prove that the rate of convergence of the attractors for the scalar reaction-diffusion equations is optimal. Inspired by the optimal rate obtained in [6] and using the framework proposed by [3] we can reduce the problem to Morse–Smale flows in finite dimension and we use the structural stability of Morse–Smale flows in a finite dimension to obtain the corresponding result in infinite dimension. As a consequence, we obtain the optimal rate of convergence of the attractors. We observe that our arguments can be carried over to the problem addressed in [3] under appropriate adaptations. Another consequence of the structural stability is the estimate of the Gromov–Hausdorff distance of the attractors \( d_{GH}(A_\varepsilon, A_0) \). This subject has been introduced by reaction-diffusion equation under perturbation of the domain in the paper [10]. Since structural stability means that there is a topological equivalence \( \kappa_\varepsilon : A_\varepsilon \to A_0 \) close to identity conjugating the flows, we have \( \kappa_\varepsilon \) a continuous \( \varepsilon \)-isometry between the attractors. This is enough requirement that we need to estimate \( d_{GH}(A_\varepsilon, A_0) \).
Consider the following family of scalar reaction-diffusion equations

\[
\begin{cases}
u^\varepsilon_t - (a^\varepsilon(x)u^\varepsilon)_x = f(u^\varepsilon), & (t, x) \in (0, \infty) \times (0, \pi) \\
u^\varepsilon(t, 0) = 0 = u^\varepsilon(t, \pi), & t \in (0, \infty), \\
u^\varepsilon(0, x) = u^\varepsilon_0(x), & x \in (0, \pi),
\end{cases}
\]

where \( \varepsilon \in [0, \varepsilon_0] \) is a parameter, \( 0 < \varepsilon_0 < 1 \), the diffusion coefficients \( a^\varepsilon \in C^1([0, \pi], [m_0, M_0]) \), \( m_0, M_0 > 0 \), are continuous functions satisfying

\[
\|a^\varepsilon - a\|_\infty := \|a^\varepsilon - a\|_{L^\infty(0, \pi)} \to 0 \quad \text{as} \quad \varepsilon \to 0
\]

and the nonlinearity \( f : \mathbb{R} \to \mathbb{R} \) is a continuously differentiable function such that,

\[
\limsup_{|s| \to \infty} \frac{f(s)}{s} < 0.
\]

It follows from [5, Theorem 14.2] that for each \( \varepsilon \in [0, \varepsilon_0] \), the solutions of (1.4) defines a nonlinear gradient semigroup \( T^\varepsilon(\cdot) \) having a global attractor \( A^\varepsilon \) such that

\[
\sup_{\varepsilon \in [0, \varepsilon_0]} \sup_{w \in A^\varepsilon} \|w\|_{H^1_0(0, \pi)} < \infty \quad \text{and} \quad \sup_{\varepsilon \in [0, \varepsilon_0]} \sup_{w \in A^\varepsilon} \|w\|_{L^\infty(0, \pi)} < \infty.
\]

Moreover, we assume that the equilibrium points of (1.4) with \( \varepsilon = 0 \) is hyperbolic. Hence, there are finitely many equilibrium points and we denote them by \( \mathcal{E}_0 = \{u_0^{1,0}, \ldots, u_0^{p,0}\} \).

Under the above assumption, we have from [5, Chapter 14] that, for \( \varepsilon_0 \) sufficiently small, the semigroup \( T^\varepsilon(\cdot) \) has exactly \( p \) equilibria that we denote \( \mathcal{E}_\varepsilon = \{u_{\varepsilon}^{1,\varepsilon}, \ldots, u_{\varepsilon}^{p,\varepsilon}\} \) and the global attractors are given by \( A_\varepsilon = \bigcup_{i=1}^p W^a(u_{\varepsilon}^{i,\varepsilon}) \) and \( A_0 = \bigcup_{i=1}^p W^a(u_0^{i,0}) \), where \( W^a \) denotes the unstable manifold. The main results of [5, Chapter 14] and [1] state that the convergence of equilibria can be estimate by

\[
\|u_{\varepsilon}^{i,\varepsilon} - u_0^{i,0}\|_{H^1_0(0, \pi)} \leq C\|a^\varepsilon - a_0\|_\infty
\]

and the continuity of the global attractors can be estimated by

\[
d_H(A_\varepsilon, A_0) \leq C\|a^\varepsilon - a_0\|_\infty^\beta,
\]

where \( C > 0 \) and \( 0 < \beta < 1 \) are constants independent of \( \varepsilon \) and \( d_H \) denotes the Hausdorff distance in \( H^1_0(0, \pi) \), that is,

\[
d_H(A_\varepsilon, A_0) = \max \left\{ \sup_{u^\varepsilon \in A_\varepsilon} \inf_{u^0 \in A_0} \|u^\varepsilon - u^0\|_{H^1_0(0, \pi)}, \sup_{u^0 \in A_0} \inf_{u^\varepsilon \in A_\varepsilon} \|u^\varepsilon - u^0\|_{H^1_0(0, \pi)} \right\}.
\]

Finally, we assume that \( T^\varepsilon(\cdot)|_{A_\varepsilon} \) is a group. It is well-known that under standard conditions the solutions of (1.4) are backward uniquely defined inside the attractor.

The main result of this paper states as follows.

**Theorem 1.1.** The equation (1.4) is structurally stable at \( \varepsilon = 0 \). That is, given \( \eta > 0 \) there is \( \varepsilon_\eta > 0 \) such that for \( \varepsilon \in (0, \varepsilon_\eta] \), there is a homeomorphism \( \kappa_\varepsilon : A_\varepsilon \to A_0 \) such that

\[
\sup_{u^\varepsilon \in A_\varepsilon} \|\kappa_\varepsilon(u^\varepsilon) - u^0\|_{H^1_0(0, \pi)} < C(\|a^\varepsilon - a_0\|_\infty + \eta) \quad \text{and} \quad \kappa_\varepsilon(T_\varepsilon(t, u^\varepsilon))u^0 = T_0(t)\kappa_\varepsilon(u^\varepsilon),
\]

where \( t \in \mathbb{R}, u^\varepsilon \in A_\varepsilon, C > 0 \) is a constant independent of \( \varepsilon \) and \( \tau_\varepsilon : \mathbb{R} \times A_\varepsilon \to \mathbb{R} \) is a function such that, \( \tau_\varepsilon(0, u^\varepsilon) = 0 \) and \( \tau_\varepsilon(\cdot, u^\varepsilon) \) is a increasing function mapping \( \mathbb{R} \) onto \( \mathbb{R} \).
Corollary 1.2. For \( \eta > 0 \) there is \( \varepsilon_{\eta} > 0 \) such that for \( \varepsilon \in (0, \varepsilon_{\eta}] \), the Hausdorff distance between the attractors can be estimated by

\[
d_H(A_\varepsilon, A_0) \leq C(\|a_\varepsilon - a_0\|_\infty + \eta),
\]

where \( C \) is a constant independent of \( \varepsilon \).

