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CONVERGENCE RATES OF THE SOLUTION OF A VOLTERRA–TYPE STOCHASTIC DIFFERENTIAL EQUATIONS TO A NON–EQUILIBRIUM LIMIT

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Abstract. This paper concerns the asymptotic behaviour of solutions of functional differential equations with unbounded delay to non–equilibrium limits. The underlying deterministic equation is presumed to be a linear Volterra integro–differential equation whose solution tends to a non–trivial limit. We show when the noise perturbation is bounded by a non–autonomous linear functional with a square integrable noise intensity, solutions tend to a non–equilibrium and non–trivial limit almost surely and in mean–square. Exact almost sure convergence rates to this limit are determined in the case when the decay of the kernel in the drift term is characterised by a class of weight functions.

1. Introduction

This paper studies the asymptotic convergence of the solution of the stochastic functional differential equation

(1.1a)
$$dX(t) = \left(AX(t) + \int_0^t K(t-s)X(s)\,ds\right)dt + G(t, X_t)\,dB(t), \quad t > 0,$$

(1.1b)
$$X(0) = X_0,$$

to a non–equilibrium limit. The paper develops recent work in Appleby, Devin and Reynolds [3, 4], which considers convergence to non–equilibrium limits in linear stochastic Volterra equations. The distinction between the works is that in [3, 4], the diffusion coefficient is independent of the solution, and so it is possible to represent the solution explicitly; in this paper, such a representation does not apply. However, we can avail of a variation of constants argument, which enables us to prove that the solution is bounded in mean square. From

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this, it can be inferred that the solution converges to a non–trivial limit in mean square, and from this the almost sure convergence can be deduced. Exact almost sure convergence rates to the non–trivial limit are also determined in this paper; we focus on subexponential, or exponentially weighted subexponential convergence.

The literature on asymptotically constant solutions to deterministic functional differential and Volterra equations is extensive; a recent contribution to this literature, which also gives a nice survey of results is presented in [11]. Motivation from the sciences for studying the phenomenon of asymptotically constant solutions in deterministic and stochastic functional differential or functional difference equations arise for example from the modelling of endemic diseases [14, 3] or in the analysis of inefficient financial markets [8].

In (1.1), the solution X is a $n \times 1$ vector-valued function on $[0,\infty)$, A is a real $n\times n$ matrix, K is a continuous and integrable $n\times n$ matrix-valued function on $[0,\infty)$, G is a continuous $n \times d$ matrix-valued functional on $[0,\infty) \times C([0,\infty),\mathbb{R}^n)$ and $B(t) = \{B_1(t), B_2(t), \dots, B_d(t)\}$, where each component of the Brownian motion $B_j(t)$ are independent. The initial condition X_0 is a deterministic constant vector.

The solution of (1.1) can be written in terms of the solution of the resolvent equation

$$R'(t) = AR(t) + \int_0^t K(t-s)R(s)\,ds, \quad t > 0, \tag{1.2a}$$

$$R(0) = I, \tag{1.2b}$$

where the matrix–valued function R is known as the resolvent or fundamental solution of (1.2). The representation of solutions of (1.1) in terms of R is given by the variation of constants formula

$$X(t) = R(t)X_0 + \int_0^t R(t-s)G(s,X_s)\,dB(s), \quad t \geq 0.$$

In this paper, it is presumed that R tends to a non–trivial limit, and that the perturbation G obeys a linear bound in the second argument, with the bound on G also satisfying a fading, time–dependent intensity. The presence of this small noise intensity ensures that the solutions of the stochastic equation (1.1), like the deterministic resolvent R, converge to a limit. Once this has been proven, we may use information about the convergence rate of solutions of R to its non–trivial limit, proven in [5], to help determine the convergence rate of solutions to the stochastic equation to the non–trivial limit. Some other papers which consider exponential and non–exponential convergence properties of solutions of stochastic Volterra equations to equilibrium solutions include [2, 16, 17].

2. Mathematical Preliminaries

We introduce some standard notation. We denote by $\mathbb{R}$ the set of real numbers. Let $M_{n,d}(\mathbb{R})$ be the space of $n\times d$ matrices with real entries. The transpose of any matrix A is denoted by A^T and the trace of a square matrix A is denoted by $\mathrm{tr}(A)$. Further denote by I_n the identity matrix in $M_{n,n}(\mathbb{R})$ and denote by $\mathrm{diag}(a_1,a_2,...,a_n)$ the $n\times n$ matrix with the scalar entries $a_1,a_2,...,a_n$ on the diagonal and 0 elsewhere.

We denote by $\langle x,y\rangle$ the standard inner product of x and $y\in\mathbb{R}^n$. Let $\|\cdot\|$ denote the Euclidian norm for any vector $x\in\mathbb{R}^n$. For $A=(a_{ij})\in M_{n,d}(\mathbb{R})$ we denote by $\|\cdot\|$ the norm defined by

$$\|A\|^2=\sum_{i=1}^n\left(\sum_{j=1}^d|a_{ij}|\right)^2,$$

and we denote by $\|\cdot\|_F$ the Frobenius norm defined by

$$\|A\|_F^2=\sum_{i=1}^n\sum_{j=1}^d|a_{ij}|^2.$$

Since $M_{n,d}(\mathbb{R})$ is a finite dimensional Banach space the two norms $\|\cdot\|$ and $\|\cdot\|_F$ are equivalent. Thus we can find universal constants $d_1(n,d)\le d_2(n,d)$ such that

$$d_1\|A\|\le\|A\|_F\le d_2\|A\|,\quad A\in M_{n,d}(\mathbb{R}).$$

The absolute value of $A=(A_{ij})$ in $M_n(\mathbb{R})$ is the matrix $|A|$ given by $(|A|)_{ij}=|A_{ij}|$. The spectral radius of a matrix A is given by $\rho(A)=\lim_{n\to\infty}\|A^n\|^{1/n}$.

If J is an interval in $\mathbb{R}$ and V a finite dimensional normed space, we denote by $C(J,V)$ the family of continuous functions $\phi:J\to V$. The space of Lebesgue integrable functions $\phi:(0,\infty)\to V$ will be denoted by $L^1((0,\infty),V)$ and the space of Lebesgue square integrable functions $\eta:(0,\infty)\to V$ will be denoted by $L^2((0,\infty),V)$. Where V is clear from the context we omit it from the notation.

We denote by $\mathbb{C}$ the set of complex numbers; the real part of z in $\mathbb{C}$ being denoted by $\mathrm{Re}\,z$ and the imaginary part by $\mathrm{Im}\,z$. If $A:[0,\infty)\to M_n(\mathbb{R})$ then the Laplace transform of A is formally defined to be

$$\hat{A}(z)=\int_0^\infty A(t)e^{-zt}dt.$$

The convolution of F and G is denoted by $F*G$ and defined to be the function given by

$$(F*G)(t)=\int_0^t F(t-s)G(s)\,ds,\qquad t\ge0.$$

The n-dimensional equation given by (1.1) is considered. We assume that the function $K : [0, \infty) \to M_{n,n}(\mathbb{R})$ satisfies

$$K \in C([0,\infty), M_{n,n}(\mathbb{R})) \cap \mathrm{L}^1((0,\infty), M_{n,n}(\mathbb{R})), \tag{2.1}$$

and $G : [0,\infty) \times C([0,\infty),\mathbb{R}^n) \to M_{n,d}(\mathbb{R})$ is continuous functional which is locally Lipschitz continuous with respect to the sup–norm topology, and obeys

(2.2) For every $n \in \mathbb{N}$ there is $K_n > 0$ such that
$$\|G(t,\phi_t) - G(t,\varphi_t)\|_F \le K_n \|\phi - \varphi\|_{[0,t]},$$
$$\text{for all } \phi,\ \varphi \in C([0,\infty),\mathbb{R}^n) \text{ with } \|\phi\|_{[0,t]} \vee \|\varphi\|_{[0,t]} \le n,$$

where we use the notation

$$\|\phi\|_{[0,t]} = \sup_{0\le s\le t} \|\phi(s)\| \text{ for } \phi \in C([0,\infty),\mathbb{R}^n) \text{ and } t \ge 0,$$

and let ϕ_t to be the function defined by $\phi_t(\theta) = \phi(t+\theta)$ for $\theta \in [-t, 0]$. Moreover, we ask that G also obey

$$\|G(t,\phi_t)\|_F^2 \le \Sigma^2(t)\left(1 + \|\phi(t)\|_2^2 + \int_0^t \kappa(t-s)\|\phi(s)\|_2^2\,ds\right), \quad t \ge 0,\ \phi \in C([0,\infty),\mathbb{R}^n) \tag{2.3}$$

where the function $\Sigma : [0,\infty) \to \mathbb{R}$ satisfies

$$\Sigma \in C([0,\infty),\mathbb{R}), \tag{2.4}$$

and κ obeys

$$\kappa \in C([0,\infty),\mathbb{R}) \cap L^1([0,\infty),\mathbb{R}). \tag{2.5}$$

Due to (2.1) we may define K_1 to be a function $K_1 \in C([0,\infty), M_{n,n}(\mathbb{R}))$ such that

$$K_1(t) = \int_t^\infty K(s)\,ds, \quad t \ge 0, \tag{2.6}$$

where this function defines the tail of the kernel K. We let $(B(t))_{t\ge0}$ denote d-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}^B, \mathbb{P})$ where the filtration is the natural one $\mathcal{F}^B(t) = \sigma\{B(s) : 0 \le s \le t\}$. We define the function $t \mapsto X(t; X_0, \Sigma)$ to be the unique solution of the initial value problem (1.1). The existence and uniqueness of solutions is covered in [9, Theorem 2E] or [21, Chapter 5] for example. Under the hypothesis (2.1), it is well-known that (1.2) has a unique continuous solution R, which is continuously differentiable. Moreover if Σ is continuous then for any deterministic initial condition X_0 there exists a unique a.s. continuous solution to (1.1) given by

$$X(t; X_0) = R(t)X_0 + \int_0^t R(t-s)G(s, X_s)\, dB(s), \quad t \geq 0. \tag{2.7}$$

We denote $\mathbb{E}\,[X^2]$ by $\mathbb{E}\,X^2$ except in cases where the meaning may be ambiguous. We now define the notion of convergence in mean square and almost sure convergence.

