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Existence of nontrivial solutions for a quasilinear Schrödinger–Poisson system in $\mathbb{R}^3$ with periodic potentials

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Abstract. In this paper, we study the following quasilinear Schrödinger–Poisson system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4\Delta_4\phi = \lambda u^2, & x \in \mathbb{R}^3, \end{cases}$$

where λ and ε are positive parameters, $\Delta_4 u = \operatorname{div}(|\nabla u|^2\nabla u)$, V is a continuous and periodic potential function with positive infimum, $f(x,t) \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ is periodic with respect to x and only needs to satisfy some superquadratic growth conditions with respect to t. One nontrivial solution is obtained for λ small enough and ε fixed by a combination of variational methods and truncation technique.

Keywords: quasilinear Schrödinger–Poisson system, periodic potential, variational methods, truncation technique, nontrivial solution.

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1 Introduction and main result

In this paper, we consider the following system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4\Delta_4\phi = \lambda u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where λ and ε are positive parameters, $\Delta_4 u = \operatorname{div}(|\nabla u|^2\nabla u)$, V is a continuous and periodic potential function with positive infimum, f is a continuous function defined on $\mathbb{R}^3 \times \mathbb{R}$ which is periodic with respect to the first variable and satisfies some superquadratic growth conditions with respect to the second variable. Precisely, we assume that

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ with $\inf_{x\in\mathbb{R}^3} V(x) = V_0 > 0$ and it is a 1-periodic potential function, that is,

$$V(x+y) = V(x), \quad \text{for every } x \in \mathbb{R}^3 \text{ and } y \in \mathbb{Z}^3.$$

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(f_1) f is 1-periodic with respect to x. There exist positive constants C and $p \in (2,6)$ such that
$$|f(x,t)| \le C(1+|t|^{p-1}), \quad \text{for } (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

(f_2) $\lim_{|t|\to 0} \frac{f(x,t)}{t} = 0$, uniformly for $x \in \mathbb{R}^3$.

(f_3) There exist $\alpha \in (2,6)$ and $R > 0$ such that
$$\inf_{x\in\mathbb{R}^3,\ |t|\ge R} F(x,t) > 0,\ f(x,t)t \ge \alpha F(x,t), \quad \text{for } (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$
where $F(x,t) = \int_0^t f(x,s)ds$.

This class of system appears by studying a quantum mechanical model of extremely small devices in semiconductor nanostructures taking into account quantum structure and the longitudinal field oscillations during the beam propagation, for more details on the physical background of this class of system see [19]. Although this class of system has been well-known among the physicists, it has never been considered before [12,13] in the mathematical literature. One of them is something of type
$$\begin{cases} -\Delta u + \omega u + (\phi + \widetilde{\phi})u = 0, \\ -\Delta\phi - \varepsilon^4 \Delta_4 \phi = u^2 - n^*, \end{cases} \tag{1.2}$$
where u and ϕ represent the modulus of the wave function and the electrostatic potential respectively, n^* and $\widetilde{\phi}$ are given data of the problem which represent respectively the dopant density and the effective external potential. System (1.2) with some periodicity conditions was studied in [12] by the Krasnoselskii genus. Under minimal summability conditions on the data n^* and $\widetilde{\phi}$, existence of ground state solutions for system (1.2) was proved in [3] by means of minimization techniques, and the behaviour of these solutions whenever $\varepsilon \to 0^+$ was studied: these solutions converge to a ground state solution of Schrödinger–Poisson system associated with $\varepsilon = 0$ in system (1.2). A quasilinear Schrödinger–Poisson system in the unitary cube under periodic boundary conditions was studied in [13], the global existence and uniqueness of solution was obtained by using Galerkin scheme. There are also some studies on quasilinear Schrödinger–Poisson system with nonlinearities by variational methods. In [8], a class of quasilinear Schrödinger–Poisson system with an asymptotically linear term was studied, the existence and behaviour of ground state solutions as $\varepsilon \to 0^+$ were given. Recently, [11] studied the existence and asymptotic behaviour of solutions for a class of quasilinear Schrödinger–Poisson system with a critical nonlinearity combining with a 4-suplinear nonlinearity. Similar results were obtained in [10] in the two-dimensional case. In [21], we also got the existence and asymptotic behaviour of solutions for a class of quasilinear Schrödinger–Poisson system with coercive potential by variational methods and a truncation technique.

Formally, system (1.1) is the well-known Schrödinger–Poisson system if $\varepsilon = 0$ which has been given extensive attention and research in the last few decades. We mention that a reduction procedure for this class of system was proposed in [2] and an eigenvalue problem in bounded domains was considered. Schrödinger–Poisson system with general nonlinearity was first studied in [6] and later studied in many literatures, see for example [1,5,7,17,20,23] and the references therein. More recently, [9] studied the following quasilinear elliptic system by variational methods
$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda\phi|u|^{p-2}u = |u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^p, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $1 < p < 3$, $p < q < \frac{3p}{3-p}$, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ and $\lambda > 0$ is a parameter. The existence of nontrivial solutions for system (1.3) is obtained by the Mountain Pass theorem. According to the range of q, the scaling technique [14] and the truncation technique [15] were used to obtain a bounded Palais–Smale sequence respectively in [9].

From a mathematical point of view, on the one hand, the main novelty of system (1.1) is that the equation of the electrostatic potential in the system is not linear, that is, it is not the classical Poisson equation. Contrast to the classical Poisson equation or the second equation in system (1.3), the solution of the second equation in system (1.1) has neither an explicit formula nor homogeneity properties. It leads to that the scaling technique [14] is no longer applicable. It is natural to ask whether the truncation technique [15] can be used to deal with system (1.1), especially for the case $\alpha \in (2, 4]$. One the other hand, since under our assumptions there is no compact embedding between the main working spaces, we can not prove that the variational functional associated to system (1.1) satisfies (PS) condition directly. Lions vanishing lemma[18] will be applied to prove that system (1.1) enjoys at least one nontrivial solutions whenever the positive parameter λ is small enough. In this process, the weak convergence property of the solutions for the second equation in system (1.1) plays an important role. However, due to the "bad" properties of those solutions, this weak convergence property of them is not apparent. We will follow the arguments of [4, 9], together with the uniqueness of the solution for the second equation in system (1.1), to solve this key technique problem, for more details, see Lemma 2.2.