We say that a map \( i_\varepsilon : A_\varepsilon \to A_0 \) is an \( \varepsilon \)-isometry between \( A_\varepsilon \) and \( A_0 \) if

\[
\left| \|i_\varepsilon(u^e) - i_\varepsilon(v^e)\|_{H_0^0(0,\pi)} - \|u^e - v^e\|_{H_0^0(0,\pi)} \right| \leq \varepsilon, \quad u^e, v^e \in A_\varepsilon
\]

and \( B(i_\varepsilon(A_\varepsilon), \varepsilon) = A_0 \), where \( B(i_\varepsilon(A_\varepsilon), \varepsilon) = \{u^0 \in A_0 : \|i_\varepsilon(u^e) - u^0\|_{H_0^0(0,\pi)} \leq \varepsilon \} \) for some \( u^e \in A_\varepsilon \). Analogously we can define a \( \varepsilon \)-isometry between \( A_0 \) and \( A_\varepsilon \). The Gromov–Hausdorff distance \( d_{GH}(A_\varepsilon, A_0) \) between \( A_\varepsilon \) and \( A_0 \) is defined as the infimum of \( \varepsilon > 0 \) for which there are \( \varepsilon \)-isometries \( i_\varepsilon : A_\varepsilon \to A_0 \) and \( i_\varepsilon : A_0 \to A_\varepsilon \).

We have the following result as an immediate consequence of the Theorem 1.1.

Corollary 1.3. For \( \eta > 0 \) there is \( \varepsilon_{\eta} > 0 \) such that for \( \varepsilon \in (0, \varepsilon_{\eta}] \), the Gromov–Hausdorff distance between the attractors can be estimated by

\[
d_{GH}(A_\varepsilon, A_0) \leq C(\|a_\varepsilon - a_0\|_\infty + \eta),
\]

where \( C \) is a constant independent of \( \varepsilon \).

This paper is organized as follows. In Section 2 we introduce the functional setting to deal with (1.4). In Section 3 we use invariant manifolds to reduce the problem to finite dimension. In Section 4 we prove the Theorem 1.1.

## 2 Functional setting and technical results

Let \( \varepsilon \in [0, \varepsilon_0] \). We define the operator \( A_\varepsilon : D(A_\varepsilon) \subset L^2(0, \pi) \to L^2(0, \pi) \) by

\[
\begin{align*}
D(A_\varepsilon) &= H^2(0, \pi) \cap H_0^1(0, \pi), \\
A_\varepsilon u &= -a_\varepsilon(x)u_x, \quad u \in D(A_\varepsilon).
\end{align*}
\]

(2.1)

It is well-known that \( A_\varepsilon \) is a self-adjoint operator with compact resolvent. Hence, we can define the fractional power spaces \( X_\varepsilon^\alpha, 0 < \alpha \leq 1 \), where \( X_\varepsilon^0 = L^2(0, \pi) \), \( X_\varepsilon^1 = D(A_\varepsilon) \) and \( X_\varepsilon^\frac{1}{2} = H_0^1(0, \pi) \) with the inner product

\[
\langle u, v \rangle_{X_\varepsilon^{\frac{1}{2}}} = \int_0^\pi a_\varepsilon(x)u_xv_x \, dx
\]

(2.2)

which produces norms uniformly equivalent to the standard \( H_0^1(0, \pi) \) norm, since \( a_\varepsilon \) is uniformly bounded in \( \varepsilon \). Therefore, estimates on \( X_\varepsilon^{\frac{1}{2}} \) are transported to \( H_0^1(0, \pi) \) uniformly in \( \varepsilon \).

Since there are many estimates in the paper, we will let \( C \) be a generic constant which is independent of \( \varepsilon \), but which may depend on \( m_0, M_0, u_0^{0}, \varepsilon_0 \).

We summarize in the next theorem some useful estimates that can be proved as in [1] and [5, Chapter 14].
Theorem 2.1. Let $\varepsilon \in [0, \varepsilon_0]$. The operators $A_{\varepsilon}$ satisfy the following properties.

(i) $\sup_{\varepsilon \in [0, \varepsilon_0]} \| A_{\varepsilon}^{-1} \|_{L^2(0, \pi), H^1_0(0, \pi)} \leq C$.

(ii) $\| A_{\varepsilon}^{-1} u^\varepsilon - A_0^{-1} u^0 \|_{H^1_0(0, \pi)} \leq C(\| u^\varepsilon - u^0 \|_{L^2(0, \pi)} + \| a_\varepsilon - a_0 \|_{\infty})$, $u^\varepsilon, u^0 \in L^2(0, \pi)$.

(iii) $\| (\mu + A_{\varepsilon})^{-1} u^\varepsilon - (\mu + A_0)^{-1} u^0 \|_{H^1_0(0, \pi)} \leq C(\| u^\varepsilon - u^0 \|_{L^2(0, \pi)} + \| a_\varepsilon - a_0 \|_{\infty})$, for $\mu$ in the resolvent set of $A_{\varepsilon}$ and $A_0$ and $u^\varepsilon, u^0 \in L^2(0, \pi)$.

Here, $C > 0$ is a constant independent of $\varepsilon$.

Proof. The proof has been done in [1, Section 3] and [5, Chapter 14]. Since there is a difference between these works due to the presence of an exponent $1/2$, we outline the proof of item (ii) here.