Definition 2.1. The $\mathbb{R}^n$-valued stochastic process $(X(t))_{t\geq 0}$ converges in mean square to X_∞ if

$$\lim_{t\to\infty} \mathbb{E}\,\|X(t) - X_\infty\|^2 = 0,$$

and we say that the difference between the stochastic process $(X(t))_{t\geq 0}$ and X_∞ is integrable in the mean square sense if

$$\int_0^\infty \mathbb{E}\,\|X(t) - X_\infty\|^2\, dt < \infty.$$

Definition 2.2. If there exists a $\mathbb{P}$-null set Ω_0 such that for every $\omega \notin \Omega_0$ the following holds

$$\lim_{t\to\infty} X(t,\omega) = X_\infty(\omega),$$

then we say X converges almost surely to X_∞. Furthermore, if

$$\int_0^\infty \|X(t,\omega) - X_\infty(\omega)\|^2\, dt < \infty,$$

we say that the difference between the stochastic process $(X(t))_{t\geq 0}$ and X_∞ is square integrable in the almost sure sense.

In this paper we are particularly interested in the case where the random variable X_∞ is nonzero almost surely.

3. Convergence to a Non-Equilibrium Limit

In this section, we consider the convergence of solutions to a non–equilibrium limit without regard to pointwise rates of convergence. Instead, we concentrate on giving conditions on the stochastic intensity (i.e., the functional G) and the Volterra drift such that solutions converge almost surely and in mean–square. The square integrability of the discrepancy between the solution and the limit is also studied. The results obtained in [3] and [4] are special cases of the ones found here, where the functional G in (1.1) is of the special form $G(t,\varphi_t) = \Theta(t)$, $t \geq 0$, where $\Theta \in C([0,\infty), M_n(\mathbb{R}))$ is in the appropriate $L^2([0,\infty), M_n(\mathbb{R}))$–weighted space.

3.1. **Deterministic results and notation.** In the deterministic case Krisztin and Terjéki [15] considered the necessary and sufficient conditions for asymptotic convergence of solutions of (1.2) to a nontrivial limit and the integrability of these solutions. Before recalling their main result we define the following notation introduced in [15] and adopted in this paper. We let $M = A + \int_0^\infty K(s)\,ds$ and T be an invertible matrix such that $T^{-1}MT$ has Jordan canonical form. Let $e_i = 1$ if all the elements of the i^{th} row of $T^{-1}MT$ are zero, and $e_i = 0$ otherwise. Put $P = T\text{diag}(e_1, e_2, ..., e_n)T^{-1}$ and $Q = I - P$.

Krisztin and Terjéki prove the following result: if K satisfies

$$\int_0^\infty t^2\|K(t)\|\,dt < \infty, \tag{3.1}$$

then the resolvent R of (1.2) satisfies

$$R - R_\infty \in L^1((0,\infty), M_{n,n}(\mathbb{R})), \tag{3.2}$$

if and only if

$$\det[zI - A - \hat{K}(z)] \neq 0 \quad \text{for } \operatorname{Re} z \geq 0 \text{ and } z \neq 0,$$

and

$$\det\left[P - M - \int_0^\infty \int_s^\infty PK(u)duds\right] \neq 0.$$

Moreover, they show that the limiting value of R is given by

$$R_\infty = \left[P - M + \int_0^\infty \int_s^\infty PK(u)\,du\,ds\right]^{-1} P. \tag{3.3}$$

Although Krisztin and Terjéki consider the case where $R - R_\infty$ exists in the space of $L^1(0,\infty)$ functions it is more natural to consider the case where $R - R_\infty$ lies in the $L^2(0,\infty)$ space for stochastic equations.

3.2. **Convergence to a non–equilibrium stochastic limit.** We are now in a position to state the first main result of the paper. It concerns conditions for the a.s. and mean–square convergence of solutions of (1.1) to a non–trivial and a.s. finite limit, without making any request on the speed at which the convergence takes place.

Theorem 3.1. *Let K satisfy* (2.1) *and*

$$\int_0^\infty t\|K(t)\|\,dt < \infty, \tag{3.4}$$

Suppose that the functional G obeys (2.2) *and* (2.3), *κ obeys* (2.5), *and Σ satisfis* (2.4) *and*

$$\int_0^\infty \Sigma^2(t)\,dt < \infty. \tag{3.5}$$

If the resolvent R of (1.2) *satisfies*

$$R - R_\infty \in L^2((0,\infty), M_{n,n}(\mathbb{R})), \tag{3.6}$$

then there exists a $\mathcal{F}^B(\infty)$–measurable and a.s. finite random variable X_∞ such that the solution X of (1.1) *satisfies*

$$\lim_{t\to\infty} X(t) = X_\infty \quad a.s.,$$

where

$$X_\infty = R_\infty \left(X_0 + \int_0^\infty G(s, X_s)\, dB(t) \right) \quad a.s..$$

Moreover,

$$\lim_{t\to\infty} \mathbb{E}[\|X(t) - X_\infty\|^2 = 0. \tag{3.7}$$

In this theorem the existence of the first moment of K is required rather than the existence of the second moment of K as required by Krisztin and Terjéki. However, the condition (3.6) is required. The condition (3.5) is exactly that required for mean–square convergence in [4], and for almost sure convergence in [3]. In both these papers, the functional G depends only on t, and not on the path of the solution. Moreover, as (3.5) was shown to be necessary for mean–square and almost sure convergence in [3, 4], it is difficult to see how it can readily be relaxed. Furthermore, in [3, 4] the necessity of the condition (3.6) has been established, if solutions of equations with path–independent G are to converge in almost sure and mean–square senses.

The condition (2.3) on G, together with the fact that Σ (3.5), makes Theorem 3.1 reminiscent of a type of Hartman–Wintner theorem, in which the asymptotic behaviour of an unperturbed linear differential equation is preserved when that equation is additively perturbed, and the perturbation can be decomposed into the product of a time–dependent, and (in some sense) rapidly decaying function, and a function which is linearly bounded in the state variable. Indeed the results in this paper suggest that a general Hartman–Wintner theorem should be available for stochastic functional differential equations which are subject to very mild nonlinearities, along the lines investigated in the deterministic case by Pituk in [18].

Sufficient conditions for the square integrability of $X - X_\infty$ in the almost sure sense and in the mean sense are considered in Theorem 3.2. As before the conditions (3.5) and (3.6) are required for convergence; in addition, (3.8) is required for integrability. This last condition has been shown to be necessary to guarantee the mean square integrability of $X - X_\infty$ in [4], for equations with path–independent G.

Theorem 3.2. *Let K satisfy* (2.1) *and* (3.4). *Suppose that there exists a constant matrix R_∞ such that the solution R of* (1.2) *satisfies* (3.6). *Suppose that the functional G obeys* (2.2) *and* (2.3), *κ obeys* (2.5) *and Σ satisfy* (2.4), (3.5) *and*

$$\int_0^\infty t\Sigma^2(t)\,dt < \infty. \tag{3.8}$$

Then for all initial conditions X_0 there exists an a.s. finite $\mathcal{F}^B(\infty)$–measurable random variable X_∞ with $\mathbb{E}[\|X_\infty\|^2] < \infty$ such that the unique continuous adapted process X which obeys (1.1) *satisfies*

$$\mathbb{E}\|X(\cdot) - X_\infty\|^2 \in L^1((0,\infty),\mathbb{R}). \tag{3.9}$$

and

$$X(\cdot) - X_\infty \in L^2((0,\infty),\mathbb{R}^n) \quad a.s. \tag{3.10}$$

4. Exact rates of convergence to the limit

In this section, we examine the rate at which X converges almost surely to X_∞, in the case when the most slowly decaying entry of the kernel K in the drift term of (1.1a) is asymptotic to a scalar function in a class of weighted functions, and the noise intensity Σ decays sufficiently quickly.

4.1. **A class of weight functions.**

We make a definition, based on the hypotheses of Theorem 3 of [13].

Definition 4.1. Let $\mu \in \mathbb{R}$, and $\gamma : [0,\infty) \to (0,\infty)$ be a continuous function. Then we say that $\gamma \in \mathcal{U}(\mu)$ if

$$\hat{\gamma}(\mu) = \int_0^\infty \gamma(t)e^{-\mu t}\,dt < \infty, \tag{4.1}$$

$$\lim_{t\to\infty} \frac{(\gamma * \gamma)(t)}{\gamma(t)} = 2\hat{\gamma}(\mu), \tag{4.2}$$

$$\lim_{t\to\infty} \frac{\gamma(t-s)}{\gamma(t)} = e^{-\mu s} \quad \text{uniformly for } 0 \le s \le S, \text{ for all } S > 0. \tag{4.3}$$

We say that a continuously differentiable $\gamma : [0,\infty) \to (0,\infty)$ is in $\mathcal{U}^1(\mu)$ if it obeys (4.1), (4.2), and

$$\lim_{t\to\infty} \frac{\gamma'(t)}{\gamma(t)} = \mu. \tag{4.4}$$

It is to be noted that the condition (4.4) implies (4.3), so $\mathcal{U}^1(\mu) \subset \mathcal{U}(\mu)$.