Before stating our main result, we give several notations. For any $q \in [1, +\infty]$, we denote by $|\cdot|_q$ the norm of the Lebesgue space $L^q(\mathbb{R}^3)$. $D^{1,2}(\mathbb{R}^3)$ is the Hilbert space defined as the completion of the test functions $C_0^\infty(\mathbb{R}^3)$ with respect to the L^2 norm of the gradient. We denote by X the completion of the functions $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $|\nabla \cdot|_2 + |\nabla \cdot|_4$, which is a reflexive Banach space. Under assumption (V), let $H_V^1(\mathbb{R}^3)$ be $H^1(\mathbb{R}^3)$ equipped with the following norm and inner product

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\, dx\right)^{\frac{1}{2}}, \qquad (u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv) dx.$$

Assumption (V) also guarantees the continuous embedding from $H_V^1(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$, $q \in [2, 6]$ and local compact embedding from $H_V^1(\mathbb{R}^3)$ to $L^q_{loc}(\mathbb{R}^3)$, $q \in [1, 6)$.

As usual, a weak solution for system (1.1) is a pair $(u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}) \in H_V^1(\mathbb{R}^3) \times X$ such that

$$\begin{cases} \int_{\mathbb{R}^3} (\nabla u_{\lambda,\varepsilon} \nabla v + V(x) u_{\lambda,\varepsilon} v + \lambda \phi_{\lambda,\varepsilon} u_{\lambda,\varepsilon} v) dx = \int_{\mathbb{R}^3} f(x, u_{\lambda,\varepsilon}) v dx, & v \in H_V^1(\mathbb{R}^3), \\ \int_{\mathbb{R}^3} (\nabla \phi_{\lambda,\varepsilon} + \varepsilon^4 |\nabla \phi_{\lambda,\varepsilon}|^2 \nabla \phi_{\lambda,\varepsilon}) \nabla \varphi dx = \lambda \int_{\mathbb{R}^3} u_{\lambda,\varepsilon}^2 \varphi dx, & \varphi \in X. \end{cases}$$

Our main result is as follows.

Theorem 1.1. *Under the assumptions (V) and (f_1)–(f_3), there exists $\lambda_0 > 0$ such that system (1.1) has at least one nontrivial solution $(u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}) \in H_V^1(\mathbb{R}^3) \times X$ for all $(\lambda, \varepsilon) \in (0, \lambda_0) \times (0, \infty)$. Moreover, $\phi_{\lambda,\varepsilon}$ is nonnegative.*

Remark 1.2. Compared with our last result in [21], the main difficulty here is the lack of compactness. In particular, the weak convergence property of the solutions for the second equation in system (1.1) is the key to obtaining a nontrivial solution for system (1.1).

Remark 1.3. The constraint on λ is mainly used to guarantee the variational functional associated to system (1.1) enjoys a (PS) sequence with a prior bound. If $\alpha \in (4, 6)$, due to (i)

of Lemma 2.2, it is easy to obtain a bounded (PS) sequence of the variational functional associated to system (1.1) with $(\lambda,\varepsilon)\in(0,\infty)\times(0,\infty)$ by using standard methods. Thus, the constraint that $\lambda<\lambda_0$ can be got rid of in this case. We leave details of the proof to the interested readers.

Throughout the paper, we denote C_q the constant of Sobolev imbedding from $H^1_V(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ for $q\in[2,6]$. $S=\inf_{\varphi\in D^{1,2}(\mathbb{R}^3))\setminus\{0\}}\frac{|\nabla\varphi|_2^2}{|\varphi|_6^2}$ is the optimal constant in the Sobolev inequality. The rest of the paper is organized as follows. We give some preliminaries in Section 2. The proof of Theorem 1.1 is given in Section 3.

2 Preliminaries

First, under our assumptions, system (1.1) has a variational structure. Formally, its corresponding functional is defined by

$$\mathcal{J}_{\lambda,\varepsilon}(u,\phi)=\frac{1}{2}\|u\|^2+\frac{\lambda}{2}\int_{\mathbb{R}^3}\phi u^2dx-\frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi|^2dx-\frac{\varepsilon^4}{8}\int_{\mathbb{R}^3}|\nabla\phi|^4dx-\int_{\mathbb{R}^3}F(x,u)dx.$$

It is not difficult to see that the critical points of $\mathcal{J}_{\lambda,\varepsilon}$ are the weak solutions of system (1.1). Since the functional $\mathcal{J}_{\lambda,\varepsilon}$ is strongly indefinite, the reduction procedure which is successfully used to study the classical Schrödinger–Poisson system will be applied to deal with system (1.1). Similar to Lemma 2.1 of [21] or Lemma 2.2 of [8], we have the following result.

Lemma 2.1. *For any $u\in H^1_V(\mathbb{R}^3)$ and $\lambda,\varepsilon>0$, there exists a unique nonnegative weak solution $\phi_{\lambda,\varepsilon}(u)\in X$ for*

$$-\Delta\phi-\varepsilon^4\Delta_4\phi=\lambda u^2,\qquad x\in\mathbb{R}^3.\tag{2.1}$$

That is, for any $\varphi\in X$, we have

$$\int_{\mathbb{R}^3}(\nabla\phi_{\lambda,\varepsilon}(u)+\varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u)|^2\nabla\phi_{\lambda,\varepsilon}(u))\nabla\varphi dx=\lambda\int_{\mathbb{R}^3}u^2\varphi dx.$$

Next, we give some properties of the weak solution $\phi_{\lambda,\varepsilon}(u)$ for equation (2.1).