Let $u^\varepsilon, u^0 \in L^2(0, \pi)$ and let $v^\varepsilon, v^0$ be the respective solution of $A_{\varepsilon} v^\varepsilon = u^\varepsilon$ and $A_0 v^0 = u^0$. Then,

$$
\int_0^\pi a_\varepsilon v^\varepsilon \varphi_x dx = \int_0^\pi u^\varepsilon \varphi dx, \quad \text{and} \quad \int_0^\pi a_0 v^0 \varphi_x dx = \int_0^\pi u^0 \varphi dx, \quad \varphi \in H^1_0(0, \pi). \quad (2.3)
$$

Taking $\varphi = v^\varepsilon - v^0$, we obtain

$$
\int_0^\pi a_\varepsilon v^\varepsilon (v^\varepsilon_x - v^0_x) dx - \int_0^\pi a_0 v^0 (v^\varepsilon_x - v^0_x) dx = \int_0^\pi (u^\varepsilon - u^0)(v^\varepsilon - v^0) dx.
$$

which implies

$$
\int_0^\pi a_\varepsilon (v^\varepsilon_x - v^0_x)^2 dx + \int_0^\pi (a_\varepsilon - a_0) v^0 (v^\varepsilon_x - v^0_x) dx = \int_0^\pi (u^\varepsilon - u^0)(v^\varepsilon - v^0) dx.
$$

By (2.2) and the uniformity between the $X^1_{\varepsilon}$ norm and $H^1_0(0, \pi)$ norm, we get

$$
\| v^\varepsilon - v^0 \|_{H^1_0(0, \pi)} \leq C(\| u^\varepsilon - u^0 \|_{L^2(0, \pi)} + \| a_\varepsilon - a_0 \|_{\infty}),
$$

for some positive constant $C$ independent of $\varepsilon$.

Finishing we notice that $A_{\varepsilon} v^\varepsilon = u^\varepsilon$ and $A_0 v^0 = u^0$ implies $v^\varepsilon = A_{\varepsilon}^{-1} u^\varepsilon$ and $v^0 = A_0^{-1} u^0$. \hfill \Box

We write (1.4) as an evolution equation in $L^2(0, \pi)$ in the following way

$$
\begin{cases}
  u^\varepsilon_t + A_{\varepsilon} u^\varepsilon = f(u^\varepsilon), \\
  u^\varepsilon(0) = u^\varepsilon_0,
\end{cases} \quad (2.4)
$$

where we have used the same notation $f$ for the nonlinearity of (1.4) and its functional $f_1 : H^1_0(0, \pi) \to L^2(0, \pi)$ given by $f_1(u)(x) = f(u(x))$.

We denote the spectra of the divergence operator $-A_{\varepsilon}$, $\varepsilon \in [0, \varepsilon_0]$, ordered and counting multiplicity by

$$
\cdots < -\lambda^\varepsilon_m < -\lambda^\varepsilon_{m-1} < \cdots < -\lambda^\varepsilon_1
$$

and we let $\left\{ \varphi^\varepsilon_j \right\}_{j=1}^{\infty}$ be the corresponding eigenfunctions.

The resolvent convergence $\| A_{\varepsilon}^{-1} - A_0^{-1} \|_{L^2(0, \pi), H^1_0(0, \pi)} \to 0$ as $\varepsilon \to 0$ imply the convergence of eigenvalues, that is, $\lambda^\varepsilon_m \to \lambda^0_m$ as $\varepsilon \to 0$, $m = 1, 2, \ldots$ as we can see in [1, Proposition 3.3]. Moreover, by [1, Corollary 3.6], we obtain a constant $C > 0$ independent of $\varepsilon$ such that

$$
|\lambda^\varepsilon_m - \lambda^0_m| \leq C \| a_\varepsilon - a \|_{\infty}, \quad m = 1, 2, \ldots
$$

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We take a closed curve $\Gamma_m$ contained in the resolvent set of $-A_0$ around $\{-\lambda^0_1, \ldots, -\lambda^0_m\}$. By (2.5) we can take $\varepsilon$ sufficiently small for that $\Gamma_m$ be contained in the resolvent set of $-A_\varepsilon$ around $\{-\lambda^\varepsilon_1, \ldots, -\lambda^\varepsilon_m\}$. Thus, we can define

$$P_\varepsilon = \frac{1}{2\pi i} \int_{\Gamma_m} (\mu + A_\varepsilon)^{-1} d\mu, \quad \varepsilon \in [0, \varepsilon_0],$$

(2.6)

which is the spectral projection onto the space generated by the first $m$ eigenfunctions of $A_\varepsilon$. It follows from (2.6) and Theorem 2.1 that there is a constant $C > 0$ independent of $\varepsilon$ such that,

$$\|P_\varepsilon u^\varepsilon - P_0 u^0\|_{H^1_0(0,\pi)} \leq C(\|u^\varepsilon - u^0\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty), \quad u^\varepsilon, u^0 \in H^1_0(0,\pi)$$

(2.7)

and

$$\sup_{\varepsilon \in [0,\varepsilon_0]} \sup_{u^\varepsilon \in L^2(0,\pi)} \|P_\varepsilon u^\varepsilon\|_{H^1_0(0,\pi)} \leq C.$$

In the next section, we will fix $m$ sufficiently large to obtain conditions for the invariant manifold theorem. Thus, to avoid heavy notation, we omit the dependency of $m$ on $P_\varepsilon$ and we denote $Q_\varepsilon = (I - P_\varepsilon)$ the projection over its orthogonal complement.

# 3 Invariant manifold and reduction of the dimension

The resolvent convergence $\|A_\varepsilon^{-1} - A_0^{-1}\|_{L^2(0,\pi), H^1(0,\pi)} \to 0$ as $\varepsilon \to 0$ guarantees the spectral convergence of the eigenvalues $\lambda^\varepsilon_m \to \lambda^0_m$ as $\varepsilon \to 0$, $m = 1, 2, \ldots$. But, the operator $A_0$ has a gap on its eigenvalues, that is, $\lambda^0_m - \lambda^0_{m+1} \to \infty$ as $m \to \infty$. Thus, for $\varepsilon_0$ sufficiently small, we have a similar gap on the eigenvalues of $A_\varepsilon$. This fact, enables us to construct inertial manifolds of the same dimension given by rank$(P_\varepsilon) = \text{span}[\varphi_1^\varepsilon, \ldots, \varphi_m^\varepsilon]$, where according with the previous section, $\varphi_i^\varepsilon$ is the associated eigenfunction to the eigenvalue $\lambda^\varepsilon_i$, $m = 1, 2, \ldots$.

For each $\varepsilon \in [0,\varepsilon_0]$, we decompose $H^1_0(0,\pi) = Y_\varepsilon \oplus Z_\varepsilon$, where $Y_\varepsilon = P_\varepsilon(H^1_0(0,\pi))$ and $Z_\varepsilon = Q_\varepsilon(H^1_0(0,\pi))$ and we define $A_\varepsilon^+ = A_\varepsilon|_{Y_\varepsilon}$ and $A_\varepsilon^- = A_\varepsilon|_{Z_\varepsilon}$. Using this decomposition we rewrite (2.4) as the following coupled equation

\[
\begin{align*}
\dot{v}_\varepsilon^\varepsilon + A_\varepsilon^+ v^\varepsilon & = P_\varepsilon f(v^\varepsilon + z^\varepsilon) = H_\varepsilon(v^\varepsilon, z^\varepsilon), \\
\dot{z}_\varepsilon^\varepsilon + A_\varepsilon^- z^\varepsilon & = Q_\varepsilon f(v^\varepsilon + z^\varepsilon) = G_\varepsilon(v^\varepsilon, z^\varepsilon).
\end{align*}
\]

(3.1)

The invariant manifold theorem whose proof can be found in [5, Chapter 8], states as follows.