If γ is in $\mathcal{U}(0)$ it is termed a *subexponential* function[1]. The nomenclature is suggested by the fact that (4.3) with $\mu = 0$ implies that, for every $\epsilon > 0$, $\gamma(t)e^{\epsilon t} \to \infty$ as $t \to \infty$. This is proved for example in [6]. As a direct consequence of this fact, it is true that $\gamma \in \mathcal{U}(\mu)$ obeys

$$\lim_{t\to\infty} \gamma(t)e^{(\epsilon-\mu)t} = \infty, \quad \text{for each } \epsilon > 0. \tag{4.5}$$

It is also true that

$$\lim_{t\to\infty} \gamma(t)e^{-\mu t} = 0. \tag{4.6}$$

It is noted in [7] that the class of subexponential functions includes all positive, continuous, integrable functions which are regularly varying at infinity[2]. The properties of $\mathcal{U}(0)$ have been extensively studied, for example in [7, 6, 12, 13].

Note that if γ is in $\mathcal{U}(\mu)$, then $\gamma(t) = e^{\mu t}\delta(t)$ where δ is a function in $\mathcal{U}(0)$. Simple examples of functions in $\mathcal{U}(\mu)$ are $\gamma(t) = e^{\mu t}(1+t)^{-\alpha}$ for $\alpha > 1$, $\gamma(t) = e^{\mu t}e^{-(1+t)^{\alpha}}$ for $0 < \alpha < 1$ and $\gamma(t) = e^{\mu t}e^{-t/\log(t+2)}$. The class $\mathcal{U}(\mu)$ therefore includes a wide variety of functions exhibiting exponential and slower–than–exponential decay: nor is the slower–than–exponential decay limited to a class of polynomially decaying functions.

If the domain of F contains an interval of the form (T, ∞) and γ is in $\mathcal{U}(\mu)$, $L_\gamma F$ denotes $\lim_{t\to\infty} F(t)/\gamma(t)$, if it exists.

Finally, if $\gamma \in \mathcal{U}(\mu)$ and for matrix–valued functions F and G the limits $L_\gamma F$ and $L_\gamma G$ exist, it follows that $L_\gamma(F * G)$ also exists, and a formula can be given for that limit. This fact was proved in [5].

Proposition 4.2. *Let $\mu \le 0$ and $\gamma \in \mathcal{U}(\mu)$. If $F, G : (0,\infty) \to M_n(\mathbb{R})$ are functions for which $L_\gamma F$ and $L_\gamma G$ both exist, $L_\gamma(F * G)$ exists and is given by*

$$L_\gamma(F*G) = L_\gamma F\left(\int_0^\infty e^{-\mu s}G(s)\,ds\right) + \left(\int_0^\infty e^{-\mu s}F(s)\,ds\right)L_\gamma G. \tag{4.7}$$

Corresponding results exist for the weighted limits of convolutions of matrix–valued functions with vector–valued functions. Proposition 4.2 is often applied in this paper.

4.2. **Almost sure rate of convergence in weighted spaces.** We are now in a position to give our main result, which is a stochastic and finite dimensional generalisation of results given in [6], and, in some

[1] In [7] the terminology positive subexponential function was used instead of just subexponential function. Because the functions in $\mathcal{U}(\mu)$ play the role here of weight functions, it is natural that they have strictly positive values.

[2] γ is *regularly varying at infinity* if $\gamma(\alpha t)/\gamma(t)$ tends to a limit as $t \to \infty$ for all $\alpha > 0$; for further details see [10].

sense a stochastic generalisation of results in [5]. Let $\beta > 0$ and define $e_\beta(t) = e^{-\beta t}$, $t \geq 0$. If (3.1) holds, the function K_2 given by

$$K_2(t) = \int_t^\infty K_1(s)\,ds = \int_t^\infty \int_s^\infty K(u)\,du\,ds, \quad t \geq 0. \tag{4.8}$$

is well–defined and integrable. We also introduce $\tilde{\Sigma}^2$ by

$$\tilde{\Sigma}^2(t) = \int_0^t e^{-2\beta(t-s)}\Sigma^2(s)\,ds \cdot \log t, \quad t \geq 2, \tag{4.9}$$

and F by

$$F(t) = -e^{-\beta t}(\beta Q + QM) + K_1(t) - \beta Q(e_\beta * K_1)(t), \quad t \geq 0. \tag{4.10}$$

The following is our main result of this section.

Theorem 4.3. *Let K satisfy (2.1) and (3.4). Suppose that the functional G obeys (2.2) and (2.3), κ obey (2.5) and Σ satisfy (2.4) and (3.5). Suppose there is a $\mu \leq 0$ and $\beta > 0$ such that $\beta + \mu > 0$ and F defined by (4.10) obeys*

$$\rho\left(\int_0^\infty e^{-\mu s}|F(s)|\,ds\right) < 1. \tag{4.11}$$

Suppose that $\gamma \in \mathcal{U}^1(\mu)$ is such that

$$L_\gamma K_1 \text{ and } L_\gamma K_2 \text{ exist,} \tag{4.12}$$

where K_1 and K_2 are defined by (2.6) and (4.8). If $\tilde{\Sigma}$ given by (4.9) obeys

$$L_\gamma \tilde{\Sigma} = 0, \quad L_\gamma \int_\cdot^\infty \tilde{\Sigma}(s)\,ds = 0, \tag{4.13}$$

then for all initial conditions X_0 there exists an a.s. finite $\mathcal{F}^B(\infty)$–measurable random variable X_∞ with $\mathbb{E}[X_\infty^2] < \infty$ such that the unique continuous adapted process X which obeys (1.1) has the property that $L_\gamma(X - X_\infty)$ exists and is a.s. finite. Furthermore

(a) *if $\mu = 0$, and R_∞ is given by (3.3), then*

$$L_\gamma(X - X_\infty) = R_\infty(L_\gamma K_2)R_\infty\left(X_0 + \int_0^\infty G(s, X_s)\,dB(s)\right) \quad a.s.; \tag{4.14}$$

(b) *if $\mu < 0$, and we define $R_\infty(\mu) := \left(I + \hat{K}_1(\mu) - \frac{1}{\mu}M\right)^{-1}$, then*

$$L_\gamma(X - X_\infty) = R_\infty(\mu)(L_\gamma K_2)R_\infty(\mu)\left(X_0 + \int_0^\infty e^{-\mu s}G(s, X_s)\,dB(s)\right), \tag{4.15}$$

almost surely.

We explain briefly the role of the new conditions in this theorem. The condition (4.11) is sufficient to ensure that $R - R_\infty$ lies in the appropriate exponentially weighted L^1 space. (4.12) characterises the rate of decay of the entries of K. In fact, by considering the limit relations (4.14) or (4.15), it may be seen that it is rate of decay of K_2 defined by (4.8) which determines the rate of convergence of solutions of (1.1) to the limit, under the condition (4.13). This last condition ensures that the noise intensity Σ decays sufficiently quickly relative to the decay rate of the kernel K so that the stochastic perturbation is sufficiently small to ensure that the rate of convergence to the limit is the same as that experienced by the deterministic resolvent R to its non–trivial limit R_∞. The fact that $L_\gamma(R-R_\infty)$ is finite is deduced as part of the proof of Theorem 4.3.

We observe that in the case when $G \equiv 0$ that the formula in case (a) is exactly that found to apply to the deterministic and scalar Volterra integrodifferential equation studied in [6].

Finally, in the case when G is deterministic and $G = G(t)$, the limit $L_\gamma(X-X_\infty)$ is in general non–zero, almost surely, because $L_\gamma(X-X_\infty)$ is a finite–dimensional Gaussian vector.

In the one–dimensional case, the following corollary is available. Suppose that $A \in \mathbb{R}$, $K \in C([0,\infty);\mathbb{R}) \cap L^1([0,\infty);\mathbb{R})$ and $A + \int_0^\infty K(s)\,ds = 0$. In this case $Q = 0$ and so $F(t) = K_1(t)$. Moreover, R_∞ defined in (3.3) reduces to

$$R_\infty = \frac{1}{1+\int_0^\infty uK(u)\,du} = \frac{1}{1+\hat{K}_1(0)}.$$

Theorem 4.4. *Let $K \in C([0,\infty);\mathbb{R}) \cap L^1([0,\infty);\mathbb{R})$. Suppose that the functional G obeys (2.2) and (2.3) and let $\Sigma \in C([0,\infty);\mathbb{R}) \cap L^2([0,\infty);\mathbb{R})$ and $\kappa \in C([,\infty);\mathbb{R}) \cap L^1([0,\infty);\mathbb{R})$. Suppose there is a $\mu \le 0$ such that*

$$\int_0^\infty e^{-\mu s}|K_1(s)|\,ds < 1.$$

Suppose that $\gamma \in \mathcal{U}^1(\mu)$ is such that $L_\gamma K_1$ and $L_\gamma K_2$ exist, where K_1 and K_2 are defined by (2.6) and (4.8). If $\tilde{\Sigma}$ given by (4.9) obeys

$$L_\gamma \tilde{\Sigma} = 0, \quad L_\gamma \int_\cdot^\infty \tilde{\Sigma}(s)\,ds = 0,$$

then for all initial conditions X_0 there exists an a.s. finite $\mathcal{F}^B(\infty)$–measurable random variable X_∞ with $\mathbb{E}[\|X_\infty\|^2] < \infty$ such that the unique continuous adapted process X which obeys (1.1) has the property *that $L_\gamma(X - X_\infty)$ exists and is a.s. finite. Furthermore*

$$L_\gamma(X - X_\infty) = \frac{L_\gamma K_2}{(1+\hat{K_1}(\mu))^2}\left(X_0 + \int_0^\infty e^{-\mu s} G(s, X_s)\, dB(s)\right), \quad a.s.$$

5. Proof of Theorems 3.1 and 3.2

We begin with a Lemma. Let X be the solution of (1.1) and define $M = \{M(t); \mathcal{F}^B(t); 0 \le t < \infty\}$ by

$$M(t) = \int_0^t G(s, X_s)\, dB(s), \quad t \ge 0. \tag{5.1}$$

The convergence of M to a finite limit and the rate at which that convergence takes place is crucial in establishing the convergence claimed in Theorems 3.1 and 3.2.

