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Uniqueness of Bounded Solutions to a Viscous Diffusion Equation

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Abstract

In this paper we prove the uniqueness of bounded solutions to a viscous diffusion equation based on approximate Holmgren's approach.

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1 Introduction

We consider the uniqueness of bounded solutions to the following viscous diffusion equation in one dimension of the form

$$\frac{\partial u}{\partial t} - \lambda \frac{\partial}{\partial t}\Big(\frac{\partial^2 u}{\partial x^2}\Big) = \frac{\partial^2 A(u)}{\partial x^2} + \frac{\partial B(u)}{\partial x} + f, \quad (x,t) \in Q_T, \tag{1.1}$$

with the initial and boundary condition

$$u(0,t) = u(1,t) = 0, \qquad t \in [0,T], \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in [0,1], \tag{1.3}$$

where $\lambda > 0$ is the viscosity coefficient, $Q_T = (0,1) \times (0,T)$, $A(s)$, $B(s) \in C^1(\mathbb{R})$, $A'(s) > -\mu$, $0 \le \mu < \dfrac{\lambda}{2T}$ is a constant, and f is a function only of x and t.

If $\lambda = 0$, then the equation (1.1) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 A(u)}{\partial x^2} + \frac{\partial B(u)}{\partial x} + f. \tag{1.4}$$

In the case that $A'(s) \ge 0$, the equation (1.4) is the one dimensional form of the well-known nonlinear diffusion equation, which is degenerate at the points where $A'(u) = 0$ and has been studied extensively. In particular, the discussion of the uniqueness of solutions can be found in many papers, see for example [1], [3]–[7]. While if $A'(s)$

is permitted to change sign, (1.4) is called the forward–backward nonlinear diffusion equation.

For the case of $\lambda > 0$, Cohen and Pego [10] considered the equation (1.1) with $B(s) = 0$ and $f = 0$, namely

$$\frac{\partial u}{\partial t} - \lambda \frac{\partial \Delta u}{\partial t} = \Delta A(u), \tag{1.5}$$

where $A(s)$ has no monotonicity. Their interests center on the steady state solution for the equation (1.5), and the uniqueness of the solution of the Neumann initial-boundary value problem and the Dirichlet initial-boundary value problem of the linear case of the equation (1.5),

$$\frac{\partial u}{\partial t} - \lambda \frac{\partial \Delta u}{\partial t} = \alpha \Delta u \tag{1.6}$$

have been discussed by Chen, Gurtin [11] and Ting, Showalter [12].

In this paper, we establish the uniqueness of the solutions to the initial-boundary problem of the equation (1.1) by using an approximate Holmgren's approach. It is worth recalling the work of [1] concerning related parabolic problems (1.4). Due to the degeneracy, the problem (1.1)–(1.3) admits only weak solutions in general. So our result is concerned with the generalized solutions to the problem (1.1)–(1.3).

Definition 1.1 *A function $u(x,t) \in L^\infty(Q_T)$ is called a generalized solution of the boundary value problem (1.1)–(1.3) if for any test function $\varphi \in C^\infty(\overline{Q}_T)$ with $\varphi(0,t) = \varphi(1,t) = \varphi(x,T) = 0$, the following integral equality holds*

$$\iint_{Q_T} u\Big[\frac{\partial \varphi}{\partial t} - \lambda \frac{\partial}{\partial t}\Big(\frac{\partial^2 \varphi}{\partial x^2}\Big)\Big] dxdt + \iint_{Q_T} \Big(A(u)\frac{\partial^2 \varphi}{\partial x^2} - B(u)\frac{\partial \varphi}{\partial x} + f\varphi\Big) dxdt$$
$$+ \int_0^1 u_0(x)\Big(\varphi(x,0) - \lambda \frac{\partial^2 \varphi(x,0)}{\partial x^2}\Big) dx = 0.$$

Our main result is the following theorem.

Theorem 1.1 *Assume that $u_0(x) \in L^\infty(0,1)$, $A(s), B(s) \in C^1(\mathbb{R})$ with $A'(s) > -\mu$, $0 \le \mu < \dfrac{\lambda}{2T}$ is a constant, then the initial-boundary value problem (1.1)–(1.3) has at most one generalized solution in the sense of Definition 1.1.*