**Theorem 3.1.** For sufficiently large $m$ and $\varepsilon_0 > 0$ small, there is an invariant manifold $\mathcal{M}_\varepsilon$ for (2.4) given by

$$\mathcal{M}_\varepsilon = \{u^\varepsilon \in H^1_0(0,\pi) ; u^\varepsilon = P_\varepsilon u^\varepsilon + s^\varepsilon(P_\varepsilon u^\varepsilon)\}, \quad \varepsilon \in [0,\varepsilon_0],$$

where $s^\varepsilon : Y_\varepsilon \to Z_\varepsilon$ is a Lipschitz continuous map satisfying

$$\|s^\varepsilon(\varphi^\varepsilon) - s^0(\varphi^0)\|_{H^1_0(0,\pi)} \leq C(\|\varphi^\varepsilon - \varphi^0\|_{H^1_0(0,\pi)} + \|a_\varepsilon - a_0\|_\infty \log(\|a_\varepsilon - a_0\|_\infty)))$$

(3.2)

where $\varphi^\varepsilon \in Y_\varepsilon$, $\varphi^0 \in Y_0$ and $C$ is a positive constant independent of $\varepsilon$. The invariant manifold $\mathcal{M}_\varepsilon$ is exponentially attracting and the global attractor $A_\varepsilon$ of the problem (2.4) lies in $\mathcal{M}_\varepsilon$. The flow of $u^\varepsilon_0 \in \mathcal{M}_\varepsilon$ is given by

$$T_\varepsilon(t) u^\varepsilon_0 = v^\varepsilon(t) + s^\varepsilon(v^\varepsilon(t)), \quad t \in \mathbb{R},$$

(3.3)
where $\psi^\varepsilon(t)$ satisfies

\[
\begin{cases}
\psi^\varepsilon_t + A^\varepsilon_1 \psi^\varepsilon = H^\varepsilon(\psi^\varepsilon, s^\varepsilon_1(\psi^\varepsilon)), & t \in \mathbb{R}, \\
\psi^\varepsilon(0) = P^\varepsilon u_0^\varepsilon \in Y^\varepsilon.
\end{cases}
\]

(3.4)

For the proof of Theorem 3.1 we refer [5]. To see how obtain the estimate (3.2), we refer [1, 4].

Now, we use the theory developed in [4] to identify (3.4) as an ordinary differential equation in $\mathbb{R}^m$. This identification is made by an isomorphism between $Y^\varepsilon$ and $\mathbb{R}^m$. Since our aim in the next section will be to construct a $\varepsilon$-isometry between the attractors, it is convenient to make the isomorphism $Y^\varepsilon \approx \mathbb{R}^m$ an isometry. To accomplish this we follow the ideas of [4] that modify the basis of $Y^\varepsilon$.

Let $\varepsilon \in [0,\varepsilon_0]$. We consider in $Y^\varepsilon$ the following set \{\(P^\varepsilon q^0_1, \ldots, P^\varepsilon q^0_m\)\}. It has been proved in [4] that this set is a basis for $Y^\varepsilon$. We define $L^\varepsilon : Y^\varepsilon \rightarrow \mathbb{R}^m$ by $L^\varepsilon(\sum_{i=1}^m \alpha_i P^\varepsilon q^0_i) = \sum_{i=1}^m \alpha_i \varepsilon^i$, where \{\(\varepsilon^i\)\}_{i=1}^m is the canonical basis of $\mathbb{R}^m$. This choices make $L^\varepsilon$ a isometry between $Y^\varepsilon$ and $\mathbb{R}^m$ and if we denote $\mathbb{R}^m_0$ the $\mathbb{R}^m$ with the norm $\|x\|_{\mathbb{R}^m} = (\sum_{i=1}^m x_i^2 \lambda_i)^{\frac{1}{2}}$, then $\|\bar{u}^0\|_{H^\varepsilon_0(0,\pi)} = \|L^\varepsilon_0 \bar{u}^0\|_{\mathbb{R}^m_0}$.

**Proposition 3.2.** The following statements hold true:

(i) If $\bar{u}^\varepsilon \in Y^\varepsilon$ and $\bar{u}_0 \in Y^\varepsilon_0$ are such that $\|\bar{u}^\varepsilon\|_{H^\varepsilon_0(0,\pi)} < \bar{C}$ and $\|\bar{u}^\varepsilon\|_{H^\varepsilon(0,\pi)} < \bar{C}$, where $\bar{C}$ is a constant independent of $\varepsilon$. Then $\|L^\varepsilon \bar{u}^\varepsilon - L^\varepsilon_0 \bar{u}^0\|_{\mathbb{R}^m} \leq C(\|\bar{u}^\varepsilon - \bar{u}_0\|_{H^\varepsilon_0(0,\pi)} + \|a^\varepsilon - a_0\|_{\infty})$, for a constant $C > 0$ independent of $\varepsilon$.

(ii) If $\bar{u}^\varepsilon, \bar{u}_0 \in \mathbb{R}^m$ are such that $\|\bar{u}^\varepsilon\|_{\mathbb{R}^m} < \bar{C}$ and $\|\bar{u}^\varepsilon\|_{\mathbb{R}^m} < \bar{C}$, where $\bar{C}$ is a constant independent of $\varepsilon$. Then $\|L^\varepsilon_0^{-1} \bar{u}^\varepsilon - L^\varepsilon_0^{-1} \bar{u}^0\|_{H^\varepsilon_0(0,\pi)} \leq C(\|\bar{u}^\varepsilon - \bar{u}_0\|_{\mathbb{R}^m} + \|a^\varepsilon - a_0\|_{\infty})$, for a constant $C > 0$ independent of $\varepsilon$.