Lemma 5.1. *Suppose that K obeys* (2.1), (3.4) *and that R defined by* (1.2) *obeys* (3.6). *Suppose that G obeys* (2.2) *and* (2.3), *κ obeys* (2.5), *and Σ obeys* (2.4) *and* (3.5). *If X is the unique continuous adapted process which satisfies* (1.1), *then*

(i) $\int_0^\infty \mathbb{E}[\|G(s, X_s)\|_F^2]\, ds < \infty$;

(ii) $\int_0^\infty \|G(s, X_s)\|_F^2\, ds < \infty$, *a.s. and there exists an almost surely finite and $\mathcal{F}^B(\infty)$–measurable random variable $M(\infty)$ such that*

$$M(\infty) := \lim_{t\to\infty} M(t), \quad a.s. \tag{5.2}$$

(iii) *There exists $x^* > 0$ such that $\mathbb{E}[\|X(t)\|_2^2] \le x^*$ for all $t \ge 0$.*

(iv) *With $M(\infty)$ defined by* (5.2), *we have $\mathbb{E}[\|M(\infty) - M(t)\|_2^2] \to 0$ as $t \to \infty$.*

(v) *In the case that Σ obeys* (3.8) *we have*

$$\int_0^\infty \mathbb{E}[\|M(\infty) - M(t)\|_2^2]\, dt < \infty, \quad \int_0^\infty \|M(\infty) - M(t)\|_2^2\, dt < \infty, \quad a.s.$$

Proof. The proof of part (i) is fundamental. (ii) is an easy consequence of it, and (iii) follows very readily from (i) also. The proofs of (iv) and (v) use part (i).

The key to the proof of (i) is to develop a linear integral inequality for the function $t \mapsto \sup_{0\le s\le t} \mathbb{E}[\|X(s)\|_2^2]$. This in turn is based on the fundamental variation of constants formula

$$X(t) = R(t)X_0 + \int_0^t R(t-s)G(s, X_s)\, dB(s), \quad t \ge 0.$$

This result has been rigorously established in [19]. This implies, with $C_{ij}(s,t) := \sum_{k=1}^n R_{ik}(t-s)G_{kj}(s,X_s)$, $0 \le s \le t$, that

$$X_i(t) = [R(t)X_0]_i + \sum_{j=1}^d \int_0^t C_{ij}(s,t)\,dB_j(s).$$

Therefore

$$\mathbb{E}[\|X(t)\|_2^2] \le 2\|R(t)X_0\|_2^2 + 2\sum_{i=1}^d \mathbb{E}\left(\sum_{j=1}^d \int_0^t C_{ij}(s,t)\,dB_j(s)\right)^2. \tag{5.3}$$

Now, as $C_{ij}(s,t)$ is $\mathcal{F}^B(s)$–measurable, we have

$$\mathbb{E}\left(\sum_{j=1}^d \int_0^t C_{ij}(s,t)\,dB_j(s)\right)^2 = \sum_{j=1}^d \int_0^t \mathbb{E}[C_{ij}^2(s,t)]\,ds = \int_0^t \sum_{j=1}^d \mathbb{E}[C_{ij}^2(s,t)]\,ds,$$

and so

$$C_{ij}^2(s,t) = \left(\sum_{k=1}^n R_{ik}(t-s)G_{kj}(s,X_s)\right)^2 \le n\sum_{k=1}^n R_{ik}^2(t-s)G_{kj}^2(s,X_s).$$

Since $R(t) \to R_\infty$ as $t \to \infty$, we have that $R_{ik}(t)^2 \le \bar{R}^2$ for all $t \ge 0$. Hence

$$\sum_{j=1}^d C_{ij}^2(s,t) \le \bar{R}^2 \cdot n \sum_{j=1}^d \sum_{k=1}^n G_{kj}^2(s,X_s) = \bar{R}^2 \cdot n\|G(s,X_s)\|_F^2.$$

Therefore, by (2.3), we have

$$\sum_{j=1}^d C_{ij}^2(s,t) \le \bar{R}^2 \cdot n\|G(s,X_s)\|_F^2 \le \bar{R}^2 \cdot n\Sigma^2(s)\left(1 + \|X(s)\|_2^2 + (\kappa * \|X\|_2^2)(s)\right),$$

and so, with $x(t) = \mathbb{E}[\|X(t)\|_2^2]$, we get

$$\sum_{j=1}^d \mathbb{E}[C_{ij}^2(s,t)] \le \bar{R}^2 \cdot n\Sigma^2(s)\left(1 + x(s) + (\kappa * x)(s)\right).$$

Therefore

$$\int_0^t \sum_{j=1}^d \mathbb{E}[C_{ij}^2(s,t)]\,ds \le \bar{R}^2 n \int_0^t \Sigma^2(s)\left(1 + x(s) + (\kappa * x)(s)\right)\,ds,$$

and so, by returning to (5.3) we obtain

$$x(t) \leq 2\|R(t)X_0\|_2^2 + 2\bar{R}^2 nd \int_0^t \Sigma^2(s)\left(1 + x(s) + (\kappa * x)(s)\right) ds.$$

Using the fact that $\Sigma \in L^2([0,\infty),\mathbb{R})$ and R is bounded, there exists a deterministic $c = c(R, X_0, \Sigma) > 0$ such that

$$x(t) \leq c + c\int_0^t \Sigma^2(s)x(s)\,ds + c\int_0^t \Sigma^2(s)\int_0^s \kappa(s-u)x(u)\,du\,ds.$$

Now, define $x^*(s) = \sup_{0\leq u\leq s} x(u)$. Then, as $\kappa \in L^1([0,\infty),\mathbb{R})$, we get

$$x(t) \leq c + c\int_0^t \Sigma^2(s)x^*(s)\,ds + c\int_0^t \Sigma^2(s)\|\kappa\|_{L^1[0,\infty)}x^*(s)\,ds.$$

Therefore

$$x^*(T) = \sup_{0\leq t\leq T} x(t) \leq c + c(1+\|\kappa\|_{L^1[0,\infty)})\int_0^T \Sigma^2(s)x^*(s)\,ds. \tag{5.4}$$

Next, define $c' = c(1+\|\kappa\|_{L^1[0,\infty)})$ and $\tilde{x}(t) = \Sigma^2(t)x^*(t)$, to get

$$\tilde{x}(t) \leq c\Sigma^2(t) + c'\Sigma^2(t)\int_0^t \tilde{x}(s)\,ds.$$

Therefore $\tilde{X}(t) = \int_0^t \tilde{x}(s)\,ds$ obeys the differential inequality

$$\tilde{X}'(t) \leq c\Sigma^2(t) + c'\Sigma^2(t)\tilde{X}(t), \quad t > 0; \quad \tilde{X}(0) = 0,$$

from which we infer

$$\int_0^t \tilde{x}(s)\,ds = \tilde{X}(t) \leq ce^{c'\int_0^t \Sigma^2(s)\,ds}\int_0^t \Sigma^2(s)e^{-c'\int_0^s \Sigma^2(u)\,du}\,ds.$$

Since $\Sigma \in L^2([0,\infty),\mathbb{R})$, we get

$$\int_0^t \tilde{x}(s)\,ds \leq ce^{c'\|\Sigma\|_{L^2(0,\infty)}}\int_0^t \Sigma^2(s)\,ds \leq ce^{c'\|\Sigma\|^2_{L^2(0,\infty)}}\|\Sigma\|^2_{L^2(0,\infty)},$$

and so

$$\int_0^\infty \Sigma^2(s)\sup_{0\leq u\leq s}\mathbb{E}[\|X(u)\|_2^2]\,ds = \int_0^\infty \Sigma^2(s)x^*(s)\,ds < \infty. \tag{5.5}$$

From this, (2.5), (3.5) and (2.3) we see that

$$\int_0^\infty \mathbb{E}[G_{ij}^2(s,X_s)]\,ds < \infty \quad \text{for all } i \in \{1,\ldots,n\},\ j \in \{1,\ldots,d\},$$

which implies (i). The last inequality and Fubini's theorem implies

$$\int_0^\infty G_{ij}^2(s,X_s)\,ds < \infty \quad \text{for all } i \in \{1,\ldots,n\},\ j \in \{1,\ldots,d\} \text{ a.s.}$$

from which (ii) follows by the martingale convergence theorem [20, Proposition IV.1.26].

To prove (iii), note that (5.5) and (5.4) imply

$$\begin{aligned} x^*(T) &\le c + c(1+\|\kappa\|_{L^1[0,\infty)})\int_0^T \Sigma^2(s)x^*(s)\,ds \\ &\le c + c(1+\|\kappa\|_{L^1[0,\infty)})\int_0^\infty \Sigma^2(s)x^*(s)\,ds =: x^*, \end{aligned}$$

as required. Hence for all $t\ge 0$, we have $\mathbb{E}[\|X(t)\|_2^2]\le x^*(t)\le x^*$, as required.

