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New monotonicity properties and oscillation of n-order functional differential equations with deviating argument

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Abstract. In this paper, we offer new technique for investigation of the even order linear differential equations of the form

$$y^{(n)}(t) = p(t)y(\tau(t)). \tag{E}$$

We establish new criteria for bounded and unbounded oscillation of (E) which improve a number of related ones in the literature. Our approach essentially involves establishing stronger monotonicities for the positive solutions of (E) than those presented in known works. We illustrate the improvement over known results by applying and comparing our technique with the other known methods on the particular examples.

Keywords: higher order differential equations, delay, advanced argument, monotonicity, oscillation.

2020 Mathematics Subject Classification: 34K11, 34C10.

1 Introduction

We consider the general higher order differential equation with deviating argument

$$y^{(n)}(t) = p(t)y(\tau(t)). \tag{E}$$

Throughout the paper, it is assumed that n is even and the following conditions hold

(H_1) $p(t) \in C^1([t_0, \infty))$, $p(t) > 0$,

(H_2) $\tau(t) \in C^1([t_0, \infty))$, $\tau'(t) > 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

By a proper solution of Eq. (E) we mean a function $y : [T_y, \infty) \to R$ which satisfies (E) for all sufficiently large t and $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We make the standing hypothesis that (E) does possess proper solutions.

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As is customary, a proper solution $y(t)$ of (E) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its proper solutions oscillate.

Oscillation phenomena appear in different models from real world applications; see, for instance, the papers [15–17] for models from mathematical biology where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. The problem of establishing oscillation criteria for differential equations with deviating arguments has been a very active research area over the past decades (see [1]–[18]) and several references and reviews of known results can be found in the monographs by Agarwal et al. [1], Došlý and Rehák [5] and Ladde et al. [18].

It is known that the set $\mathcal{N}$ of all nonoscillatory solutions of (E) has the following decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_n,$$

where $y(t) \in \mathcal{N}_\ell$ means that there exists $t_0 \geq T_y$ such that

$$\begin{aligned} y(t)y^{(i)}(t) > 0 \quad &\text{on } [t_0,\infty) \ \text{ for } 0 \leq i \leq \ell, \\ (-1)^i y(t)y^{(i)}(t) > 0 \quad &\text{on } [t_0,\infty) \ \text{ for } \ell \leq i \leq n. \end{aligned} \tag{1.1}$$

Such a $y(t)$ is said to be a solution of degree ℓ.

Following Kiguradze [7], we say that equation (E) enjoys property (B) if $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_n$. The reason for such definition is the observation that (E) with $\tau(t) \equiv t$ always possesses solutions of degrees 0 and n, that is $\mathcal{N}_0 \neq \emptyset$ and $\mathcal{N}_n \neq \emptyset$. The situation when $\tau(t) \not\equiv t$ is different. In fact, it may happen that $\mathcal{N}_0 = \emptyset$ or $\mathcal{N}_n = \emptyset$ when the deviation $|t - \tau(t)|$ is sufficiently large. This remarkable fact was first observed by Ladas et al. [18]. Later Koplatadze and Chanturia [11] have shown that (E) does not allow solutions of degree 0 if $\tau(t) \leq t$ and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} (s - \tau(t))^{n-1} p(s)\,\mathrm{d}s > (n-1)! \tag{1.2}$$

and (E) does not allow solutions of degree n provided that $\tau(t) \geq t$ and

$$\limsup_{t\to\infty} \int_{t}^{\tau(t)} (\tau(t) - s)^{n-1} p(s)\,\mathrm{d}s > (n-1)!. \tag{1.3}$$