**Proof.** The proof of item (i) follows as Lemma 5.4 of [4]. We prove (ii) using similar arguments.

Let $\bar{u}^\varepsilon = (\alpha^\varepsilon_1, \ldots, \alpha^\varepsilon_m)$ and $\bar{u}_0 = (\alpha^0_1, \ldots, \alpha^0_m)$ in $\mathbb{R}^m$. Then,

\[
L^\varepsilon_0^{-1} \bar{u}^\varepsilon - L^\varepsilon_0^{-1} \bar{u}^0 = \sum_{i=1}^m \alpha^\varepsilon_i P^\varepsilon q^0_i - \sum_{i=1}^m \alpha^0_i P^\varepsilon q^0_i
\]

\[
= (P^\varepsilon - P^0) \sum_{i=1}^m \alpha^\varepsilon_i q^0_i + \sum_{i=1}^m (\alpha^\varepsilon_i - \alpha^0_i) P^\varepsilon q^0_i
\]

which implies,

\[
\|L^\varepsilon_0^{-1} \bar{u}^\varepsilon - L^\varepsilon_0^{-1} \bar{u}^0\|_{H^\varepsilon_0(0,\pi)} \leq C\|a^\varepsilon - a_0\|_{\infty} + \|\bar{u}^\varepsilon - \bar{u}_0\|_{\mathbb{R}^m_0} \tag{3.4} \]

The map $s^\varepsilon : Y^\varepsilon \rightarrow Z^\varepsilon$ is obtained as the fixed point of the contraction $\Phi^\varepsilon : \Sigma^\varepsilon \rightarrow \Sigma^\varepsilon$ given by

\[
\begin{cases}
\Sigma^\varepsilon = \left\{ s^\varepsilon : Y^\varepsilon \rightarrow Z^\varepsilon ; \|s^\varepsilon\|_{\infty} \leq D \text{ and } \|s^\varepsilon(v) - s^\varepsilon(\bar{v})\|_{H^\varepsilon_0(0,\pi)} \leq \Delta \|v - \bar{v}\|_{H^\varepsilon_0(0,\pi)} \right\}, \\
\Phi^\varepsilon(s^\varepsilon)(\eta) = \int_{-\infty}^{0} e^{-\lambda_\eta r} G^\varepsilon(\psi^\varepsilon(r), s^\varepsilon(\psi^\varepsilon(r))) dr,
\end{cases}
\]

where $D$ and $\Delta$ are positive constants independent of $\varepsilon$ and $\psi^\varepsilon(r) \in Y^\varepsilon$ is the global solution of (3.4) with $\eta = P^\varepsilon u_0^\varepsilon$. With the aid of $L^\varepsilon$, we can define new invariant manifolds $N^\varepsilon$, given by

\[
N^\varepsilon = \{ L^\varepsilon^{-1}(x) + \theta^\varepsilon(x) : x \in \mathbb{R}^m \},
\]
where $\theta : \mathbb{R}^m \to Z_\varepsilon$ is given by $\theta^\varepsilon = s^\varepsilon \circ L_\varepsilon^{-1}$. Therefore, $\theta^\varepsilon$ is a fixed point of

$$
\theta^\varepsilon(x) = \int_{-\infty}^{0} e^{-A^\varepsilon r} G_{\varepsilon}(v^\varepsilon(r), \theta^\varepsilon(L_\varepsilon v^\varepsilon(r))) \, dr,
$$

such that

$$
\|\theta^\varepsilon - \theta^0\|_\infty \leq C\|a_\varepsilon - a_0\|_\infty \log(\|a_\varepsilon - a_0\|_\infty),
$$

for some constant $C > 0$ independent of $\varepsilon$.

By Theorem 3.1 the semigroup $T_\varepsilon(\cdot)$ restrict to $\mathcal{M}_\varepsilon$ is a flow whose behavior is dictate by solutions of (3.4) that can be transposed to $\mathbb{R}^m$ as

$$
x^\varepsilon(t) + L_\varepsilon A^\varepsilon_\varepsilon L_\varepsilon^{-1}(x^\varepsilon) = L_\varepsilon H_\varepsilon(L_\varepsilon^{-1}(x^\varepsilon), \theta^\varepsilon(x^\varepsilon)), \quad t \in \mathbb{R},
$$

(3.5)

\[\begin{align*}
& x^\varepsilon(0) = L_\varepsilon P_\varepsilon u_0^\varepsilon := x_0^\varepsilon \in \mathbb{R}^m.
\end{align*}\]

**Theorem 3.3.** The solutions of (3.5) generate a Morse–Smale flow in $\mathbb{R}^m$.

**Proof.** Since all equilibrium points of (1.4) are hyperbolic, the author in [9] has proved that the semigroup $T_\varepsilon(\cdot)$ is Morse–Smale. Therefore, $T_\varepsilon(\cdot)|_{\mathcal{M}_\varepsilon}$ is a Morse–Smale semigroup. Following [12, Chapter 3] we obtain that the projected semiflow $\tilde{T}_\varepsilon(\cdot)$ of $T_\varepsilon(\cdot)$ in $\mathbb{R}^m$ is Morse–Smale. \(\square\)

In what follows we prove several technical results that will be essential to prove the results in the next section. Here is the moment that we take a different way of [3].

**Proposition 3.4.** The projection $P_\varepsilon$ restrict to $\mathcal{M}_\varepsilon$ is an injective map and $P_\varepsilon^{-1}|_{\mathcal{M}_\varepsilon}$ restrict to the set $\bar{\mathcal{A}}_\varepsilon := P_\varepsilon\mathcal{A}_\varepsilon$ is uniformly bounded in $\varepsilon$ and

$$
\|P_\varepsilon^{-1} u^\varepsilon - P_0^{-1} q_\varepsilon(u^\varepsilon)\| \leq C(\|u^\varepsilon - q_\varepsilon(u^\varepsilon)\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty), \quad u^\varepsilon \in \bar{\mathcal{A}}_\varepsilon,
$$

(3.6)

for any homeomorphism $q_\varepsilon : \bar{\mathcal{A}}_\varepsilon \to \bar{A}_0$.

**Proof.** Let $u^\varepsilon, v^\varepsilon \in \mathcal{M}_\varepsilon$ such that $P_\varepsilon u^\varepsilon = P_\varepsilon v^\varepsilon$, then $u^\varepsilon = P_\varepsilon u^\varepsilon + s^\varepsilon(P_\varepsilon u^\varepsilon) = P_\varepsilon v^\varepsilon + s^\varepsilon(P_\varepsilon v^\varepsilon) = v^\varepsilon$. By (1.7), we have a positive constant $C$ independent of $\varepsilon$ such that

$$
\sup_{\varepsilon \in [0,\varepsilon_0]} \sup_{\tilde{\tilde{\alpha}} \in \bar{\mathcal{A}}_\varepsilon} \|P_\varepsilon^{-1} u^\varepsilon\|_{H^1(0,\pi)} \leq \sup_{\varepsilon \in [0,\varepsilon_0]} \sup_{u^\varepsilon \in \bar{\mathcal{A}}_\varepsilon} \|u^\varepsilon\|_{H^1(0,\pi)} \leq C.
$$