We now turn to the proof of (iv). The proof is standard, but an estimate on $\mathbb{E}[\|M(\infty)-M(t)\|_2^2]$ furnished by the argument is of utility in proving stronger convergence results under condition (3.8), so we give the proof of convergence and the estimate. To do this, note that M is a n–dimensional martingale with $M_i(t)=\langle M(t),\mathbf{e}_i\rangle$ given by

$$M_i(t)=\sum_{j=1}^d \int_0^t G_{ij}(s,X_s)\,dB_j(s).$$

Let $t\ge 0$ be fixed and $u>0$. Then

$$\mathbb{E}[(M_i(t+u)-M_i(t))^2]=\sum_{j=1}^d\int_t^{t+u}\mathbb{E}[G_{ij}^2(s,X_s)]\,ds.$$

Therefore

$$\begin{aligned} &\mathbb{E}[\|M(t+u)-M(t)\|^2] \\ &\quad=\sum_{i=1}^d\sum_{j=1}^d\int_t^{t+u}\mathbb{E}[G_{ij}^2(s,X_s)]\,ds=\int_t^{t+u}\mathbb{E}[\|G(s,X_s)\|_F^2]\,ds. \end{aligned}$$

By Fatou's lemma, as $M(t)\to M(\infty)$ as $t\to\infty$ a.s., we have

$$\begin{aligned} \mathbb{E}[\|M(\infty)-M(t)\|_2^2] &\le \liminf_{u\to\infty}\mathbb{E}[\|M(t+u)-M(t)\|_2^2] \\ &=\int_t^\infty \mathbb{E}[\|G(s,X_s)\|_F^2]\,ds. \qquad (5.6) \end{aligned}$$

The required convergence is guaranteed by the fact that $\mathbb{E}[\|G(\cdot,X_\cdot)\|_F^2]$ is integrable.

To prove part (v), we notice that (5.6), (2.3) and the fact that $\mathbb{E}[\|X(t)\|_2^2] \le x^*$ for all $t \ge 0$ together imply

$$\begin{aligned}&\mathbb{E}[\|M(\infty) - M(t)\|_2^2]\\ &\le \int_t^\infty \Sigma^2(s)\left(1 + \mathbb{E}[\|X(s)\|_2^2] + (\kappa * \mathbb{E}[\|X(\cdot)\|_2^2])(s)\right) ds\\ &\le \left(1 + x^* + \|\kappa\|_{L^1(0,\infty)} x^*\right) \int_t^\infty \Sigma^2(s)\, ds.\end{aligned}$$

Now, if Σ obeys (3.8), it follows that

$$\int_0^\infty \mathbb{E}[\|M(\infty) - M(t)\|_2^2]\, dt < \infty.$$

Applying Fubini's theorem gives $\int_0^\infty \|M(\infty) - M(t)\|_2^2\, dt < \infty$ a.s., proving both elements in part (v). □

We next need to know show that $R' \in L^2([0,\infty), M_n(\mathbb{R}))$.

Lemma 5.2. *Suppose that K obeys* (2.1) *and* (3.4) *and that R defined by* (1.2) *obeys* (3.6)*. Then $R' \in L^2([0,\infty), M_n(\mathbb{R}))$.*

Proof. It can be deduced from e.g., [3, Lemma 5.1], that the conditions on R imply that R_∞ obeys $(A + \int_0^\infty K(s)\, ds) R_\infty = 0$. Since

$$R'(t) = A(R(t) - R_\infty) + (K * (R - R_\infty))(t) + \left(A + \int_0^t K(s)\, ds\right) R_\infty,$$

it follows that

$$R'(t) = A(R(t) - R_\infty) + \int_0^t K(t-s)(R(s) - R_\infty)\, ds - \int_t^\infty K(s)\, ds \cdot R_\infty.$$

Since $K_1(t) \to 0$ as $t \to \infty$ and $K_1 \in L^1([0,\infty), M_n(\mathbb{R}))$ by (3.4), it follows that the last term on the righthand side is in $L^2([0,\infty), M_n(\mathbb{R}))$. The first term is also in $L^2([0,\infty), M_n(\mathbb{R}))$ on account of (3.6). As for the second term, as K is integrable, by the Cauchy–Schwartz inequality, we get

$$\begin{aligned}&\left\|\int_0^t K(t-s)(R(s) - R_\infty)\, ds\right\|_2^2\\ &\le c\left(\int_0^t \|K(t-s)\|_2 \|R(s) - R_\infty\|_2\, ds\right)^2\\ &\le c' \int_0^\infty \|K(s)\|\, ds \cdot \int_0^t \|K(t-s)\|_2 \|R(s) - R_\infty\|_2^2\, ds.\end{aligned}$$

The right hand side is integrable, since $R - R_\infty \in L^2([0,\infty), M_n(\mathbb{R}))$ and K is integrable. Therefore $K * (R - R_\infty)$ is in $L^2([0,\infty), M_n(\mathbb{R}))$, and so $R' \in L^2([0,\infty), M_n(\mathbb{R})$, as needed. □

Lemma 5.3. *Suppose that K obeys* (2.1) *and* (3.4) *and the resolvent R which is defined by* (1.2) *also obeys* (3.6). *Suppose further that*

$$\int_0^\infty \mathbb{E}[\|G(s,X_s)\|_F^2]\,ds < \infty.$$

Then U defined by

$$U(t) = \int_0^t (R(t-s) - R_\infty)G(s,X_s)\,dB(s), \quad t \geq 0 \tag{5.7}$$

obeys

$$\lim_{t\to\infty} \mathbb{E}[\|U(t)\|_2^2] = 0, \quad \lim_{t\to\infty} U(t) = 0, \quad a.s.$$

and

$$\int_0^\infty \mathbb{E}[\|U(t)\|_2^2]\,dt < \infty, \quad \int_0^\infty \|U(t)\|_2^2\,dt < \infty, \quad a.s.$$

Proof. By Itô's isometry, there is a constant $c_2 > 0$ independent of t such that

$$\mathbb{E}[\|U(t)\|_2^2] \leq c_2 \int_0^t \|R(t-s) - R_\infty\|_2^2 \mathbb{E}[\|G(s,X_s)\|_F^2]\,ds.$$

Therefore, as $\mathbb{E}[\|G(\cdot,X_\cdot)\|_F^2]$ is integrable, and $R(t) - R_\infty \to 0$ as $t \to \infty$, it follows that

$$\lim_{t\to\infty} \mathbb{E}[\|U(t)\|_2^2] = 0. \tag{5.8}$$

Moreover, as $R - R_\infty \in L^2([0,\infty), M_n(\mathbb{R}))$, we have that

$$\int_0^\infty \mathbb{E}[\|U(t)\|_2^2]\,dt < \infty. \tag{5.9}$$

By Fubini's theorem, we must also have

$$\int_0^\infty \|U(t)\|_2^2\,dt < \infty, \quad \text{a.s.} \tag{5.10}$$

Due to (5.9) and the continuity and non–negativity of $t \mapsto \mathbb{E}[\|U(t)\|_2^2]$, for every $\mu > 0$, there exists an increasing sequence $(a_n)_{n\geq 0}$ with $a_0 = 0$, $a_n \to \infty$ as $n \to \infty$, and $a_{n+1} - a_n < \mu$ such that

$$\sum_{n=0}^\infty \mathbb{E}[\|U(a_n)\|_2^2] < \infty. \tag{5.11}$$

This fact was proven in [1, Lemma 3]. The next part of the proof is modelled after the argument used to prove [1, Theorem 4]. We want to show that $U(t) \to 0$ as $t \to \infty$. Defining $\rho(t) = R(t) - R_\infty$, we get $\rho'(t) = R'(t)$ and

$$\begin{aligned}
U(t) &= \int_0^t \rho(t-s)G(s,X_s)\,dB(s) \\
&= \int_0^t \left(\rho(0) + \int_0^{t-s} R'(u)\,du\right) G(s,X_s)\,dB(s) \\
&= \int_0^t \rho(0)G(s,X_s)\,dB(s) + \int_0^t \int_s^t R'(v-s)G(s,X_s)\,dv\,dB(s) \\
&= \int_0^t \rho(0)G(s,X_s)\,dB(s) + \int_0^t \int_0^v R'(v-s)G(s,X_s)\,dB(s)\,dv.
\end{aligned}$$

Therefore, for $t \in [a_n, a_{n+1}]$

$$\begin{aligned}
U(t) = U(a_n) + \int_{a_n}^t \rho(0)G(s,X_s)\,dB(s) \\
+ \int_{a_n}^t \int_0^v R'(v-s)G(s,X_s)\,dB(s)\,dv.
\end{aligned}$$

Taking norms both sides of this equality, using the triangle inequality, the scalar inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, taking suprema on each side of this inequality, and then taking the expectations of both sides, we get

$$\begin{aligned}
\text{(5.12)}\quad \mathbb{E}\left[\max_{a_n \le t \le a_{n+1}} \|U(t)\|_2^2\right] \le 3\Big(\mathbb{E}[\|U(a_n)\|_2^2] \\
+ \mathbb{E}\left[\max_{a_n \le t \le a_{n+1}} \left\|\int_{a_n}^t \int_0^s R'(s-u)G(u,X_u)\,dB(u)\,ds\right\|_2^2\right] \\
+ \mathbb{E}\left[\max_{a_n \le t \le a_{n+1}} \left\|\int_{a_n}^t \rho(0)G(s,X_s)\,dB(s)\right\|_2^2\right]\Big).
\end{aligned}$$

Let us obtain estimates for the second and third terms on the righthand side of (5.12). For the second term, by applying the Cauchy–Schwartz inequality, and then taking the maximum over $[a_n, a_{n+1}]$, we get

$$\begin{aligned}
\mathbb{E}\left[\max_{a_n \le t \le a_{n+1}} \left\|\int_{a_n}^t \int_0^s R'(s-u)G(u,X_u)\,dB(u)\,ds\right\|_2^2\right] \\
\le (a_{n+1}-a_n)\int_{a_n}^{a_{n+1}} \mathbb{E}\left[\left\|\int_0^s R'(s-u)G(u,X_u)\,dB(u)\right\|_2^2\right]\,ds.
\end{aligned}$$

By Itô's isometry, and the fact that $a_{n+1} - a_n < \mu$, we have

$$\sum_{n=0}^{\infty} \mathbb{E}\left[\max_{a_n \le t \le a_{n+1}} \left\| \int_{a_n}^{t} \int_0^s R'(s-u) G(u, X_u)\, dB(u)\, ds \right\|_2^2 \right] \tag{5.13}$$
$$\le \mu \int_0^\infty \int_0^s \|R'(s-u)\|_2^2 \mathbb{E}[\|G(u,X_u)\|_F^2]\, du\, ds.$$

This quantity on the righthand side is finite, due to the fact that $R' \in L^2([0,\infty), M_n(\mathbb{R}))$ and $\mathbb{E}[\|G(\cdot, X.)\|_2^2]$ is integrable.