On the other hand, Koplatadze et al. [12] proved that (E) enjoys property (B) if $\tau(t) \leq t$ and

$$\begin{aligned} \limsup_{t\to\infty} \bigg\{ \tau(t) \int_t^\infty s^{n-3}\tau(s)p(s)\,\mathrm{d}s &+ \int_{\tau(t)}^{t} s(\tau(s))^{n-2}p(s)\,\mathrm{d}s \\ &+ \frac{1}{\tau(t)} \int_0^{\tau(t)} s^2(\tau(s))^{n-2}p(s)\,\mathrm{d}s \bigg\} > 2(n-2)! \end{aligned} \tag{1.4}$$

or $\tau(t) \geq t$ and

$$\begin{aligned} \limsup_{t\to\infty} \bigg\{ \tau(t) \int_t^\infty s^{n-3}\tau(s)p(s)\,\mathrm{d}s &+ \int_{t}^{\tau(t)} s^{n-2}\tau(s)p(s)\,\mathrm{d}s \\ &+ \frac{1}{\tau(t)} \int_0^{t} s^{n-2}(\tau(s))^2p(s)\,\mathrm{d}s \bigg\} > 2(n-2)!. \end{aligned} \tag{1.5}$$

Therefore conditions (1.2)–(1.5) yield stronger asymptotic behavior than property (B) claims, namely (1.3) together with (1.5) guarantees that $\mathcal{N} = \mathcal{N}_0$ for (E) with $\tau(t) \geq t$, i.e., every unbounded solution is oscillatory, while (1.2) together with (1.4) are sufficient for $\mathcal{N} = \mathcal{N}_n$ for (E) with $\tau(t) \le t$, i.e., roughly speaking every bounded solution is oscillatory.

In this paper, we establish new technique that essentially improves (1.2) and (1.3), which leads to qualitative better criteria for bounded or unbounded oscillation of (E). Our approach essentially involves establishing stronger monotonicities for the positive solutions of (E) than those presented in known works.

2 Main results

Now we are introduce new monotonicity for nonoscillatory solution $y(t) \in \mathcal{N}_0$ of (E).

Lemma 2.1. *Assume that* $y(t) \in \mathcal{N}_0$ *and*

$$\int_{t_0}^{\infty} p(s) s^{n-1} \,\mathrm{d}s = \infty. \tag{2.1}$$

Then $\lim_{t\to\infty} y(t) = 0$.

Proof. Assume on the contrary that $y(t)$ is an eventually positive solution of (E) such that $y(t) \in \mathcal{N}_0$, and $\lim_{t\to\infty} y(t) = \ell > 0$. Then $y(\tau(t)) > \ell$, eventually, let us say for $t \ge t_1$. An integration of (E) from t to ∞ yields

$$-y^{(n-1)}(t) \ge \int_t^{\infty} p(s) y(\tau(s)) \,\mathrm{d}s \ge \ell \int_t^{\infty} p(s) \,\mathrm{d}s.$$

Integrating again from t to ∞ and changing the order of integration, we have

$$y^{(n-2)}(t) \ge \ell \int_t^{\infty} \int_u^{\infty} p(s) \,\mathrm{d}s \,\mathrm{d}u = \ell \int_t^{\infty} p(s)(s-t) \,\mathrm{d}s.$$

Repeating this procedure, we are led to

$$y(t_1) \ge \ell \int_{t_1}^{\infty} \frac{p(s)(s-t_1)^{n-1}}{(n-1)!} \,\mathrm{d}s, \tag{2.2}$$

where the last integration was from t_1 to ∞. Condition (2.2) contradicts (2.1) and we conclude that $y(t) \to 0$ as $t \to \infty$. □

Corollary 2.2. *For* $y(t) \in N_0$, *it follows from* $y(t) \to 0$ *as* $t \to \infty$ *that* $y'(t) \to 0$, $y''(t) \to 0, \dots, y^{(n-1)}(t) \to 0$ *as* $t \to \infty$.