Lemma 2.2. *For every $\lambda,\varepsilon>0$, $\phi_{\lambda,\varepsilon}(u)$ enjoys the following properties.*

(i) For every $u\in H^1_V(\mathbb{R}^3)$,

$$|\nabla\phi_{\lambda,\varepsilon}(u)|_2^2+\varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u)|_4^4=\lambda\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u)u^2dx\le\lambda^2S^{-1}C^4_{\frac{12}{5}}\|u\|^4;$$

(ii) if $\{u_n\}$ is bounded in $H^1_V(\mathbb{R}^3)$, then there exist a subsequence still denoted by $\{u_n\}$ and $u\in H^1_V(\mathbb{R}^3)$ such that

$$\phi_{\lambda,\varepsilon}(u_n)\rightharpoonup\phi_{\lambda,\varepsilon}(u)\text{ in }X,\ \int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u_n)u_nvdx\to\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u)uvdx,\quad\text{for }v\in H^1_V(\mathbb{R}^3);$$

(iii) $\phi_{\lambda,\varepsilon}(u_y)(\cdot)=\phi_{\lambda,\varepsilon}(u)(\cdot+y)$, for every $y\in\mathbb{R}^3$, where $u_y(\cdot)=u(\cdot+y)$.

Proof. (1) By the definition of $\phi_{\lambda,\varepsilon}(u)$, the first equality in (i) is true. Then by the Hölder inequality and the Sobolev embedding theorem, we can get that the first conclusion is true.

(2) Since $\{u_n\}$ is bounded in $H^1_V(\mathbb{R}^3)$, going if necessary to a subsequence, there exists $u \in H^1_V(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H^1_V(\mathbb{R}^3)$. By the Sobolev embedding theorem and the local compact embedding theorem, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } L^p(\mathbb{R}^3),\ p \in [2,6]; \\ u_n &\to u && \text{in } L^p_{loc}(\mathbb{R}^3),\ p \in [1,6); \\ u_n(x) &\to u(x), && a.e.\ x \in \mathbb{R}^3. \end{aligned}$$

Since (i) leads to that $\{\phi_{\lambda,\varepsilon}(u_n)\}$ is bounded in X, going if necessary to a subsequence, there exists $\phi_{\lambda,\varepsilon} \in X$ such that

$$\phi_{\lambda,\varepsilon}(u_n) \rightharpoonup \phi_{\lambda,\varepsilon} \quad \text{in } X \quad (\text{which is also valid in } D^{1,2}(\mathbb{R}^3)).$$

Furthermore, we can also assume that

$$\begin{aligned} \phi_{\lambda,\varepsilon}(u_n) &\rightharpoonup \phi_{\lambda,\varepsilon} && \text{in } L^6(\mathbb{R}^3); \\ \phi_{\lambda,\varepsilon}(u_n) &\to \phi_{\lambda,\varepsilon} && \text{in } L^p_{loc}(\mathbb{R}^3),\ p \in [1,6); \\ \phi_{\lambda,\varepsilon}(u_n)(x) &\to \phi_{\lambda,\varepsilon}(x), && a.e.\ x \in \mathbb{R}^3. \end{aligned}$$

On the one hand, by Lemma 2.1, we have

$$\int_{\mathbb{R}^3} (\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\varphi + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\varphi)dx = \lambda\int_{\mathbb{R}^3} u_n^2\varphi dx, \tag{2.2}$$

and

$$\int_{\mathbb{R}^3} (\nabla\phi_{\lambda,\varepsilon}(u)\nabla\varphi + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u)|^2\nabla\phi_{\lambda,\varepsilon}(u)\nabla\varphi)dx = \lambda\int_{\mathbb{R}^3} u^2\varphi dx, \quad \text{for } \varphi \in X. \tag{2.3}$$

Set $\varphi = (\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R$ in (2.2), where $\psi_R \in C_0^\infty(\mathbb{R}^3,\ [0,1])$ is a cut-off function such that $\psi_R|_{B_R(0)} = 1$, $\operatorname{supp}\psi_R \subset B_{2R}(0)$ and $|\nabla\psi_R| \le \frac{2}{R}$, we can get

$$\begin{aligned} 0 = &\int_{\mathbb{R}^3} (1 + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2)\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx \\ &+ \int_{\mathbb{R}^3} (1 + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2)\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \\ &- \lambda\int_{\mathbb{R}^3} u_n^2(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx. \end{aligned} \tag{2.4}$$

On the other hand, by the definition of weak convergence in X, we have

$$\int_{\mathbb{R}^3} (1 + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}|^2)\nabla\phi_{\lambda,\varepsilon}\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx \to 0, \quad \text{as } n \to \infty.$$

The local compact embedding theorem implies that

$$\int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0, \qquad \int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}|^2\nabla\phi_{\lambda,\varepsilon}\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0,$$

as $n \to \infty$. Then

$$\begin{aligned} o_n(1) = &\int_{\mathbb{R}^3} (1 + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}|^2)\nabla\phi_{\lambda,\varepsilon}\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx \\ &+ \int_{\mathbb{R}^3} (1 + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}|^2)\nabla\phi_{\lambda,\varepsilon}\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx. \end{aligned} \tag{2.5}$$

By calculating (2.4) minus (2.5), we deduce that

$$\begin{aligned} o_n(1) = &\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u_n) - \nabla\phi_{\lambda,\varepsilon}|^2\psi_R dx \\ &+ \varepsilon^4 \int_{\mathbb{R}^3} (|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n) - |\nabla\phi_{\lambda,\varepsilon}|^2\nabla\phi_{\lambda,\varepsilon})\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx \\ &+ \int_{\mathbb{R}^3} (\nabla\phi_{\lambda,\varepsilon}(u_n) - \nabla\phi_{\lambda,\varepsilon})\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \\ &+ \varepsilon^4 \int_{\mathbb{R}^3} (|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n) - |\nabla\phi_{\lambda,\varepsilon}|^2\nabla\phi_{\lambda,\varepsilon})\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \\ &- \lambda \int_{\mathbb{R}^3} u_n^2(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx. \end{aligned} \tag{2.6}$$

Since $\phi_{\lambda,\varepsilon}(u_n) \to \phi_{\lambda,\varepsilon}$ in $L^p_{loc}(\mathbb{R}^3),\ p \in [1,6)$, by the Hölder inequality and the definition of ψ_R, we can get that

$$\int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0, \quad \text{as } n \to \infty, \tag{2.7}$$

$$\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0, \quad \text{as } n \to \infty. \tag{2.8}$$