We now seek an estimate on the third term on the right-hand side of (5.12). Using Doob's inequality, we obtain

$$\mathbb{E}\left[\max_{a_n \le t \le a_{n+1}} \left\| \int_{a_n}^{t} \rho(0) G(s, X_s)\, dB(s) \right\|_2^2 \right]$$
$$\le C_2 \int_{a_n}^{a_{n+1}} \mathbb{E}\left[\|\rho(0) G(s, X_s)\|_F^2\right]\, ds.$$

Therefore there exists $C_2' > 0$ which is independent of n and the stochastic integrand such that

$$\mathbb{E}\left[\max_{a_n \le t \le a_{n+1}} \left\| \int_{a_n}^{t} \rho(0) G(s, X_s)\, dB(s) \right\|_2^2 \right]$$
$$\le C_2' \|\rho(0)\|_F^2 \int_{a_n}^{a_{n+1}} \mathbb{E}\left[\|G(s, X_s)\|_F^2\right]\, ds,$$

and so

$$\sum_{n=0}^{\infty} \mathbb{E}\left[\max_{a_n \le t \le a_{n+1}} \left\| \int_{a_n}^{t} \rho(0) G(s, X_s)\, dB(s) \right\|_2^2 \right] \tag{5.14}$$
$$\le C_2' \|\rho(0)\|_F^2 \int_0^\infty \mathbb{E}\left[\|G(s, X_s)\|_F^2\right]\, ds.$$

Using the estimates (5.14), (5.13) and (5.11) in (5.12), we see that

$$\sum_{n=0}^{\infty} \mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} \|U(t)\|_2^2\right] < \infty.$$

Note that this also implies $\sum_{n=0}^{\infty} \max_{a_n \le t \le a_{n+1}} \|U(t)\|_2^2 < \infty$, a.s., and therefore $\lim_{n\to\infty} \max_{a_n \le t \le a_{n+1}} \|U(t)\|_2^2 = 0$, a.s. Therefore, as $a_n \to \infty$ as $n \to \infty$, $\|U(t)\|_2^2 \to 0$ as $t \to \infty$, a.s. Therefore, by (5.15), it follows that $X(t) - X_\infty \to 0$ as $t \to \infty$, because $R' \in L^2([0,\infty), M_n(\mathbb{R})$, by Lemma 5.2. □

5.1. **Proof of Theorem 3.1.** By Lemma 5.1 and the martingale convergence theorem, we see that

$$\int_0^\infty G(s, X_s)\, dB(s) := \lim_{t\to\infty} \int_0^t G(s, X_s)\, dB(s) \quad \text{exists and is finite a.s.}$$

Define $X_\infty = R_\infty(X_0 + \int_0^\infty G(s, X_s)\, dB(s))$, and note that the definition of M and U gives

$$X(t) - X_\infty = (R(t) - R_\infty)X_0 + U(t) - R_\infty\,(M(\infty) - M(t))\,. \tag{5.15}$$

By part (ii) of Lemma 5.1 and Lemma 5.3, the third and second terms on the righthand side of (5.15) tend to zero as $t \to \infty$ a.s. The first term also tends to zero, so $X(t) \to X_\infty$ as $t \to \infty$ a.s. As to the convergence in mean–square, observe by (5.8), and part (iv) of Lemma 5.1, that $\lim_{t\to\infty} \mathbb{E}\,[\|X_\infty - X(t)\|_2^2] = 0$.

5.2. **Proof of Theorem 3.2.** Under the conditions of the theorem, we have that $X(t) \to X(\infty)$ as $t \to \infty$ a.s. and $\mathbb{E}[\|X(t) - X(\infty)\|^2] \to 0$ as $t \to \infty$. Now by (3.8), it follows from part (v) of Lemma 5.1 that $\int_0^\infty \mathbb{E}[\|M(\infty) - M(t)\|_2^2]\, dt < \infty$. From Lemma 5.3 we also have $\int_0^\infty \mathbb{E}[\|U(t)\|_2^2]\, dt < \infty$. Finally, $R - R_\infty \in L^2([0,\infty), M_n(\mathbb{R}))$. Therefore from (5.15) it follows that $\int_0^\infty \mathbb{E}[\|X_\infty - X(t)\|_2^2]\, dt < \infty$, and by Fubini's theorem, we have $\int_0^\infty \|X_\infty - X(t)\|_2^2\, dt < \infty$, as required.

6. Proof of Theorem 4.3

Let $\beta > 0$ and define $e_\beta(t) = e^{-\beta t}$ and F by (4.10). Next, we introduce the process Y by

$$Y(t) = e^{-\beta t} \int_0^t e^{\beta s} G(s, X_s)\, dB(s), \quad t \geq 0 \tag{6.1}$$

and the process J by

$$J(t) = Qe^{-\beta t} X_0 - \beta P \int_t^\infty Y(s)\, ds + Y(t) + \int_t^\infty F(s)\, ds \cdot X_\infty, \quad t \geq 0. \tag{6.2}$$

Proposition 6.1. *Let $V = X - X_\infty$. Then V obeys the integral equation*

$$V(t) + (F * V)(t) = J(t), \quad t \geq 0, \tag{6.3}$$

where F and J are as defined in (4.10) *and* (6.2) *above, and the process Y is defined by* (6.1).

Proof. Define $\Phi(t) = P + e^{-\beta t}Q$, for $t \geq 0$. Fix $t \geq 0$, then from (1.1), we obtain for any $T \geq 0$

$$\int_0^T \Phi(t-s)\,dX(s) = \int_0^T \Phi(t-s)AX(s)\,ds$$
$$+ \int_0^T \Phi(t-s)\int_0^s K(s-u)X(u)\,du\,ds + \int_0^T \Phi(t-s)G(s,X_s)\,dB(s).$$

Integration by parts on the integral on the lefthand side gives

$$\Phi(t-T)X(T) - \Phi(t)X_0 = \int_0^T \Phi(t-s)\,dX(s) - \int_0^T \Phi'(t-s)X(s)\,ds.$$

Now, we set $T = t$ and rearrange these identities to obtain

(6.4)
$$\Phi(0)X(t) - \Phi(t)X_0 + \int_0^t \Phi'(t-s)X(s)\,ds = \int_0^t \Phi(t-s)AX(s)\,ds$$
$$+ \int_0^t \Phi(t-s)\int_0^s K(s-u)X(u)\,du\,ds + \int_0^t \Phi(t-s)G(s,X_s)\,dB(s).$$

Since $\Phi(0) = I$, by applying Fubini's theorem to the penultimate term on the righthand side, we arrive at

$$X(t) + (F * X)(t) = H(t), \quad t \geq 0 \tag{6.5}$$

where H is given by

$$H(t) = \Phi(t)X_0 + \int_0^t \Phi(t-s)G(s,X_s)\,dB(s), \tag{6.6}$$

and

$$F(t) = \Phi'(t-s) - \Phi(t)A - \int_0^t \Phi(t-s)K(s)\,ds, \quad t \geq 0.$$

Integrating the convolution term in F by parts yields

$$\int_0^t \Phi(t-s)K(s)\,ds = -\Phi(0)K_1(t) + \Phi(t)K_1(0) - \int_0^t \Phi'(t-s)K_1(s)\,ds,$$

and so F may be rewritten to give

$$F(t) = \Phi'(t) - \Phi(t)A + K_1(t) - \Phi(t)\int_0^\infty K(s)\,ds + (\Phi' * K_1)(t).$$

Therefore F obeys the formula given in (4.10).