2 Preliminaries

Let u_1, $u_2 \in L^\infty(Q_T)$ be solutions of the boundary value problem (1.1)–(1.3). By the definition of generalized solutions, we have

$$\iint_{Q_T} (u_1 - u_2)\Big(\frac{\partial \varphi}{\partial t} - \lambda \frac{\partial}{\partial t}(\frac{\partial^2 \varphi}{\partial x^2}) + \tilde{A}\frac{\partial^2 \varphi}{\partial x^2} - \mu \frac{\partial^2 \varphi}{\partial x^2} - \tilde{B}\frac{\partial \varphi}{\partial x}\Big) dxdt = 0,$$

where

$$\tilde{A} = \tilde{A}(u_1, u_2) = \int_0^1 A'(\theta u_1 + (1-\theta)u_2)d\theta + \mu,$$
$$\tilde{B} = \tilde{B}(u_1, u_2) = \int_0^1 B'(\theta u_1 + (1-\theta)u_2)d\theta.$$

For small $\eta > 0$, let

$$\lambda_\eta = (\eta + \tilde{A})^{-1/2}\tilde{B} \quad \text{on } Q_T.$$

Since $A(s) \in C^1(\mathbb{R})$, $A'(s) > -\mu$ and u_1, $u_2 \in L^\infty(Q_T)$, there must be constants $L > 0, K > 0$, such that

$$\tilde{A} = \frac{A(u_1) - A(u_2)}{u_1 - u_2} + \mu \geq L,$$
$$|\lambda_\eta| \leq K.$$

Let $\tilde{A}_\varepsilon$ and $\lambda_{\eta,\varepsilon}$ be a C^∞ approximation of $\tilde{A}$ and λ_η respectively, such that

$$\lim_{\varepsilon\to 0} \tilde{A}_\varepsilon = \tilde{A}, \qquad \text{a.e.} \quad \text{in } Q_T,$$
$$\lim_{\varepsilon\to 0} \lambda_{\eta,\varepsilon} = \lambda_\eta, \qquad \text{a.e.} \quad \text{in } Q_T,$$
$$\tilde{A}_\varepsilon \leq C,$$
$$|\lambda_{\eta,\varepsilon}| \leq K.$$

Denote

$$\tilde{B}_{\eta,\varepsilon} = \lambda_{\eta,\varepsilon}(\eta + \tilde{A}_\varepsilon)^{1/2}.$$

For given $g \in C_0^\infty(\overline{Q_T})$, consider the approximate adjoint problem

$$\frac{\partial\varphi}{\partial t} - \lambda\frac{\partial}{\partial t}(\frac{\partial^2\varphi}{\partial x^2}) + (\eta + \tilde{A}_\varepsilon)\frac{\partial^2\varphi}{\partial x^2} - \mu\frac{\partial^2\varphi}{\partial x^2} - \tilde{B}_{\eta,\varepsilon}\frac{\partial\varphi}{\partial x} = g, \tag{2.1}$$
$$\varphi(0,t) = \varphi(1,t) = 0, \tag{2.2}$$
$$\varphi(x,T) = 0. \tag{2.3}$$

It is easily to see that the solution to the problem (2.1)–(2.3) is in C^∞ from the smooth of g in (2.1).

Lemma 2.1 *The solution φ of the problem (2.1)–(2.3) satisfies*

$$\iint_{Q_T} \Big(\frac{\partial\varphi}{\partial x}\Big)^2 dxdt \leq C\eta^{-1}. \tag{2.4}$$

Here and in the sequel, we use C to denote a universal constant, indenpent of η and ε, which may take different value on different occasions.

Proof. Denote

$$\Phi(t) = \int_0^1 (\frac{\partial^2\varphi}{\partial x^2})^2 dx,$$

and assume

$$\Phi(t_0) = \max_{(0,T)} \Phi(t).$$

First we show

$$\iint_{Q_T^{t_0}} (\eta + \tilde{A}_\varepsilon)\Big(\frac{\partial^2 \varphi}{\partial x^2}\Big)^2 dxdt \le C\eta^{-1}, \tag{2.5}$$

where $Q_T^{t_0} = (0,1) \times (t_0, T)$. Multiply (2.1) by $\dfrac{\partial^2 \varphi}{\partial x^2}$, integrate it over $Q_T^{t_0}$ by parts and use (2.2), (2.3), we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left(\frac{\partial \varphi(x,t_0)}{\partial x}\right)^2 dx + \frac{1}{2}\lambda\Phi(t_0) - \mu\int_{t_0}^T \Phi(t)dt \\
&\quad + \iint_{Q_T^{t_0}} \left(\eta + \tilde{A}_\varepsilon\right)\left(\frac{\partial^2 \varphi}{\partial x^2}\right)^2 dxdt \\
&\quad - \iint_{Q_T^{t_0}} \tilde{B}_{\eta,\varepsilon}\frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x^2} dxdt = \iint_{Q_T^{t_0}} g\frac{\partial^2 \varphi}{\partial x^2} dxdt.
\end{aligned} \tag{2.6}$$