Finally, if $\tilde{u}^\varepsilon \in \bar{\mathcal{A}}_\varepsilon$ and $q_\varepsilon : \bar{\mathcal{A}}_\varepsilon \to \bar{A}_0$ is a homeomorphism, then

$$
\|P_\varepsilon^{-1} \tilde{u}^\varepsilon - P_0^{-1} q_\varepsilon(\tilde{u}^\varepsilon)\|_{H^1(0,\pi)} = \|P_0^{-1} P_\varepsilon - P_0^{-1} q_\varepsilon(\tilde{u}^\varepsilon)\|_{H^1(0,\pi)}
\leq \|P_0^{-1} ||L(H^1_\varepsilon(0,\pi),L^2(0,\pi))| |P_0 P_\varepsilon^{-1} \tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{H^1_\varepsilon(0,\pi)}
\leq \|P_0^{-1} ||L(H^1_\varepsilon(0,\pi),L^2(0,\pi))| |P_0 P_\varepsilon^{-1} \tilde{u}^\varepsilon - P_\varepsilon P_\varepsilon^{-1} \tilde{u}^\varepsilon + P_\varepsilon P_\varepsilon^{-1} \tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{H^1_\varepsilon(0,\pi)}
\leq \|P_0^{-1} ||L(H^1_\varepsilon(0,\pi),L^2(0,\pi))| |(P_0 - P_\varepsilon) P_\varepsilon^{-1} \tilde{u}^\varepsilon\|_{H^1_\varepsilon(0,\pi)} + \|P_\varepsilon^{-1} ||L(H^1_\varepsilon(0,\pi),L^2(0,\pi))| |\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{H^1_\varepsilon(0,\pi)}
\leq C(\|a_\varepsilon - a_0\|_\infty + \|\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{L^2(0,\pi)}),
$$

where we have used (2.7) to obtain a positive constant $C$ independent of $\varepsilon$. \(\square\)

In what follows, we denote $P_\varepsilon^{-1}$ the inverse of $P_\varepsilon|_{\mathcal{M}_\varepsilon} : \mathcal{M}_\varepsilon \to Y_\varepsilon$.

**Proposition 3.5.** Let $\tilde{T}_\varepsilon(\cdot)$ be the flow given by solutions of (3.5) and $\bar{T}_\varepsilon(\cdot)$ be the flow given by solutions of (3.4). Then, it is valid the following properties
(i) \( L^{-1}_e \tilde{T}_e(t) \bar{u}^e = \tilde{T}_e(t)L^{-1}_e \bar{u}^e, \quad \bar{u}^e \in \mathbb{R}^m, \quad t \in \mathbb{R}. \)

(ii) \( \tilde{T}_e(t)L_e \bar{u}^e = L_e \tilde{T}_e(t) \bar{u}^e, \quad \bar{u}^e \in Y_e, \quad t \in \mathbb{R}. \)

(iii) \( P_eT_e(t)u^e = \tilde{T}_e(t)P_eu^e, \quad u^e \in H^1_0(0, \pi), \quad t \geq 0. \)

(iv) \( P_eL^{-1}_e \tilde{T}_e(t) \bar{u}^e = T_e(t)L^{-1}_e \bar{u}^e, \quad \bar{u}^e \in Y_e, \quad t \geq 0. \)

(v) Given a function \( \tau_e : \mathbb{R} \times \bar{A}_e \to \mathbb{R} \) such that, \( \tau_e(0,u^e) = 0 \) and \( \tau_e(\cdot,u^e) \) is a increasing function mapping \( \mathbb{R} \) onto \( \mathbb{R} \), there exist a function \( \bar{\tau}_e : \mathbb{R} \times \bar{A}_e \to \mathbb{R} \) such that, \( \bar{\tau}_e(0,P_eu^e) = 0 \) and \( \bar{\tau}_e(\cdot,P_eu^e) \) is a increasing function mapping \( \mathbb{R} \) onto \( \mathbb{R} \) such that

\[
P_eT_e(\bar{\tau}_e(t,P_eu^e))u^e = \tilde{T}_e(\bar{\tau}_e(t,P_eu^e))P_eu^e, \quad u^e \in \mathcal{A}_e, \quad t \in \mathbb{R}.
\]

(vi) Given a function \( \bar{\tau}_e : \mathbb{R} \times \bar{A}_e \to \mathbb{R} \) such that, \( \bar{\tau}_e(0,u^e) = 0 \) and \( \bar{\tau}_e(\cdot,u^e) \) is a increasing function mapping \( \mathbb{R} \) onto \( \mathbb{R} \), there exist a function \( \bar{\tau}_e : \mathbb{R} \times \bar{A}_e \to \mathbb{R} \) such that, \( \bar{\tau}_e(0,L_eu^e) = 0 \) and \( \bar{\tau}_e(\cdot,L_eu^e) \) is a increasing function mapping \( \mathbb{R} \) onto \( \mathbb{R} \) such that

\[
L_e \tilde{T}_e(\bar{\tau}_e(t,L_eu^e))u^e = \tilde{T}_e(\bar{\tau}_e(t,L_eu^e))L_eu^e, \quad u^e \in \bar{A}_e, \quad t \in \mathbb{R}.
\]

**Proof.** Let \( \bar{u}^e \in \mathbb{R}^m \), then \( L^{-1}_e \bar{u}^e \in Y_e \) and \( \tilde{T}_e(t)L^{-1}_e \bar{u}^e \) is a solution of

\[
\begin{align*}
\dot{v}^e_t + A^+_e v^e &= H_e(v^e,s^e_0(v^e)), \quad t \in \mathbb{R}, \\
v^e(0) &= L^{-1}_e \bar{u}^e \in Y_e.
\end{align*}
\]

(3.7)

Defining \( q^e(t) = L^{-1}_e \tilde{T}_e(t) \bar{u}^e \), we have \( q^e(0) = L^{-1}_e \tilde{T}_e(0) \bar{u}^e = L^{-1}_e \bar{u}^e \) and

\[
q^e_t + A^+_e q^e(t) = L^{-1}_e \frac{\partial}{\partial t} \tilde{T}_e(t) \bar{u}^e + A^+_e L^{-1}_e \tilde{T}_e(t) \bar{u}^e
\]

\[
= L^{-1}_e \left( \frac{\partial}{\partial t} \tilde{T}_e(t) \bar{u}^e + L_eA^+_e L^{-1}_e \tilde{T}_e(t) \bar{u}^e \right).
\]

Since \( x^e_t := \tilde{T}_e(t) \bar{u}^e \) is a solution of

\[
\begin{align*}
x^e_t + L_e A^+_e L^{-1}_e (x^e) &= L_e H_e(L^{-1}_e (x^e), \theta^e_0 (x^e)), \quad t \in \mathbb{R}, \\
x^e(0) &= \bar{u}^e \in \mathbb{R}^m,
\end{align*}
\]

(3.8)

we obtain

\[
q^e_t + A^+_e q^e(t) = H_e(q^e(t), \theta^e_0 (q^e(t))).
\]

The bijection between \( \theta^e_0 \) and \( s^e_0 \) enables us to conclude that \( q^e(t) \) is also a solution of (3.7).