To simplify our notation we introduce the following couple of functions

$$\alpha_0(t) = -\frac{p'(t)}{p(t)} + \tau'(t) \int_{\tau(t)}^{t} p(s) \frac{(s-\tau(t))^{n-2}}{(n-2)!} \,\mathrm{d}s,$$
$$\beta_0'(t) = \alpha_0(t).$$

Theorem 2.3. *Let* $y(t) \in \mathcal{N}_0$, $\tau(t) \le t$ *and* (2.1) *hold. Then*

$$|y(\tau(t))| p(t) e^{\beta_0(t)} \quad \textit{is decreasing}.$$

Proof. Assume that $y(t) \in \mathcal{N}_0$ is an eventually positive solution of (*E*). It follows from (*E*) that

$$y^{(n+1)}(t) = p'(t)y(\tau(t)) + p(t)y'(\tau(t))\tau'(t). \tag{2.3}$$

In view of Corollary 2.2, an integration of (*E*) from t to ∞ yields

$$-y^{(n-1)}(t) = \int_t^\infty p(s)y(\tau(s))\,\mathrm{d}s.$$

Integrating again from t to ∞ and changing the order of integration, we have

$$y^{(n-2)}(t) = \int_t^\infty \int_u^\infty p(s)y(\tau(s))\,\mathrm{d}s\,\mathrm{d}s = \int_t^\infty p(s)y(\tau(s))(s-t)\,\mathrm{d}s.$$

Repeated reusing of this procedure yields

$$-y'(t) = \int_t^\infty p(s)y(\tau(s))\frac{(s-t)^{n-2}}{(n-2)!}\,\mathrm{d}s.$$

Since $y(\tau(t))$ is decreasing, this implies

$$\begin{aligned}
-y'(\tau(t)) &= \int_{\tau(t)}^\infty p(s)y(\tau(s))\frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s \\
&\geq \int_{\tau(t)}^t p(s)y(\tau(s))\frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s \\
&\geq y(\tau(t))\int_{\tau(t)}^t p(s)\frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s.
\end{aligned} \tag{2.4}$$

Setting (2.4) into (2.3) and taking (*E*) into account, one gets

$$\begin{aligned}
y^{(n+1)}(t) &\leq y(\tau(t))\left[p'(t) - p(t)\tau'(t)\int_{\tau(t)}^t p(s)\frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s\right] \\
&= y^{(n)}(t)\left[\frac{p'(t)}{p(t)} - \tau'(t)\int_{\tau(t)}^t p(s)\frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s\right].
\end{aligned}$$

Therefore

$$y^{(n+1)}(t) + \alpha_0(t)y^{(n)}(t) \leq 0$$

which means that

$$\left(e^{\beta_0(t)}y^{(n)}(t)\right)' \leq 0$$

and we conclude that $e^{\beta_0(t)}y^{(n)}(t)$ is decreasing, which is, in view of (*E*), equivalent to the fact that $p(t)y(\tau(t))e^{\beta_0(t)}$ is decreasing. □

Employing the above-mentioned monotonicity we are prepared to present criterion for bounded oscillation of (*E*).

Theorem 2.4. *Assume that* (2.1) *holds,* $\tau(t) \leq t$*, and*

$$\limsup_{t\to\infty} p(t)e^{\beta_0(t)}\int_{\tau(t)}^t e^{-\beta_0(s)}(s-\tau(t))^{n-1}\,\mathrm{d}s > (n-1)!. \tag{2.5}$$

Then $\mathcal{N}_0 = \emptyset$*. If in addition* (1.4) *holds, then all nonoscillatory solutions of* (*E*) *are of degree* n*, i.e.,* $\mathcal{N} = \mathcal{N}_n$*.*

Proof. We argue by contradiction. Assume that (E) possesses an eventually positive solution $y(t) \in \mathcal{N}_0$. Integrating (E) from u to t ($u \le t$) and using the monotonicity of $p(t)y(\tau(t))e^{\beta_0(t)}$, we have

$$-y^{(n-1)}(u) \ge \int_u^t p(s)y(\tau(s))\,\mathrm{d}s \ge y(\tau(t))p(t)e^{\beta_0(t)} \int_u^t e^{-\beta_0(s)}\,\mathrm{d}s.$$