If Y is the process defined by (6.1), then Y obeys the stochastic differential equation

$$Y(t) = -\beta\int_0^t Y(s)\,ds + \int_0^t G(s,X_s)\,dB(s) \quad t \geq 0, \tag{6.7}$$

and because $\int_0^\infty \mathbb{E}[\|G(t,X_t)\|_F^2]\,dt < \infty$, we have

$$\int_0^\infty \mathbb{E}[\|Y(t)\|_2^2]\,dt < \infty$$

and hence $\int_0^\infty \|Y(t)\|_2^2\,dt < \infty$. The technique used to prove Lemma 5.3 enables us to prove that $\lim_{t\to\infty} Y(t) = 0$ a.s. Therefore, by re–expressing H according to

$$\begin{aligned} H(t) &= Qe^{-\beta t}X_0 + PX_0 + P\int_0^t G(s,X_s)\,dB(s) \\ &\quad + Q\int_0^t e^{-\beta(t-s)}G(s,X_s)\,dB(s) \\ &= Qe^{-\beta t}X_0 + PX_0 + P\int_0^\infty G(s,X_s)\,dB(s) \\ &\quad - P\int_t^\infty G(s,X_s)\,dB(s) + QY(t), \end{aligned}$$

we see that $H(t) \to H(\infty)$ as $t\to\infty$ a.s., where

$$H(\infty) = PX_0 + P\int_0^\infty G(t,X_t)\,dB(t), \tag{6.8}$$

Therefore

$$H(t) - H(\infty) = Qe^{-\beta t}X_0 - P\int_t^\infty G(s,X_s)\,dB(s) + QY(t). \tag{6.9}$$

Since $X(t)\to X_\infty$ as $t\to\infty$ a.s. and $F\in L^1([0,\infty),M_n(\mathbb{R}))$ it follows from (6.5) that

$$X_\infty + \int_0^\infty F(s)\,ds \cdot X_\infty = H(\infty).$$

Therefore

$$\begin{aligned} X(t) - X_\infty + \int_0^t F(t-s)(X(s)-X_\infty)\,ds - \int_t^\infty F(s)\,dsX_\infty \\ = H(t) - X_\infty - \int_0^t F(s)\,dsX_\infty - \int_t^\infty F(s)\,dsX_\infty, \end{aligned}$$

and so , with $V = X - X_\infty$, we get (6.3) where $J(t) = H(t) - H(\infty) + \int_t^\infty F(s)\,ds\cdot X_\infty$. This implies

$$J(t) = Qe^{-\beta t}X_0 - P\int_t^\infty G(s,X_s)\,dB(s) + QY(t) + \int_t^\infty F(s)\,ds\cdot X_\infty.$$

We now write J entirely in terms of Y. By (6.7), and the fact that $Y(t)\to 0$ as $t\to\infty$ a.s. and $G(\cdot,X_\cdot)\in L^2([0,\infty),M_{n,d}(\mathbb{R})$ a.s., it follows that

$$0 = -\beta \int_0^\infty Y(s)\,ds + \int_0^\infty G(s, X_s)\,dB(s).$$

Combining this with (6.7) gives

$$\int_t^\infty G(s, X_s)\,dB(s) = \beta \int_t^\infty Y(s)\,ds - Y(t),$$

and so J is given by (6.2), as claimed. $\square$

6.1. **Proof of Theorem 4.3.** We start by noticing that $\gamma \in \mathcal{U}(\mu)$ implies that $\gamma(t)e^{(\epsilon-\mu)t} \to \infty$ as $t \to \infty$ for any $\epsilon > 0$. Therefore with $\epsilon = \beta + \mu > 0$ we have

$$\lim_{t\to\infty} \frac{e^{-\beta t}}{\gamma(t)} = \lim_{t\to\infty} \frac{e^{(\epsilon-\mu)t}}{\gamma(t)e^{\beta t}e^{(\epsilon-\mu)t}} = \lim_{t\to\infty} \frac{e^{\beta t}}{\gamma(t)e^{\beta t}e^{(\epsilon-\mu)t}} = 0.$$

Hence $L_\gamma e_\beta = 0$.

Next, define $\Lambda = L_\gamma K_1$. If $\gamma \in \mathcal{U}^1(\mu)$, by L'Hôpital's rule, we have

$$\lim_{t\to\infty} \frac{\int_t^\infty \gamma(s)\,ds}{\gamma(t)} = -\frac{1}{\mu}.$$

Therefore

$$\lim_{t\to\infty} \frac{\int_t^\infty K_1(s)\,ds}{\gamma(t)} = \lim_{t\to\infty} \frac{\int_t^\infty K_1(s)\,ds}{\int_t^\infty \gamma(s)\,ds} \cdot \frac{\int_t^\infty \gamma(s)\,ds}{\gamma(t)} = \frac{-1}{\mu}\Lambda.$$

If $\mu = 0$, suppose $\Lambda \neq 0$. Then $L_\gamma K_2$ is not finite, contradicting (4.12). Therefore, if $\mu = 0$, then $\Lambda = 0$. Hence for all $\mu \le 0$, we have

$$L_\gamma K_1 = -\mu L_\gamma K_2.$$

Let Y be defined by (6.1). Then for each $i = 1, \dots, n$, $Y_i(t) := \langle Y(t), \mathbf{e}_i\rangle$ is given by

$$e^{\beta t} Y_i(t) = \sum_{j=1}^d \int_0^t e^{\beta s} G_{ij}(s, X_s)\,dB_j(s) =: \tilde{Y}_i(t), \quad t \ge 0.$$

Then $\tilde{Y}_i$ is a local martingale with quadratic variation given by

$$\langle \tilde{Y}_i\rangle(t) = \int_0^t e^{2\beta s} \sum_{j=1}^d G_{ij}^2(s, X_s)\,ds \le \int_0^t e^{2\beta s} \|G(s, X_s)\|_F^2\,ds.$$

Hence, as $X(t) \to X_\infty$ as $t \to \infty$ a.s., and X is continuous, it follows from (2.3) and (2.5) that there is an a.s. finite and $\mathcal{F}^B(\infty)$—measurable random variable $C > 0$ such that

$$\langle \tilde{Y}_i\rangle(t) \le C \int_0^t e^{2\beta s} \Sigma^2(s)\,ds, \quad t \ge 0.$$

We now consider two possibilities. The first possibility is that $\langle\tilde{Y}_i\rangle(t)$ tends to a finite limit as $t\to\infty$. In this case $\lim_{t\to\infty} e^{\beta t}Y_i(t)$ exists and is finite. Then $L_\gamma Y_i = 0$ and $L_\gamma[\int_\cdot^\infty Y(s)\,ds]=0$ on the event on which the convergence takes place.

On the other hand, if $\langle\tilde{Y}_i\rangle(t)\to\infty$ as $t\to\infty$, then by the Law of the Iterated Logarithm, for all $t\geq 0$ sufficiently large, there exists an a.s. finite and $\mathcal{F}^B(\infty)$—measurable random variable $C_1>0$ such that

$$e^{2\beta t}Y_i^2(t)=\tilde{Y}_i^2(t)\leq C_1\int_0^t e^{2\beta s}\Sigma^2(s)\,ds\,\log_2\left(e+\int_0^t e^{2\beta s}\Sigma^2(s)\,ds\right).$$

If $\int_0^\infty e^{2\beta t}\Sigma^2(t)\,dt<\infty$, once again $L_\gamma Y_i=0$ and $L_\gamma[\int_\cdot^\infty Y(s)\,ds]=0$. If not, then the fact that $\Sigma\in L^2([0,\infty),\mathbb{R})$ yields the estimate $\int_0^t e^{2\beta s}\Sigma^2(s)\,ds\leq e^{2\beta t}\int_0^\infty \Sigma^2(s)\,ds$, and so

$$\log\int_0^t e^{2\beta s}\Sigma^2(s)\,ds\leq 2\beta t+\log\int_0^\infty \Sigma^2(s)\,ds,$$

and so, when $\int_0^\infty e^{2\beta t}\Sigma^2(t)\,dt=\infty$, we have

$$|Y_i(t)|^2\leq C_2\int_0^t e^{-2\beta(t-s)}\Sigma^2(s)\,ds\;\cdot\log t=C_2\tilde{\Sigma}^2(t).$$

Hence, if the first part of (4.13) holds, we have $L_\gamma Y_i=0$ and the second part implies (4.13). Therefore we have

$$L_\gamma Y=0,\quad L_\gamma[\int_\cdot^\infty Y(s)\,ds]=0,\quad \text{a.s.} \tag{6.10}$$

Next, by (4.10) and (4.12) and the fact that $L_\gamma e_\beta=0$, we have

$$\begin{aligned}L_\gamma F&=-L_\gamma e_\beta(\beta Q+QM)+L_\gamma K_1-\beta Q L_\gamma(e_\beta * K_1)\\ &=\left(I-\frac{\beta}{\beta+\mu}Q\right)L_\gamma K_1.\end{aligned} \tag{6.11}$$

Next, we compute $\int_t^\infty F(s)\,ds$ as a prelude to evaluating $L_\gamma\int_\cdot^\infty F(s)\,ds$. By (4.10)

$$\int_t^\infty F(s)\,ds=-\frac{1}{\beta}e^{-\beta t}(\beta Q+QM)+K_2(t)-\beta Q\int_t^\infty (e_\beta * K_1)(s)\,ds.$$

Now

$$\int_t^\infty (e_\beta * K_1)(s)\,ds=\frac{1}{\beta}(e_\beta * K_1)(t)+\frac{1}{\beta}K_2(t).$$

Therefore, as $L_\gamma e_\beta=0$, and (4.12) holds we have

$$\begin{aligned}L_\gamma\int_\cdot^\infty F(s)\,ds&=L_\gamma K_2-\beta Q L_\gamma\left(\frac{1}{\beta}(e_\beta * K_1)+\frac{1}{\beta}K_2\right)\\ &=PL_\gamma K_2-QL_\gamma[e_\beta * K_1].\end{aligned}$$

Hence

$$L_\gamma \int_\cdot^\infty F(s)\,ds = PL_\gamma K_2 - Q\frac{1}{\beta+\mu}L_\gamma K_1,$$

and so

$$L_\gamma J = L_\gamma \int_\cdot^\infty F(s)\,ds \cdot X_\infty = \left(P + Q\frac{\mu}{\beta+\mu}\right) L_\gamma K_2 \cdot X_\infty. \tag{6.12}$$

Hence, as $\gamma \in \mathcal{U}(\mu)$, (6.12) and (6.11) and (4.11) all hold, it follows that from (6.3) that $L_\gamma V = L_\gamma(X - X_\infty)$ exists and is a.s. finite, and is given by

$$L_\gamma V + [L_\gamma F]\hat{V}(\mu) + \hat{F}(\mu)L_\gamma V = L_\gamma J.$$

Hence $(I+\hat{F}(\mu))L_\gamma V = L_\gamma J - [L_\gamma F]\hat{V}(\mu)$. Since (4.11) holds, it follows that $I + \hat{F}(\mu)$ is invertible, and so

$$L_\gamma(X - X_\infty) = (I+\hat{F}(\mu))^{-1}\left(L_\gamma J - [L_\gamma F]\int_0^\infty (X(s) - X_\infty)e^{-\mu s}\,ds\right). \tag{6.13}$$

This establishes the existence of the finite a.s. limit $L_\gamma(X - X_\infty)$.