The result follows from the well-posedness of (3.7).

In the same way, we proof item (ii).

Item (iii) is immediate from (3.3) and (3.4) by noticing that \( P_eT_e(t)u^e = v^e(t) \) and we are denoting \( v^e(t) = \tilde{T}_e(t)P_eu^e \). Item (iv) follows from (iii) using that \( P_eu^e = \bar{u}^e \) if only if \( u^e = P_e^{-1} \bar{u}^e \), for some \( \bar{u}^e \in Y_e \). Item (v) follows from (iii) defining \( \bar{\tau}_e(t,P_eu^e) = \tau_e(t,u^e) \). In the same way, we obtain (vi).

**Proposition 3.6.** The set \( \bar{A}_e = P_e A_e \) is the global attractor for the semigroup \( \tilde{T}_e(\cdot) \) given by solutions of (3.4).
Proof. Since $A_ε$ is compact and $P_ε$ is continuous, we have $\bar{A}_ε = P_εA_ε$ a compact set in $Y_ε$. Proving the attraction, let $B \subset Y_ε$ a bounded set and let $v^f \in B$. Then $v^f + s_ε^*(v^f) \in M_ε$ and $T_ε(t)v^f = T_ε(t)v^f + s_ε^*(T_ε(t)v^f)$ for $t > 0$ and $v^f \in P_ε^{-1}(v^f)$. But $T_ε(t)$ is a gradient semigroup, then there is $u^f \in A_ε$ such that, $\|T_ε(t)v^f - u^f\|_{H^2(0, π)} \to 0$ as $t \to \infty$. In fact, the attraction property of the global attractor is uniform for the solutions starting at bounded sets. Hence, $\bar{A}_ε$ is a neighborhood of $A_ε$ containing all trajectory starting at $B$ after a time $t_B$. We take $\bar{u}^f \in \bar{A}_ε$ such that $\bar{u}^f = P_εu^f$. Thus,

$$\|T_ε(t)v^f - \bar{u}^f\|_{H^2(0, π)} \leq C\|T_ε(t)v^f - \bar{u}^f\|_{H^2(0, π)} + \|s_ε^*(T_ε(t)v^f) - s_ε^*(\bar{u}^f)\|_{H^2(0, π)} \leq C\|T_ε(t)v^f + s_ε^*(T_ε(t)v^f) - P_εu^f + s_ε^*(P_εu^f)\|_{H^2(0, π)} \to 0 \text{ as } t \to \infty,$$

for a constant $C > 0$ independent of $ε$, where the attraction property is also uniform for the solutions starting at bounded sets.

It remains to prove that $\bar{A}_ε$ is invariant. Let $\bar{u}^f \in \bar{A}_ε$ and $t \geq 0$. Writing $w^f = P_t\bar{u}^f$ for some $w^f \in A_ε$, we have by the invariance of $A_ε$, that there is $\bar{w}^f \in A_ε$ such that $T_ε(t)\bar{w}^f = w^f$, for some $t \geq 0$.

$$\bar{u}^f + s_ε^*(\bar{u}^f) = P_εw^f + s_ε^*(P_εw^f) = w^f = T_ε(t)\bar{w}^f = \hat{T}_ε(t)P_t\bar{w}^f + s_ε^*(\hat{T}_ε(t)P_t\bar{w}^f),$$

which implies $\bar{u}^f = \hat{T}_ε(t)P_t\hat{w}^f$, where $P_t\hat{w}^f \in \bar{A}_ε$.

Proposition 3.7. The set $\bar{A}_ε = L_εP_εA_ε$ is the global attractor for the semigroup $\hat{T}_ε(\cdot)$ given by solutions of (3.5).

Proof. Since $L_ε$ is continuous and $P_εA_ε$ is compact, we have $\bar{A}_ε = L_εP_εA_ε$ a compact set in $\mathbb{R}^n$. Let $B$ a bounded set in $\mathbb{R}^n$ and $\bar{u}^f \in B$, then $L_ε^{-1}\bar{u}^f \in L_ε^{-1}B$ which is a bounded set in $Y_ε$. Since $\hat{T}_ε(\cdot)$ is gradient, there is $\hat{w}^f \in \bar{A}_ε$ such that, $\|\hat{T}_ε(t)L_ε^{-1}\hat{w}^f - \hat{w}^f\|_{H^2(0, π)} \to 0$ as $t \to \infty$, where the attraction property is uniform for the solutions starting at bounded sets. Hence, $L_ε\hat{w}^f \in \bar{A}_ε$ is such that,

$$\|\hat{T}_ε(t)\hat{w}^f - L_ε\hat{w}^f\|_{\mathbb{R}^n} = \|L_ε^{-1}\hat{T}_ε(t)\hat{w}^f - \hat{w}^f\|_{H^2(0, π)} \to 0, \text{ as } t \to \infty,$$

where we have used that $L_ε$ is a isometry and Proposition 3.5.

It remains to prove that $\bar{A}_ε$ is invariant. Let $\bar{u}^f \in \bar{A}_ε$. Then $L_ε^{-1}\bar{u}^f \in \bar{A}_ε$ which is invariant. Thus, there is $\bar{w}^f \in \bar{A}_ε$ and $\bar{t} > 0$ such that $\hat{T}_ε(\bar{t})\bar{w}^f = L_ε^{-1}\bar{u}^f$. Thus, $L_ε\hat{T}_ε(\bar{t})\bar{w}^f = \bar{u}^f$ and by Proposition 3.5, we have $\hat{T}_ε(\bar{t})L_ε\bar{w}^f = \bar{u}^f$. 

4 Proof of Theorem 1.1

In this section, we prove the main result of this paper, the Theorem 1.1.