Integrating the above inequality from u to t and changing the order of integration leads to

$$\begin{aligned} y^{(n-2)}(u) &\ge y(\tau(t))p(t)e^{\beta_0(t)} \int_u^t \int_x^t e^{-\beta_0(s)}\,\mathrm{d}s\,\mathrm{d}x \\ &= y(\tau(t))p(t)e^{\beta_0(t)} \int_u^t e^{-\beta_0(s)}(s-u)\,\mathrm{d}s. \end{aligned} \tag{2.6}$$

Proceeding in the same way $(n-2)$-times, we finally get

$$y(u) \ge y(\tau(t))p(t)e^{\beta_0(t)} \int_u^t e^{-\beta_0(s)} \frac{(s-u)^{n-1}}{(n-1)!}\,\mathrm{d}s.$$

Setting $u = \tau(t)$, we obtain

$$y(\tau(t)) \ge y(\tau(t))p(t)e^{\beta_0(t)} \int_{\tau(t)}^t e^{-\beta_0(s)} \frac{(s-\tau(t))^{n-1}}{(n-1)!}\,\mathrm{d}s$$

which is contraction with (2.5) and we conclude, that class N_0 is empty. Moreover, thanks to (1.4) every nonoscillatory solution of (E) is of degree n. □

Example 2.5. Consider the delay differential equation

$$y^{(n)}(t) = p_0 y(t-\tau), \quad p_0 > 0, \quad \tau > 0. \tag{E_{x1}}$$

It is easy to see that (1.4) holds true. Since $\alpha_0(t) = \frac{p_0}{(n-1)!}\tau^{n-1} = \omega$ and $\beta_0(t) = \omega t$, condition (2.5) takes the form

$$\lim_{t\to\infty} p_0 e^{\omega t} \int_{t-\tau}^t e^{-\omega s} \frac{(s-t+\tau)^{n-1}}{(n-1)!}\,\mathrm{d}s > 1 \tag{2.7}$$

which after substitution $s - t + \tau = x$ reduces to

$$\frac{p_0}{(n-1)!} e^{\omega\tau} \int_0^\tau e^{-\omega x} x^{n-1}\,\mathrm{d}s > 1.$$

Let us denote

$$I(n) = e^{\omega\tau} \int_0^\tau e^{-\omega x} x^{n-1}\,\mathrm{d}s.$$

Then

$$I(n) = -\frac{\tau^{n-1}}{\omega} + \frac{n-1}{\omega} I(n-1), \quad I(1) = -\frac{1}{\omega} + \frac{e^{\omega\tau}}{\omega}$$

which implies

$$I(n) = \frac{(n-1)!e^{\omega\tau}}{\omega^n} - \frac{\tau^{n-1}}{\omega} - \frac{(n-1)\tau^{n-2}}{\omega^2} - \cdots - \frac{(n-1)!}{\omega^n}.$$

Therefore, (2.7) is equivalent to

$$\frac{p_0}{(n-1)!}\left[\frac{(n-1)!e^{\omega\tau}}{\omega^n} - \frac{\tau^{n-1}}{\omega} - \frac{(n-1)\tau^{n-2}}{\omega^2} - \cdots - \frac{(n-1)!}{\omega^n}\right] > 1. \tag{2.8}$$

By Theorem 2.4 condition (2.8) guarantees that every nonoscillatory solution of (E) is of degree n or in other words, every bounded solution of (E) is oscillatory. If $p_0 = \left(\frac{\pi(4k+1-(-1)^{n/2})}{2\tau}\right)^n$ where k is a positive integer such that (2.8) holds, then a bounded oscillatory solution of (E) is $y(t) = \sin(\sqrt[n]{p_0})t$.