We now determine formulae for $L_\gamma(X - X_\infty)$ in the cases where $\mu = 0$ and $\mu < 0$. From the formula for F, we can readily compute

$$I + \hat{F}(\mu) = \left(I - \frac{\beta}{\beta+\mu}Q\right)(I + \hat{K}_1(\mu)) - \frac{1}{\beta+\mu}M. \tag{6.14}$$

When $\mu = 0$, we have that $L_\gamma K_1 = 0$, so $L_\gamma F = 0$ and $L_\gamma J$ simplifies to give $L_\gamma J = P \cdot L_\gamma K_2 \cdot X_\infty$. Hence

$$L_\gamma(X - X_\infty) = (I + \hat{F}(0))^{-1} \cdot P \cdot L_\gamma K_2 \cdot X_\infty.$$

Now $I + \hat{F}(0) = P(I + \int_0^\infty K_1(s)\,ds) - \beta^{-1}M =: C_0$. The formula for R_∞ implies that

$$\left(P - M + P\int_0^\infty K_1(s)\,ds\right)R_\infty = P,$$

and as $MR_\infty = 0$, we have

$$P\left(I + \int_0^\infty K_1(s)\,ds\right)R_\infty = P. \tag{6.15}$$

Thus

$$\begin{aligned} L_\gamma(X - X_\infty) &= C_0^{-1} \cdot P\left(I + \int_0^\infty K_1(s)\,ds\right)R_\infty \cdot L_\gamma K_2 \cdot X_\infty \\ &= C_0^{-1} \cdot [C_0 + \beta^{-1}M]R_\infty \cdot L_\gamma K_2 \cdot X_\infty \\ &= R_\infty(L_\gamma K_2)X_\infty, \end{aligned}$$

where we have used the fact that $MR_\infty = 0$ at the last step. This proves part (i).

As for part (ii), from (6.13), we have

$$\begin{aligned}&L_\gamma(X-X_\infty)\\&\quad=(I+\hat F(\mu))^{-1}\left(I-\frac{\beta}{\mu+\beta}Q\right)L_\gamma K_2\left\{X_\infty+\mu(\widehat{X-X_\infty})(\mu)\right\}.\end{aligned}$$

Since $QM=M$, by (6.14), we get

$$\left(I-\frac{\beta}{\beta+\mu}Q\right)\left(I+\hat K_1(\mu)-\frac{1}{\mu}M\right)=I+\hat F(\mu).$$

Due to (4.11), $I+\hat F(\mu)$ is invertible, and so

$$(I+\hat F(\mu))^{-1}\left(I-\frac{\beta}{\beta+\mu}Q\right)\left(I+\hat K_1(\mu)-\frac{1}{\mu}M\right)=I. \tag{6.16}$$

Therefore

$$\begin{aligned}&L_\gamma(X-X_\infty)\\&\quad=\left(I+\hat K_1(\mu)-\frac{1}{\mu}M\right)^{-1}L_\gamma K_2\left\{X_\infty+\mu(\widehat{X-X_\infty})(\mu)\right\}.\end{aligned}$$

It remains to prove that

$$\begin{aligned}&X_\infty+\int_0^\infty(X(s)-X_\infty)\mu e^{-\mu s}\,ds\\&\quad=\left(I+\hat K_1(\mu)-\frac{1}{\mu}M\right)^{-1}\left(X_0+\int_0^\infty e^{-\mu s}G(s,X_s)\,dB(s)\right).\end{aligned}$$

First, we note from (6.3) that $(\widehat{X-X_\infty})(\mu)=(I+\hat F(\mu))^{-1}\hat J(\mu)$. Using (6.2), we obtain

$$\begin{aligned}\mu\hat J(\mu)=\frac{\mu}{\beta+\mu}QX_0+(\mu I+\beta P)\hat Y(\mu)-\beta P\int_0^\infty Y(s)\,ds\\+\left(\int_0^\infty F(s)\,ds-\hat F(\mu)\right)X_\infty.\end{aligned} \tag{6.17}$$

By (6.14), and as $MX_\infty=0$, the last term equals

$$\begin{aligned}&\left(\int_0^\infty F(s)\,ds-\hat F(\mu)\right)X_\infty\\&\quad=\left\{(I-Q)(I+\hat K_1(0))-\left(I-\frac{\beta}{\beta+\mu}Q\right)(I+\hat K_1(\mu))\right\}X_\infty.\end{aligned} \tag{6.18}$$

Using integration by parts on (6.7) yields

$$\int_0^t e^{-\mu s}Y(s)\,ds=\frac{1}{\beta+\mu}\int_0^t e^{-\mu s}G(s,X_s)\,dB(s)-\frac{1}{\beta+\mu}e^{-\mu t}Y(t).$$

Since $\gamma \in \mathcal{U}(\mu)$, we have $\gamma(t)e^{-\mu t} \to 0$ as $t \to \infty$. Therefore, as $L_\gamma Y = 0$,

$$e^{-\mu t}Y(t) = \frac{Y(t)}{\gamma(t)} \cdot \gamma(t)e^{-\mu t} \to 0, \quad \text{as } t \to \infty.$$

Moreover $L_\gamma Y = 0$ and $\gamma \in \mathcal{U}(\mu)$ implies that $\hat{Y}(\mu)$ is finite. Therefore we have

$$\hat{Y}(\mu) = \frac{1}{\beta+\mu}\int_0^\infty e^{-\mu s}G(s, X_s)\, dB(s).$$

Since $\beta\int_0^\infty Y(s)\, ds = \int_0^\infty G(s, X_s)\, dB(s)$, we have

$$\text{(6.19)} \quad (\mu I + \beta P)\,\hat{Y}(\mu) - \beta P\int_0^\infty Y(s)\, ds$$
$$= (\mu I + \beta P)\frac{1}{\beta+\mu}\int_0^\infty e^{-\mu s}G(s, X_s)\, dB(s) - P\int_0^\infty G(s, X_s)\, dB(s).$$

Collecting (6.17), (6.18) and (6.19), and using the fact that $P = I - Q$, we obtain

$$\begin{aligned} X_\infty + \int_0^\infty (X(s) - X_\infty)\mu e^{-\mu s}\, ds &= \frac{\mu}{\beta+\mu}(I+\hat{F}(\mu))^{-1}QX_0 \\ &+ (I+\hat{F}(\mu))^{-1}\Big\{\Big(I - \frac{\beta}{\beta+\mu}Q\Big)\int_0^\infty e^{-\mu s}G(s, X_s)\, dB(s) \\ &\qquad - P\int_0^\infty G(s, X_s)\, dB(s)\Big\} \\ &+ (I+\hat{F}(\mu))^{-1}\Big\{(I-Q)(I+\hat{K}_1(0)) \\ &\qquad - \Big(I - \frac{\beta}{\beta+\mu}Q\Big)(I+\hat{K}_1(\mu))\Big\}X_\infty + X_\infty. \end{aligned}$$

Using (6.14), we get

$$I + \hat{F}(\mu) + \frac{1}{\beta+\mu}M = \Big(I - \frac{\beta}{\beta+\mu}Q\Big)(I+\hat{K}_1(\mu)).$$

and since $MX_\infty = 0$, we may use the last identity, and the formula for X_∞ to get

$$\begin{aligned}
&(I+\hat{F}(\mu))^{-1}\Big\{(I-Q)(I+\hat{K}_1(0)) \\
&\qquad -\left(I-\frac{\beta}{\beta+\mu}Q\right)(I+\hat{K}_1(\mu))\Big\}X_\infty + X_\infty \\
&= (I+\hat{F}(\mu))^{-1}(I-Q)(I+\hat{K}_1(0))X_\infty \\
&= (I+\hat{F}(\mu))^{-1}P(I+\hat{K}_1(0))R_\infty\left(X_0+\int_0^\infty G(s,X_s)\,dB(s)\right) \\
&= (I+\hat{F}(\mu))^{-1}P\left(X_0+\int_0^\infty G(s,X_s)\,dB(s)\right)
\end{aligned}$$

where we have used (6.15) at the last step. Combining this identity with the last expression for $X_\infty + \int_0^\infty (X(s)-X_\infty)\mu e^{-\mu s}\,ds$, we get

$$\begin{aligned}
X_\infty + \int_0^\infty (X(s)-X_\infty)\mu e^{-\mu s}\,ds &= \frac{\mu}{\beta+\mu}(I+\hat{F}(\mu))^{-1}QX_0 \\
&\quad + (I+\hat{F}(\mu))^{-1}\left(I-\frac{\beta}{\beta+\mu}Q\right)\int_0^\infty e^{-\mu s}G(s,X_s)\,dB(s) \\
&\qquad + (I+\hat{F}(\mu))^{-1}PX_0.
\end{aligned}$$

The first and third terms on the righthand side combine to give

$$(I+\hat{F}(\mu))^{-1}\left(I-\frac{\beta}{\beta+\mu}Q\right)X_0,$$

so by (6.16) we have

$$\begin{aligned}
&X_\infty + \int_0^\infty (X(s)-X_\infty)\mu e^{-\mu s}\,ds = \\
&\qquad \left(I+\hat{K}_1(\mu)-\frac{1}{\mu}M\right)^{-1}\left(X_0+\int_0^\infty e^{-\mu s}G(s,X_s)\,dB(s)\right),
\end{aligned}$$

as required to prove part (ii).

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