Theorem 4.1. The equation (3.5) is structurally stable at $ε = 0$. That is, for each $η > 0$ there is $ε_η > 0$ and for $ε \in (0, ε_η]$ there is a homeomorphism $h_ε : \bar{A}_ε \to \bar{A}_0$ such that,

$$\sup_{\bar{w}^f \in \bar{A}_ε} \|h_ε(\bar{w}^f) - \bar{u}^f\|_{\mathbb{R}^n} < η \quad \text{and} \quad h_ε(\hat{T}_ε(\bar{t}, \bar{u}^f)) \bar{u}^f = \hat{T}_0(\bar{t})h_ε(\bar{u}^f),$$

(4.1)

where $\bar{u}^f \in \bar{A}_ε$, $t \in \mathbb{R}$ and $\bar{t}_ε : \mathbb{R} \times \bar{A}_ε \to \mathbb{R}$ is a function such that, $\bar{t}_ε(0, \bar{u}^f) = 0$ and $\bar{t}_ε(\cdot, \bar{u}^f)$ is a increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$. 
Proof. The works [1] and [5, Chapter 14] have obtained the continuity of the semigroups $T_\varepsilon(\cdot) \to T_0(\cdot)$ as $\varepsilon \to 0$ in the $H^1_0(0,\pi)$ norm. Following [2] we obtain $\hat{T}_\varepsilon(\cdot) \to \hat{T}_0(\cdot)$ as $\varepsilon \to 0$ in the $C^1$ norm, since the invariant manifolds $M_\varepsilon$ and $M_0$ are close in the $C^1$ topology. Thus, $\hat{T}_\varepsilon(\cdot)$ is a small $C^1$ perturbation of $\hat{T}_0(\cdot)$ which is a Morse–Smale semigroup $\mathbb{R}^m$. The main property of Morse–Smale flows in finite dimension stated in [11, 14] and [13] is the structural stability, that is, for each $\eta > 0$ there is $\varepsilon_\eta > 0$ and for $\varepsilon \in (0,\varepsilon_\eta]$ there is a homeomorphism $h_\varepsilon : \tilde{A}_\varepsilon \to \tilde{A}_0$ such that, (4.1) is valid.

**Theorem 4.2.** The equation (3.4) is structurally stable at $\varepsilon = 0$. That is, for each $\eta > 0$ there is $\varepsilon_\eta > 0$ and for $\varepsilon \in (0,\varepsilon_\eta]$ there is a homeomorphism $j_\varepsilon : \tilde{A}_\varepsilon \to \tilde{A}_0$ such that,

$$\sup_{\tilde{u}^\varepsilon \in \tilde{A}_\varepsilon} ||j_\varepsilon(\tilde{u}^\varepsilon) - \tilde{u}^\varepsilon||_{H^1_0(0,\pi)} < C(||a_\varepsilon - a_0||_\infty + \eta) \quad \text{and} \quad j_\varepsilon(\tilde{T}_\varepsilon(t,\tilde{u}^\varepsilon)) = \tilde{T}_0(t)j_\varepsilon(\tilde{u}^\varepsilon), \quad (4.2)$$

where $\tilde{u}^\varepsilon \in \tilde{A}_\varepsilon$, $t \in \mathbb{R}$ and $\tilde{T}_\varepsilon : \mathbb{R} \times \tilde{A}_\varepsilon \to \mathbb{R}$ is function such that, $\tilde{T}_\varepsilon(0,\tilde{u}^\varepsilon) = 0$ and $\tilde{T}_\varepsilon(\cdot,\tilde{u}^\varepsilon)$ is a increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$.

Proof. We define the map $j_\varepsilon : \tilde{A}_\varepsilon \to \tilde{A}_0$ by $j_\varepsilon = L_0^{-1} \circ h_\varepsilon \circ L_\varepsilon$. Then, for $\tilde{u}^\varepsilon \in \tilde{A}_\varepsilon$ it follows from Proposition 3.2 and (4.1) that

$$||j_\varepsilon(\tilde{u}^\varepsilon) - \tilde{u}^\varepsilon||_{H^1_0(0,\pi)} = ||L_0^{-1}h_\varepsilon(L_\varepsilon(\tilde{u}^\varepsilon)) - \tilde{u}^\varepsilon||_{H^1_0(0,\pi)}$$

$$= ||L_0^{-1}h_\varepsilon(L_\varepsilon(\tilde{u}^\varepsilon)) - L_\varepsilon^{-1}L_\varepsilon\tilde{u}^\varepsilon||_{H^1_0(0,\pi)}$$

$$\leq C(||h_\varepsilon(L_\varepsilon(\tilde{u}^\varepsilon)) - L_\varepsilon\tilde{u}^\varepsilon||_{\mathbb{R}^m} + ||a_\varepsilon - a_0||_\infty)$$

$$\leq C(\eta + ||a_\varepsilon - a_0||_\infty).$$

Moreover, by (4.1) and Proposition 3.5, we obtain

$$j_\varepsilon(\tilde{T}_\varepsilon(t,\tilde{u}^\varepsilon)) = L_0^{-1} \circ h_\varepsilon \circ L_\varepsilon(\tilde{T}_\varepsilon(t,\tilde{u}^\varepsilon))$$

$$= L_0^{-1}(h_\varepsilon(\tilde{T}_\varepsilon(t,L_\varepsilon(\tilde{u}^\varepsilon)))L_\varepsilon(\tilde{u}^\varepsilon))$$

$$= L_0^{-1}\tilde{T}_0(t)h_\varepsilon(L_\varepsilon\tilde{u}^\varepsilon)$$

$$= \tilde{T}_0(t)L_0^{-1}h_\varepsilon(L_\varepsilon\tilde{u}^\varepsilon)$$

$$= \tilde{T}_0(t)L_0^{-1}h_\varepsilon(L_\varepsilon\tilde{u}^\varepsilon).$$

Hence, $j_\varepsilon$ is a homeomorphism between $\tilde{A}_\varepsilon$ and $\tilde{A}_0$ satisfying (4.2).

Now, we are in a condition to prove the Theorem 1.1.

**Proof. of Theorem 1.1.** We define the map $\kappa_\varepsilon : \mathcal{A}_\varepsilon \to \mathcal{A}_0$ by $\kappa_\varepsilon = P_0^{-1} \circ j_\varepsilon \circ P_\varepsilon$. Similarly to the proof of Theorem 4.2, we can prove that $\kappa_\varepsilon$ is a homeomorphism between $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ satisfying

$$||\kappa_\varepsilon(u^\varepsilon) - u^\varepsilon||_{H^1_0(0,\pi)} \leq C(\eta + ||a_\varepsilon - a_0||_\infty)$$

and

$$\kappa_\varepsilon(T_\varepsilon(\tau_\varepsilon(t,u^\varepsilon))) = T_0(t)\kappa_\varepsilon(u^\varepsilon).$$
References