In fact, by the Hölder inequality, we have

$$\begin{aligned} \left|\int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx\right| &\le C|\nabla\phi_{\lambda,\varepsilon}(u_n)|_2 \left(\int_{B_{2R}(0)} |\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon}|^2 dx\right)^{\frac{1}{2}} \\ &\to 0, \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx\right| \\ &\le C|\nabla\phi_{\lambda,\varepsilon}(u_n)|_4^3 \left(\int_{B_{2R}(0)} |\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon}|^4 dx\right)^{\frac{1}{4}} \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Similarly, we can also get that

$$\int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0,\ \int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}|^2\nabla\phi_{\lambda,\varepsilon}\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0,$$

and

$$\int_{\mathbb{R}^3} u_n^2(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx \to 0, \quad \text{as } n \to \infty.$$

Thus, it follows from (2.6)–(2.8) that

$$\begin{aligned} o_n(1) = \int_{\mathbb{R}^3} &[\varepsilon^4(|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n) - |\nabla\phi_{\lambda,\varepsilon}|^2\nabla\phi_{\lambda,\varepsilon})\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon}) \\ &+ |\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})|^2]\psi_R dx. \end{aligned}$$

Then the Simon inequality leads to that

$$\int_{\mathbb{R}^3} (|\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})|^2 + \varepsilon^4|\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})|^4)\psi_R dx \to 0, \quad \text{as } n \to \infty.$$

Thus,

$$\int_{B_R(0)} |\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})|^2 dx \to 0, \quad \text{as } n \to \infty.$$

Up to a subsequence, we have

$$\nabla\phi_{\lambda,\varepsilon}(u_n)(x) \to \nabla\phi_{\lambda,\varepsilon}(x), \quad \text{a.e. } x \in B_R(0), \text{ as } n \to \infty.$$

The arbitrariness of R implies that, going to a subsequence,

$$\nabla\phi_{\lambda,\varepsilon}(u_n)(x) \to \nabla\phi_{\lambda,\varepsilon}(x), \quad \text{a.e. } x \in \mathbb{R}^3, \text{ as } n \to \infty.$$

The boundedness of $\{|\nabla\phi_{\lambda,\varepsilon}(u_n)|\}$ in $L^4(\mathbb{R}^3)$ ensures that $\{|\nabla\phi_{\lambda,\varepsilon}(u_n)|^3\}$ is also bounded in $L^{\frac{4}{3}}(\mathbb{R}^3)$. Thus, it follows from [22, Proposition 5.4.7] that

$$|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2 D_i\phi_{\lambda,\varepsilon}(u_n) \rightharpoonup |\nabla\phi_{\lambda,\varepsilon}|^2 D_i\phi_{\lambda,\varepsilon} \quad \text{in } L^{\frac{4}{3}}(\mathbb{R}^3),\ i = 1,2,3.$$

Therefore, for every $\varphi \in X$,

$$\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u_n)|^2 D_i\phi_{\lambda,\varepsilon}(u_n) D_i\varphi dx \to \int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}|^2 D_i\phi_{\lambda,\varepsilon} D_i\varphi dx, \qquad i = 1,2,3.$$

Then

$$\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u_n)|^2 \nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\varphi dx \to \int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}|^2 \nabla\phi_{\lambda,\varepsilon}\nabla\varphi dx, \quad \text{as } n \to \infty.$$

It follows from $\phi_{\lambda,\varepsilon}(u_n) \rightharpoonup \phi_{\lambda,\varepsilon}$ in $D^{1,2}(\mathbb{R}^3)$ that

$$\int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\varphi dx \to \int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}\nabla\varphi dx, \quad \text{as } n \to \infty.$$

Since $\varphi \in L^6(\mathbb{R}^3)$ and $u_n^2 \rightharpoonup u^2$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$ by [22, Proposition 5.4.7], we have

$$\int_{\mathbb{R}^3} u_n^2\varphi dx \to \int_{\mathbb{R}^3} u^2\varphi dx, \quad \text{as } n \to \infty.$$

Therefore, by taking limits as $n \to \infty$ on both sides of (2.2), we can obtain that

$$\int_{\mathbb{R}^3} (\nabla\phi_{\lambda,\varepsilon}\nabla\varphi + \varepsilon^4 |\nabla\phi_{\lambda,\varepsilon}|^2 \nabla\phi_{\lambda,\varepsilon}\nabla\varphi) dx = \lambda \int_{\mathbb{R}^3} u^2\varphi dx, \quad \text{for } \varphi \in X.$$

The uniqueness of solution for equation (2.1) with given u and (2.3) result that $\phi_{\lambda,\varepsilon} = \phi_{\lambda,\varepsilon}(u)$.

By [22, Proposition 5.4.7] again, we can get that $\phi_{\lambda,\varepsilon}(u_n)u_n \rightharpoonup \phi_{\lambda,\varepsilon}(u)u$ in $L^{\frac{3}{2}}(\mathbb{R}^3)$. Then for every $v \in H_V^1(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(u_n)u_n v dx \to \int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(u)uv dx, \quad \text{as } n \to \infty.$$

(3) The uniqueness of solution for equation (2.1) and the translation invariance of Lebesgue integral on $\mathbb{R}^3$ also guarantee that (iii) is true. In fact, for every $\varphi \in X$ and $y \in \mathbb{R}^3$,

$$\int_{\mathbb{R}^3} (1 + \varepsilon^4 |\nabla\phi_{\lambda,\varepsilon}(u)(x)|^2) \nabla\phi_{\lambda,\varepsilon}(u)(x)\nabla\varphi(x-y) dx = \lambda \int_{\mathbb{R}^3} u^2(x)\varphi(x-y) dx.$$

By the translation invariance of Lebesgue integral on $\mathbb{R}^3$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} (1 + \varepsilon^4 |\nabla\phi_{\lambda,\varepsilon}(u)(x+y)|^2) \nabla\phi_{\lambda,\varepsilon}(u)(x+y)\nabla\varphi(x) dx \\
&\quad = \lambda \int_{\mathbb{R}^3} u^2(x+y)\varphi(x) dx \\
&\quad = \lambda \int_{\mathbb{R}^3} u_y^2(x)\varphi(x) dx.
\end{aligned}$$