Example 2.6. We consider the delayed Euler differential equation

$$y^{(n)}(t) = \frac{p_0}{t^n} y(\lambda t), \quad p_0 > 0, \quad \lambda \in (0,1). \tag{E_{x2}}$$

It is easy to see that (1.4) reduces to

$$p_0 \left(\lambda^2 - \lambda^{n-2} \ln \lambda + \lambda^{n-3}\right) > 2(n-2)!. \tag{2.9}$$

On the other hand,

$$\alpha_0(t) = \frac{1}{t}\left[\frac{p_0(1-\lambda)^{n-1}}{(n-1)!} + n\right].$$

Using notation

$$\frac{p_0(1-\lambda)^{n-1}}{(n-1)!} + n = \delta_0,$$

we obtain

$$\beta_0(t) = \delta_0 \ln t.$$

Therefore (2.5) is equivalent to

$$\limsup_{t\to\infty} \frac{p_0 t^{\delta_0 - n}}{(n-1)!} \int_{\lambda t}^{t} \frac{(s-\lambda t)^{n-1}}{s^{\delta_0}} \, \mathrm{d}s > 1.$$

Since

$$\int_{\lambda t}^{t} \frac{(s-\lambda t)^{n-1}}{s^{\delta_0}} \, \mathrm{d}s = \sum_{i=0}^{n-1} \frac{(n-1)!(-\lambda t)^i s^{n-\delta_0-i}}{(n-1-i)!\, i!\, (n-\delta_0-i)} \Bigg|_{\lambda t}^{t},$$

condition (2.5) takes the form

$$p_0 \sum_{i=0}^{n-1} (-1)^i \frac{\lambda^i - \lambda^{n-\delta_0}}{(n-1-i)!\, i!\, (n-\delta_0-i)} > 1 \tag{2.10}$$

which guarantees that $\mathcal{N}_0 = \emptyset$ for (E_{x2}). If in addition (2.9) holds, then every nonoscillatory solution of (E_{x2}) is of degree n.

For $n = 2$ $(n = 4)$ and $\lambda = 0.5$ condition (2.10) is satisfied when

$$p_0 > 3.3198 \quad (p_0 > 135.77)$$

while (1.2) requires $p_0 > 5.1774$ $(p_0 > 226.58)$. So our progress is significant.

Now we turn our attention to bounded oscillation of (E). We set

$$\alpha_n(t) = \frac{p'(t)}{p(t)} + \tau'(t) \int_t^{\tau(t)} p(s) \frac{(\tau(t)-s)^{n-2}}{(n-2)!} \, \mathrm{d}s,$$
$$\beta_n'(t) = \alpha_n(t).$$

Theorem 2.7. *Let $y(t) \in \mathcal{N}_n$, $\tau(t) \geq t$. Then*

$$|y(\tau(t))| p(t) e^{-\beta_n(t)} \quad \textit{is increasing.}$$

Proof. Assume that $y(t) \in \mathcal{N}_n$ is an eventually positive solution of (E). An integration of (E) from t_1 to t yields

$$y^{(n-1)}(t) \geq \int_{t_1}^{t} p(s) y(\tau(s))\, \mathrm{d}s.$$

Integrating the last inequality from t_1 to t and and changing the order of integration, we obtain

$$y^{(n-2)}(t) \geq \int_{t_1}^{t} \int_{t_1}^{u} p(s) y(\tau(s))\, \mathrm{d}s\, \mathrm{d}u = \int_{t_1}^{t} p(s) y(\tau(s))(t-s)\, \mathrm{d}s.$$

Repeating this procedure, we have

$$y'(t) \geq \int_{t_1}^{t} p(s) y(\tau(s)) \frac{(t-s)^{n-2}}{(n-2)!}\, \mathrm{d}s.$$