The uniqueness of solution for equation (2.1) leads to $\phi_{\lambda,\varepsilon}(u_y)(\cdot) = \phi_{\lambda,\varepsilon}(u)(\cdot + y)$. □

As shown in [11], the functional

$$\begin{aligned} J_{\lambda,\varepsilon}(u) &:= \mathcal{J}_{\lambda,\varepsilon}(u, \phi_{\lambda,\varepsilon}(u)) \\ &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} |\nabla \phi_{\lambda,\varepsilon}(u)|^2 dx + \frac{3\varepsilon^4}{8}\int_{\mathbb{R}^3} |\nabla \phi_{\lambda,\varepsilon}(u)|^4 dx - \int_{\mathbb{R}^3} F(x,u)dx, \quad u \in H_V^1(\mathbb{R}^3) \end{aligned}$$

is of class C^1. Its Fréchet derivative at $u \in H_V^1(\mathbb{R}^3)$ is given by

$$\begin{aligned} \langle J'_{\lambda,\varepsilon}(u), v\rangle &= \langle \partial_u \mathcal{J}_{\lambda,\varepsilon}(u, \phi_{\lambda,\varepsilon}(u)), v\rangle \\ &= \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv + \lambda \phi_{\lambda,\varepsilon}(u)uv)dx - \int_{\mathbb{R}^3} f(x,u)vdx. \end{aligned}$$

Lemma 2.3 ([11, Lemma 4])**.** *Let $\lambda, \varepsilon > 0$ be fixed, the following statements are equivalent:*

(i) the pair $(u_{\lambda,\varepsilon},\ \phi_{\lambda,\varepsilon}) \in H_V^1(\mathbb{R}^3) \times X$ is a critical point of $\mathcal{J}_{\lambda,\varepsilon}$;

(ii) $u_{\lambda,\varepsilon} \in H_V^1(\mathbb{R}^3)$ is a critical point of $J_{\lambda,\varepsilon}$ and $\phi_{\lambda,\varepsilon} = \phi_{\lambda,\varepsilon}(u_{\lambda,\varepsilon})$.

For convenience, we set the functional

$$I_{\lambda,\varepsilon}(u) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla \phi_{\lambda,\varepsilon}(u)|^2 dx + \frac{3\varepsilon^4}{8}\int_{\mathbb{R}^3} |\nabla \phi_{\lambda,\varepsilon}(u)|^4 dx, \qquad u \in H_V^1(\mathbb{R}^3).$$

It follows from [3, Proposition 4.1] that $I_{\lambda,\varepsilon} \in C^1(H_V^1(\mathbb{R}^3), \mathbb{R})$ and

$$\langle I'_{\lambda,\varepsilon}(u), v\rangle = \lambda \int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(u)uvdx, \quad \text{for } v \in H_V^1(\mathbb{R}^3).$$

In such a way, $J_{\lambda,\varepsilon}$ can be rewritten as

$$J_{\lambda,\varepsilon}(u) = \frac{1}{2}\|u\|^2 + I_{\lambda,\varepsilon}(u) - \int_{\mathbb{R}^3} F(x,u)dx.$$

In view of the above facts, in order to obtain a weak solution for system (1.1), it is sufficient to find a critical point of the functional $J_{\lambda,\varepsilon}$ in $H_V^1(\mathbb{R}^3)$.

3 Proof of our main result

In this section, we complete the proof of our main result. It is a difficult task to get a bounded Palais–Smale sequence for the functional $J_{\lambda,\varepsilon}$ directly due to the presence of nonlocal term for the case $\alpha \in (2,4]$ in (f_3). We use a truncation method which has been widely used [1,9,15–17] to deal with it. Precisely, we define a truncation for the functional $J_{\lambda,\varepsilon}$ in the following way. Let $\chi \in C^\infty([0,+\infty), [0,1])$ satisfy

$$\begin{cases} \chi(s) = 1, \ s \in [0,1], \\ 0 \le \chi(s) \le 1, \ s \in (1,2), \\ \chi(s) = 0, \ s \in [2,+\infty), \\ -2 \le \chi'(s) \le 0. \end{cases}$$

For each $T > 0$, we define $h_T(u) = \chi(\frac{\|u\|^2}{T^2})$ for $u \in H_V^1(\mathbb{R}^3)$ and the truncated functional

$$J^T_{\lambda,\varepsilon}(u) = \frac{1}{2}\|u\|^2 + h_T(u) I_{\lambda,\varepsilon}(u) - \int_{\mathbb{R}^3} F(x,u)dx. \tag{3.1}$$

The functional $J^T_{\lambda,\varepsilon} \in C^1(H^1_V(\mathbb{R}^3), \mathbb{R})$ with Fréchet derivative at u given by

$$\langle {J^T_{\lambda,\varepsilon}}'(u), v\rangle = \left(1 + \frac{2}{T^2}\chi'(\frac{\|u\|^2}{T^2})I_{\lambda,\varepsilon}(u)\right)\int_{\mathbb{R}^3}(\nabla u\nabla v + V(x)uv)dx + \lambda h_T(u)\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u)uvdx$$
$$- \int_{\mathbb{R}^3} f(x,u)vdx,\ v \in H^1_V(\mathbb{R}^3).$$

Then $u_{\lambda,\varepsilon} \in H^1_V(\mathbb{R}^3)$ is a critical point of $J^T_{\lambda,\varepsilon}$ if and only if $(u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}(u_{\lambda,\varepsilon})) \in H^1_V(\mathbb{R}^3) \times X$ is a weak solution of

$$\begin{cases} \left(1 + \frac{2}{T^2}\chi'(\frac{\|u\|^2}{T^2})I_{\lambda,\varepsilon}(u)\right)(-\Delta u + V(x)u) + \lambda h_T(u)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4\Delta_4\phi = \lambda u^2, & x \in \mathbb{R}^3. \end{cases}$$

From the definition of χ, for given T, we have

$$J^T_{\lambda,\varepsilon}(u) = J_{\lambda,\varepsilon}(u) \quad \text{and} \quad {J^T_{\lambda,\varepsilon}}'(u) = J'_{\lambda,\varepsilon}(u), \quad \text{if } \|u\| \le T.$$

Thus, if $\{u_n\}$ is a (PS) sequence of $J^T_{\lambda,\varepsilon}$ with $\|u_n\| \le T$, then it is also a bounded (PS) sequence of $J_{\lambda,\varepsilon}$.