Consequently,

$$\begin{aligned} y'(\tau(t)) &\geq \int_{t}^{\tau(t)} p(s) y(\tau(s)) \frac{(\tau(t)-s)^{n-2}}{(n-2)!}\, \mathrm{d}s \\ &\geq y(\tau(t)) \int_{t}^{\tau(t)} p(s) \frac{(\tau(t)-s)^{n-2}}{(n-2)!}\, \mathrm{d}s, \end{aligned} \tag{2.11}$$

where we have used that $y(\tau(t))$ is increasing. By combining inequalities (2.3) and (2.11), we conclude that

$$y^{(n+1)}(t) \geq y(\tau(t)) \left[p'(t) + p(t)\tau'(t) \int_{t}^{\tau(t)} p(s) \frac{(\tau(t)-s)^{n-2}}{(n-2)!}\, \mathrm{d}s \right]$$

which in view of (E) implies

$$y^{(n+1)}(t) \geq y^{(n)}(t) \left[\frac{p'(t)}{p(t)} + \tau'(t) \int_{t}^{\tau(t)} p(s) \frac{(\tau(t)-s)^{n-2}}{(n-2)!}\, \mathrm{d}s \right],$$

that is

$$y^{(n+1)}(t) - \alpha_n(t) y^{(n)}(t) \geq 0.$$

Consequently,

$$\left(e^{-\beta_n(t)} y^{(n)}(t) \right)' \geq 0$$

and we conclude that $e^{-\beta_n(t)} y^{(n)}(t)$ is increasing, which is in view of (E) means that $p(t) y(\tau(t)) e^{-\beta_n(t)}$ is increasing function. The proof is completed. □

We use the above-mentioned monotonicity to establish criterion for unbounded oscillation of (E).

Theorem 2.8. *Let $\tau(t) \geq t$ and*

$$\limsup_{t\to\infty} p(t) e^{-\beta_n(t)} \int_{t}^{\tau(t)} e^{\beta_n(s)} (\tau(t)-s)^{n-1}\, \mathrm{d}s > (n-1)!, \tag{2.12}$$

then $\mathcal{N}_n = \emptyset$. If in addition (1.5) holds, then all nonoscillatory solutions of (E) are of degree 0, i.e., $\mathcal{N} = \mathcal{N}_0$.

Proof. Assume on the contrary that (E) possesses an eventually positive solution $y(t) \in \mathcal{N}_n$. Integrating (E) from t to u $(t \le u)$ and using the monotonicity of $p(t)y(\tau(t))e^{-\beta_n(t)}$, we have

$$y^{(n-1)}(u) \ge y(\tau(t))p(t)e^{-\beta_n(t)} \int_t^u e^{\beta_n(s)}\,\mathrm{d}s.$$

Integrating again from t to u and changing order of integration, we get

$$\begin{aligned} y^{(n-2)}(u) &\ge y(\tau(t))p(t)e^{-\beta_n(t)} \int_t^u \int_t^x e^{\beta_n(s)}\,\mathrm{d}s\,\mathrm{d}x \\ &= y(\tau(t))p(t)e^{-\beta_n(t)} \int_t^u e^{\beta_n(s)}(u-s)\,\mathrm{d}s. \end{aligned} \tag{2.13}$$

Proceeding in the same way $(n-2)$-times, we finally obtain

$$y(u) \ge y(\tau(t))p(t)e^{-\beta_n(t)} \int_t^u e^{\beta_n(s)} \frac{(u-s)^{n-1}}{(n-1)!}\,\mathrm{d}s.$$

Setting $u = \tau(t)$, we have

$$y(\tau(t)) \ge y(\tau(t))p(t)e^{-\beta_n(t)} \int_t^{\tau(t)} e^{\beta_n(s)} \frac{(\tau(t)-s)^{n-1}}{(n-1)!}\,\mathrm{d}s.$$