We firstly prove that the truncated functional $J^T_{\lambda,\varepsilon}$ enjoys the mountain pass geometry structure.

Lemma 3.1. *For every fixed $(\lambda, \varepsilon) \in (0, \infty) \times (0, \infty)$, there exist $\rho > 0$ and $e_T \in H^1_V(\mathbb{R}^3)$ such that $\|e_T\| > \rho$ and*

$$\inf_{u \in H^1_V(\mathbb{R}^3),\ \|u\| = \rho} J^T_{\lambda,\varepsilon}(u) > J^T_{\lambda,\varepsilon}(0) = 0 > J^T_{\lambda,\varepsilon}(e_T).$$

Proof. On the one hand, it follows from (f_1) and (f_2) that there exists $a_1 > 0$ such that

$$|f(x,t)| \le \frac{V_0}{2}|t| + a_1|t|^{p-1},\ |F(x,t)| \le \frac{V_0}{4}t^2 + \frac{a_1}{p}|t|^p, \quad \text{for } (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.2}$$

Then (3.1) and (3.2) imply that

$$\begin{aligned} J^T_{\lambda,\varepsilon}(u) &= \frac{1}{2}\|u\|^2 + h_T(u)I_{\lambda,\varepsilon}(u) - \int_{\mathbb{R}^3} F(x,u)dx \\ &\ge \frac{1}{4}\|u\|^2 - \frac{a_1}{p}\int_{\mathbb{R}^3}|u|^p dx \\ &\ge \frac{1}{4}\|u\|^2 - \frac{a_1}{p}C_p^p\|u\|^p. \end{aligned}$$

We conclude that there exists $\rho > 0$ small enough such that for any $u \in H^1_V(\mathbb{R}^3)$ with $0 < \|u\| \le \rho$, it results that $J^T_{\lambda,\varepsilon}(u) > 0$. In particular, we have

$$J^T_{\lambda,\varepsilon}(u) \ge \frac{1}{4}\rho^2 - \frac{a_1}{p}C_p^p\rho^p > 0, \quad \text{for } u \in H^1_V(\mathbb{R}^3) \text{ with } \|u\| = \rho.$$

On the other hand, by (f_1)–(f_3), there exist $a_2, a_3 > 0$ such that

$$F(x,t) \ge a_2|t|^\alpha - a_3t^2, \quad \text{for } (x,t) \in \mathbb{R}^3 \times \in \mathbb{R}. \tag{3.3}$$

Then for $\overline{u} \in H_V^1(\mathbb{R}^3)$ with $\|\overline{u}\| = 1$ fixed and $s > \sqrt{2}T$, by (3.3) and the definition of h_T, we have

$$\begin{aligned} J_{\lambda,\varepsilon}^T(s\overline{u}) &= \frac{s^2}{2}\|\overline{u}\|^2 + h_T(s\overline{u})I_{\lambda,\varepsilon}(s\overline{u}) - \int_{\mathbb{R}^3} F(x, s\overline{u})dx \\ &\leq \left(\frac{1}{2} + a_3\right) s^2\|\overline{u}\|^2 - a_2|s|^\alpha \int_{\mathbb{R}^3} |\overline{u}|^\alpha dx \\ &\to -\infty, \qquad s \to +\infty. \end{aligned}$$

Thus, by choosing $s_T > \max\{\rho, \sqrt{2}T\}$ large enough, we can get $J_{\lambda,\varepsilon}^T(s_T\overline{u}) < 0$. So we can set $e_T = s_T\overline{u}$. □

Then it follows from Lemma 3.1 and the Mountain Pass lemma that there exists a $(PS)_{c_T}$ sequence $\{u_n\}$ for $J_{\lambda,\varepsilon}^T$ in $H_V^1(\mathbb{R}^3)$, where

$$c_T = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{\lambda,\varepsilon}^T(\gamma(t)),$$

with

$$\Gamma := \{\gamma \in C([0,1], H_V^1(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = e_T\}.$$

From the proof of Lemma 3.1, we can also get that $c_T > 0$, for every $T > 0$.

Second, we study the boundedness of the $(PS)_{c_T}$ sequence $\{u_n\}$ of $J_{\lambda,\varepsilon}^T$ which has been obtained by the Mountain Pass lemma. In this process, the truncation of the nonlocal term plays an important role.

Lemma 3.2. *For $T > 0$ sufficiently large, there exists $\lambda_T > 0$ such that for any $\lambda \in (0, \lambda_T)$ and $\varepsilon > 0$,*

$$\limsup_{n\to\infty} \|u_n\| < T$$

holds, where $\{u_n\}$ is the $(PS)_{c_T}$ sequence of $J_{\lambda,\varepsilon}^T$ obtained above.

Proof. If $\|u_n\| \to \infty$, as $n \to \infty$, then $h_T(u_n) = \chi(\frac{\|u_n\|^2}{T^2}) \to 0$, as $n \to \infty$. Thus, for all $n \in \mathbb{N}$ large enough

$$J_{\lambda,\varepsilon}^T(u_n) = \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^3} F(x, u_n)dx, \ \langle J_{\lambda,\varepsilon}^{T\,\prime}(u_n), u_n\rangle = \|u\|^2 - \int_{\mathbb{R}^3} f(x, u_n)u_n dx.$$

Then, by (f_3), for $n \in \mathbb{N}$ large enough

$$\begin{aligned} c_T + 1 + \|u_n\| &\geq J_{\lambda,\varepsilon}^T(u_n) - \frac{1}{\alpha}\langle J_{\lambda,\varepsilon}^{T\,\prime}(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\alpha}\right) \|u_n\|^2 - \int_{\mathbb{R}^3} \left(F(x, u_n) - \frac{1}{\alpha}f(x, u_n)u_n\right) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right) \|u_n\|^2, \end{aligned}$$

which is impossible, since $\|u_n\| \to \infty$, $n \to \infty$. Therefore, $\{u_n\}$ is bounded in $H_V^1(\mathbb{R}^3)$ which may be dependent on T.

On the contrary, we assume that $\limsup_{n\to\infty}\|u_n\| \geq T$. Up to a subsequence and still denoted by $\{u_n\}$, we have $\lim_{n\to\infty}\|u_n\| \geq T$. By (f_3), we obtain that

$$\begin{aligned}
J^T_{\lambda,\varepsilon}(u_n) - \frac{1}{\alpha}\langle {J^T_{\lambda,\varepsilon}}'(u_n), u_n\rangle &= \left[\frac{1}{2} - \frac{1}{\alpha}\left(1 + \frac{2}{T}\chi'(\frac{\|u_n\|^2}{T^2})I_{\lambda,\varepsilon}(u_n)\right)\right]\|u_n\|^2 \\
&\quad + h_T(u_n)\left(I_{\lambda,\varepsilon}(u_n) - \frac{\lambda}{\alpha}\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u_n)u_n^2dx\right) \\
&\quad - \int_{\mathbb{R}^3}\left(F(x,u_n) - \frac{1}{\alpha}f(x,u_n)u_n\right)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|^2 - \frac{\lambda}{\alpha}h_T(u_n)\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u_n)u_n^2dx.
\end{aligned}$$

Then

$$\left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|^2 + \frac{1}{\alpha}\langle {J^T_{\lambda,\varepsilon}}'(u_n), u_n\rangle \leq J^T_{\lambda,\varepsilon}(u_n) + \frac{\lambda}{\alpha}h_T(u_n)\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u_n)u_n^2dx. \tag{3.4}$$

By the definition of $I_{\lambda,\varepsilon}$ and (i) of Lemma 2.2,

$$0 \leq I_{\lambda,\varepsilon}(v) \leq \frac{3}{8}\lambda^2 S^{-1}C^4_{\frac{12}{5}}\|v\|^4, \qquad v \in H^1_V(\mathbb{R}^3). \tag{3.5}$$

By (3.3), (3.5) and the definitions of c_T and e_T, we have

$$\begin{aligned}
c_T &\leq \max_{t\in[0,1]} J^T_{\lambda,\varepsilon}(te_T) \\
&\leq \max_{t\in[0,1]}\left(\frac{t^2}{2}\|e_T\|^2 - \int_{\mathbb{R}^N}F(x,te_T)dx\right) + \max_{t\in[0,1]} h_T(te_T)I_{\lambda,\varepsilon}(te_T) \\
&\leq \max_{t\in[0,1]}\left(\frac{(s_Tt)^2}{2}(1+2a_3C_2^2)\|\overline{u}\|^2 - a_2|\overline{u}|^\alpha_\alpha(s_Tt)^\alpha\right) + \frac{3}{2}\lambda^2S^{-1}C^4_{\frac{12}{5}}T^4 \\
&\leq \max_{t\in[0,\infty)}\left(\frac{t^2}{2}(1+2a_3C_2^2)\|\overline{u}\|^2 - a_2|\overline{u}|^\alpha_\alpha t^\alpha\right) + \frac{3}{2}\lambda^2S^{-1}C^4_{\frac{12}{5}}T^4 \\
&=: c_* + \frac{3}{2}\lambda^2S^{-1}C^4_{\frac{12}{5}}T^4.
\end{aligned} \tag{3.6}$$

It should be pointed out that $c_* > 0$ is independent of T and λ. It follows from the definition of h_T and (i) of Lemma 2.2 that

$$h_T(u_n)\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u_n)u_n^2dx \leq 4\lambda S^{-1}C^4_{\frac{12}{5}}T^4. \tag{3.7}$$

By taking upper limits as $n\to\infty$ on both sides of (3.4), (3.6) and (3.7) lead to

$$\left(\frac{1}{2} - \frac{1}{\alpha}\right)T^2 \leq c_* + \left(\frac{3}{2} + \frac{4}{\alpha}\right)\lambda^2S^{-1}C^4_{\frac{12}{5}}T^4.$$

For every T large enough such that $\left(\frac{1}{2} - \frac{1}{\alpha}\right)T^2 > c_* + 1$, we can obtain $\lambda_T > 0$ small such that $\left(\frac{3}{2} + \frac{4}{\alpha}\right)\lambda_T{}^2S^{-1}C^4_{\frac{12}{5}}T^4 \leq 1$. Therefore, we can get a contradiction for every $\lambda \in (0,\lambda_T)$. □

It follows from Lemma 3.2 that there exists a $(\mathrm{PS})_{c_T}$ sequence of $J^T_{\lambda,\varepsilon}$ still denoted by $\{u_n\}$ with $\|u_n\| \leq T$ for every $T > \sqrt{\frac{2\alpha(c_*+1)}{\alpha-2}}$ and $\lambda \in (0,\lambda_T)$. By the definition of h_T again, we can get that

$$J_{\lambda,\varepsilon}(u_n) = J^T_{\lambda,\varepsilon}(u_n) \to c_T, \qquad J'_{\lambda,\varepsilon}(u_n) = {J^T_{\lambda,\varepsilon}}'(u_n) \to 0, \quad \text{as } n\to\infty.$$

That is, for every fixed $T > \sqrt{\frac{2\alpha(c_*+1)}{\alpha-2}}$, $\{u_n\}$ is also a bounded $(\mathrm{PS})_{c_T}$ of $J_{\lambda,\varepsilon}$ for $\lambda \in (0, \lambda_T)$.

By using Lemma 2.2, we can obtain the following lemma which plays a crucial role in finding a nontrivial solution of system (1.1).