This contradiction establishes the desired result and the proof is completed. □

Example 2.9. Consider the advanced differential equation

$$y^{(n)}(t) = p_0 y(t+\tau), \quad p_0 > 0, \quad \tau > 0. \tag{E_{x3}}$$

It is easy to see that (1.5) holds, $\alpha_n(t) = \frac{p_0}{(n-1)!}\tau^{n-1} = \omega$ and $\beta_n(t) = \omega t$. Condition (2.12) yields

$$\lim_{t\to\infty} p_0 e^{-\omega t} \int_t^{t+\tau} e^{\omega s} \frac{(t+\tau-s)^{n-1}}{(n-1)!}\,\mathrm{d}s > 1. \tag{2.14}$$

Employing substitution $t+\tau-s = x$, one gets

$$\frac{p_0}{(n-1)!} e^{\omega\tau} \int_0^\tau e^{-\omega x} x^{n-1}\,\mathrm{d}s > 1.$$

Proceeding exactly as in Example 2.5 we are led to (2.8) which by Theorem 2.8 ensures that every nonoscillatory solution of (E_{x3}) is of degree 0 or in other words, every unbounded solution (if exists) of (E_{x3}) is oscillatory.

Example 2.10. We consider the advanced Euler differential equation

$$y^{(n)}(t) = \frac{p_0}{t^n} y(\lambda t), \quad p_0 > 0, \quad \lambda > 1. \tag{E_{x4}}$$

Simple calculation shows that (1.5) reduces to

$$p_0\lambda\,(2 + \ln\lambda) > 2(n-2)! \tag{2.15}$$

and

$$\alpha_n(t) = \frac{1}{t}\left[\frac{p_0(\lambda-1)^{n-1}}{(n-1)!} - n\right].$$

Les us denote

$$\frac{p_0(\lambda-1)^{n-1}}{(n-1)!} - n = \delta_n > 0,$$

Then

$$\beta_n = \delta_n \ln t.$$

Therefore (2.12) is equivalent to

$$\limsup_{t\to\infty} \frac{p_0 t^{-\delta_n - n}}{(n-1)!} \int_t^{\lambda t} s^{\delta_n}(\lambda t - s)^{n-1}\,\mathrm{d}s > 1.$$

On the other hand, as

$$\int_t^{\lambda t} \frac{(\lambda t - s)^{n-1}}{s^{-\delta_n}}\,\mathrm{d}s = -\sum_{i=0}^{n-1} \frac{(n-1)!(-\lambda t)^i s^{n+\delta_n - i}}{(n-1-i)\,!i\,!(n+\delta_n-1)}\bigg|_t^{\lambda t}$$

condition (2.12), which guaranties $N_n = \emptyset$ for equation (E_{x4}), takes the form

$$p_0 \sum_{i=0}^{n-1} (-1)^{i+1} \frac{\lambda^{n+\delta_n} - \lambda^i}{(n-1-i)!\,i!\,(n+\delta_n-1)} > 1. \tag{2.16}$$

Moreover, if (2.15) holds, then every nonoscillatory solution of (E_{x2}) is of degree 0. To see the progress which our criteria brings, let us consider $n = 2$ ($n = 4$) and $\lambda = 1.5$. The condition (2.16) is satisfied when

$$p_0 > 6.56 \quad (p_0 > 304.48)$$

while (1.3) requires $p_0 > 10.58$ ($p_0 > 535.64$).

Remark 2.11. In this paper, we have introduce new technique for investigation of monotonicity for nonoscillatory solutions of higher order differential equations. The monotonicities obtained have been applied to establish new criteria for all solutions to be of degree 0 or to be of degree n.