Lemma 3.3. *Let $\{u_n\}$ be a bounded $(PS)_c$ sequence of $J_{\lambda,\varepsilon}$ with $c > 0$, then there exists $\widetilde{u} \in H^1_V(\mathbb{R}^3) \setminus \{0\}$ such that $J'_{\lambda,\varepsilon}(\widetilde{u}) = 0$.*

Proof. Let $\{u_n\}$ be a bounded $(\mathrm{PS})_c$ sequence of $J_{\lambda,\varepsilon}$. That is,

$$J_{\lambda,\varepsilon}(u_n) \to c > 0, \qquad J'_{\lambda,\varepsilon}(u_n) \to 0 \quad \text{in } H^{-1}_V(\mathbb{R}^3), \text{ as } n \to \infty. \tag{3.8}$$

It is clear that $\{u_n\}$ is either

(*i*) *vanishing*: for each $r > 0$, $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} u_n^2 dx = 0$, or

(*ii*) *non-vanishing*: there exist $r, \eta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\limsup_{n\to\infty} \int_{B_r(y_n)} u_n^2 dx \geq \eta.$$

If $\{u_n\}$ is vanishing, then it follows from Lemma I.1 in [18] that $u_n \to 0$ in $L^s(\mathbb{R}^3)$ whenever $2 < s < 6$. By [11, Lemma 2], we have

$$\int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(u_n) u_n^2 dx \to 0, \qquad n \to \infty. \tag{3.9}$$

It follows from (f_1) and (f_2) that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|f(x,t)| \leq \epsilon|t| + C_\epsilon|t|^{p-1}, \quad \text{for } (x,t) \in \mathbb{R}^3 \times \in \mathbb{R}. \tag{3.10}$$

Then

$$\left| \int_{\mathbb{R}^3} f(x,u_n)u_n dx \right| \leq \int_{\mathbb{R}^3} (\epsilon u_n^2 + C_\epsilon |u_n|^p) dx.$$

By the arbitrariness of ϵ and $u_n \to 0$ in $L^p(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} f(x,u_n)u_n dx \to 0, \qquad n \to \infty. \tag{3.11}$$

It follows from (3.9), (3.11) and $\langle J'_{\lambda,\varepsilon}(u_n), u_n\rangle \to 0$ that $u_n \to 0$ in $H^1_V(\mathbb{R}^3)$. Then $J_{\lambda,\varepsilon}(u_n) \to 0$, which is a contradiction with the fact that $c > 0$ in (3.8). Therefore, $\{u_n\}$ must be non-vanishing. Furthermore, we can assume that $\{y_n\} \subset \mathbb{Z}^3$ since $B_r(y_n) \subset B_{r+1}(z_n)$ for some $z_n \in \mathbb{Z}^3$.

Let $\widetilde{u}_n(x) := u_n(x + y_n)$. (*iii*) of Lemma 2.2 and the periodic assumptions of V and f guarantee that $\|\widetilde{u}_n\| = \|u_n\|$ and $\|J'_{\lambda,\varepsilon}(\widetilde{u}_n)\| = \|J'_{\lambda,\varepsilon}(u_n)\|$. Since $\{\widetilde{u}_n\}$ is bounded in $H^1_V(\mathbb{R}^3)$, there exists $\widetilde{u} \in H^1_V(\mathbb{R}^3)$, which is nonzero due to the fact that $\limsup_{n\to\infty} \int_{B_r(0)} \widetilde{u}_n^2 dx \geq \eta$, such that $\widetilde{u}_n \rightharpoonup \widetilde{u}$ in $H^1_V(\mathbb{R}^3)$ after passing to a subsequence. A direct computation shows that $J'_{\lambda,\varepsilon}(\widetilde{u}) = 0$. In fact, for every $v \in H^1_V(\mathbb{R}^3)$,

$$o_n(1) = \langle J'_{\lambda,\varepsilon}(\widetilde{u}_n), v\rangle = \int_{\mathbb{R}^3} (\nabla\widetilde{u}_n \nabla v + V(x)\widetilde{u}_n v + \phi_{\lambda,\varepsilon}(\widetilde{u}_n)\widetilde{u}_n v - f(x,\widetilde{u}_n)v) dx.$$

The weak convergence in $H^1_V(\mathbb{R}^3)$ leads to

$$\int_{\mathbb{R}^3} (\nabla\widetilde{u}_n \nabla v + V(x)\widetilde{u}_n v) dx \to \int_{\mathbb{R}^3} (\nabla\widetilde{u} \nabla v + V(x)\widetilde{u} v) dx, \quad \text{as } n \to \infty.$$

By (*ii*) of Lemma 2.2, we can get that $\phi_{\lambda,\varepsilon}(\widetilde{u}_n) \rightharpoonup \phi_{\lambda,\varepsilon}(\widetilde{u})$ in X and

$$\int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(\widetilde{u}_n)\widetilde{u}_n v dx \to \int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(\widetilde{u})\widetilde{u} v dx, \quad \text{as } n \to \infty.$$

It follows from (3.10) that

$$|f(x, \widetilde{u}_n)v| \le |\widetilde{u}_n||v| + C|\widetilde{u}_n|^{p-1}|v|, \quad \text{for some } C > 0.$$

By the definitions of weak convergence in $L^2(\mathbb{R}^3)$ and $L^{\frac{p}{p-1}}(\mathbb{R}^3)$, we can get that

$$\int_{\mathbb{R}^3} (|\widetilde{u}_n||v| + C|\widetilde{u}_n|^{p-1}|v|)dx \to \int_{\mathbb{R}^3} (|\widetilde{u}||v| + C|\widetilde{u}|^{p-1}|v|)dx, \quad \text{for } v \in H^1_V(\mathbb{R}^3).$$

Then, by applying the Fatou lemma twice, we have

$$\int_{\mathbb{R}^3} f(x, \widetilde{u}_n) v dx \to \int_{\mathbb{R}^3} f(x, \widetilde{u}) v dx, \qquad n \to \infty.$$

Thus, $\langle J'_{\lambda,\varepsilon}(\widetilde{u}), v\rangle = 0$. That is, $\widetilde{u}$ is a nontrivial critical point of $J_{\lambda,\varepsilon}$. □

Proof of Theorem 1.1. Let $T_0 > \sqrt{\frac{2\alpha(c_*+1)}{\alpha-2}}$ and $\lambda_0 := \lambda_{T_0}$ be chosen as in Lemma 3.2. By Lemma 3.2 and Lemma 3.3, for every $\lambda \in (0, \lambda_0)$ and $\varepsilon > 0$, $J_{\lambda,\varepsilon}$ has at least one nontrivial critical point $u_{\lambda,\varepsilon} \in H^1_V(\mathbb{R}^3)$. Lemma 2.3 indicates that $(u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}))$ is a nontrivial solution of system (1.1). The proof is completed. □

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