References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations*, Kluwer Academic Publishers, Dordrecht, 2002. https://doi.org/10.1007/978-94-017-2515-6; MR2091751; Zbl 1073.34002

[2] B. Baculíková, Oscillation of second-order nonlinear noncanonical differential equations with deviating argument, *Appl. Math. Lett.* **91**(2019), 68–75. https://doi.org/10.1016/j.aml.2018.11.021; MR3892107; Zbl 1408.34046

[3] B. Baculíková, Oscillatory behavior of the second order noncanonical differential equation, *Electron. J. Qual. Theory Differ. Equ.* **2019**, No. 89, 1–17. https://doi.org/10.14232/ejqtde.2019.1.89; MR4040408; Zbl 1449.34220

[4] B. Baculíková, Oscillation and asymptotic properties of second order half-linear differential equations with mixed deviating arguments, *Mathematics* **9**(2021), 1–12. https://doi.org/10.3390/math9202552

[5] O. Došly, P. Řehák, *Half-linear differential equations*, North-Holland Mathematics Studies, Vol. 202, 2005. MR2158903 Zbl 1090.34001

[6] J. Dzurina, B. Baculikova, Oscillation of half-linear differential equation with mixed type of argument, *Electron. J. Qual. Theory Differ. Equ.* **2022**, No. 10, 1–8. https://doi.org/10.14232/ejqtde.2022.1.10; MR4400928; Zbl 1499.34346

[7] I. T. Kiguradze, On the oscillation of solutions of the equation $\frac{d^m u}{dt^m} + a(t)|u|^n \operatorname{sign} u = 0$ (in Russian), *Mat. Sb.* **65**(1964), 172–187. Zbl 0135.14302

[8] I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Acad. Publ., Dordrecht 1993. https://doi.org/10.1007/978-94-011-1808-8; MR1220223 Zbl 0782.34002

[9] R. Koplatadze, On differential equations with a delayed argument having properties A and B, *Differentialnye Uravneniya* **25**(1989), 1897–1909. MR1034096; Zbl 0693.34068

[10] R. Koplatadze, On oscillatory properties of solutions of functional differential equations, *Mem. Differential Equations Math. Phys.* **3**(1994), 1–179. https://doi.org/1375838; Zbl 0843.34070

[11] R. Koplatadze, T. A. Chanturia, *On oscillatory properties of differential equations with deviating arguments*, Tbilisi Univ. Press, Tbilisi, 1977. MR0492725

[12] R. Koplatadze, G. Kvinkadze, I. P. Stavroulakis, Properties A and B of nth order linear differential equations with deviating argument, *Georgian Math. J.* **6**(1999), 553–566. https://doi.org/10.1023/A:1022962129926; MR1693798; Zbl 0939.34062

[13] T. Kusano, On even order functional differential equations with advanced and retarded arguments, *J. Differential Equations* **45**(1982), 75–84. https://doi.org/10.1016/0022-0396(82)90055-9; MR662487; Zbl 0512.34059

[14] T. Kusano, Oscillation of even order linear functional differential equations with deviating arguments of mixed type, *J. Math. Anal. Appl.* **98**(1984), 341–347. https://doi.org/10.1016/0022-247X(84)90253-1; MR730511; Zbl 0546.34056

[15] T. Li, N. Pintus, G. Viglialoro, Properties of solutions to porous medium problems with different sources and boundary conditions, *Z. Angew. Math. Phys.* **70**(2019), 1–18. https://doi.org/10.1007/s00033-019-1130-2; MR3952687; Zbl 1415.35156

[16] T. Li, Y. Rogovchenko, Oscillation criteria for second-order superlinear Emden–Fowler neutral differential equations, *Monatsh. Math.* **184**(2017), 489–500. https://doi.org/10.1007/s00605-017-1039-9; MR3709308; Zbl 1386.34118

[17] T. Li, G. Viglialoro, Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime, *Differential Integral Equations* **34**(2021), 315–336. MR4297724; Zbl 1474.35004

[18] G. Ladas, V. Lakshmikantham, J. S. Papadakis, Oscillation of higher-order retarded differential equations generated by retarded argument, in: *Delay and functional differential equations and their applications (Proc. Conf., Park City, Utah, 1972)*, Academic Press, New York, 1972, pp. 219–231. MR0387776; Zbl 0273.34052