Noticing $\mu < \lambda/(2T)$ and $|\lambda_{\eta,\varepsilon}| \le K$, then the Young's inequality yields

$$\begin{aligned}
&\iint_{Q_T^{t_0}} (\eta + \tilde{A}_\varepsilon)\left(\frac{\partial^2 \varphi}{\partial x^2}\right)^2 dxdt \\
\le& \iint_{Q_T^{t_0}} \lambda_{\eta,\varepsilon}(\eta + \tilde{A}_\varepsilon)^{1/2}\frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x^2} dxdt + \iint_{Q_T^{t_0}} g\frac{\partial^2 \varphi}{\partial x^2} dxdt \\
\le& \frac{1}{4}\iint_{Q_T^{t_0}} (\eta + \tilde{A}_\varepsilon)\left(\frac{\partial^2 \varphi}{\partial x^2}\right)^2 dxdt + C\iint_{Q_T^{t_0}} (\frac{\partial \varphi}{\partial x})^2 dxdt + C\eta^{-1}.
\end{aligned} \tag{2.7}$$

Using (2.2) and the Young's inequality again, it gives

$$\begin{aligned}
&\iint_{Q_T^{t_0}} (\frac{\partial \varphi}{\partial x})^2 dxdt = -\iint_{Q_T^{t_0}} \varphi\frac{\partial^2 \varphi}{\partial x^2} dxdt \\
\le& \alpha\iint_{Q_T^{t_0}} (\eta + \tilde{A}_\varepsilon)\left(\frac{\partial^2 \varphi}{\partial x^2}\right)^2 dxdt + C\alpha^{-1}\eta^{-1}
\end{aligned} \tag{2.8}$$

for any $\alpha > 0$. Substituting this into (2.7) and choosing $\alpha > 0$ small enough, we obtain (2.5).

In (2.6), the first term and the forth term are nonnegative, by using (2.5) and (2.8), it follows

$$
\begin{aligned}
&\frac{1}{2}\lambda\Phi(t_0)-\mu\int_{t_0}^{T}\Phi(t)dt\\
\leq&\iint_{Q_T^{t_0}}\lambda_{\eta,\varepsilon}(\eta+\tilde{A}_\varepsilon)^{1/2}\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}dxdt+\iint_{Q_T^{t_0}}g\frac{\partial^2\varphi}{\partial x^2}dxdt\\
\leq&\frac{1}{4}\iint_{Q_T^{t_0}}(\eta+\tilde{A}_\varepsilon)\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2dxdt+C\iint_{Q_T^{t_0}}(\frac{\partial\varphi}{\partial x})^2dxdt+C\eta^{-1}\\
\leq&C\eta^{-1}.
\end{aligned}
$$

Furthermore,

$$\Phi(t_0)\leq C\eta^{-1}. \tag{2.9}$$

we multiply (2.1) by $\frac{\partial^2\varphi}{\partial x^2}$ again, integrate it over Q_T by parts and use (2.2), (2.3), then

$$
\begin{aligned}
&\frac{1}{2}\int_0^1\left(\frac{\partial\varphi(x,0)}{\partial x}\right)^2dx+\frac{\lambda}{2}\int_0^1\big(\frac{\partial^2\varphi(x,0)}{\partial x^2}\big)^2dx-\mu\int_0^T\Phi(t)dt\\
&\quad+\iint_{Q_T}\left(\eta+\tilde{A}_\varepsilon\right)\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2dxdt-\iint_{Q_T}\tilde{B}_{\eta,\varepsilon}\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}dxdt\\
=&\iint_{Q_T}g\frac{\partial^2\varphi}{\partial x^2}dxdt.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&\iint_{Q_T}(\eta+\tilde{A}_\varepsilon)\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2dxdt\\
\leq&\frac{1}{4}\iint_{Q_T}(\eta+\tilde{A}_\varepsilon)\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2dxdt+C\iint_{Q_T}(\frac{\partial\varphi}{\partial x})^2dxdt\\
&\quad+\mu T\Phi(t_0)+C\eta^{-1}.
\end{aligned}
$$

Noticing the fact that

$$
\begin{aligned}
&\iint_{Q_T}(\frac{\partial\varphi}{\partial x})^2dxdt=-\iint_{Q_T}\varphi\frac{\partial^2\varphi}{\partial x^2}dxdt\\
\leq&\alpha\iint_{Q_T}(\eta+\tilde{A}_\varepsilon)\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2dxdt+C\alpha^{-1}\eta^{-1}
\end{aligned}
\tag{2.10}
$$

and (2.9), we get

$$\iint_{Q_T}(\eta+\tilde{A}_\varepsilon)\big(\frac{\partial^2\varphi}{\partial x^2}\big)^2dxdt\leq C\eta^{-1}, \tag{2.11}$$

the above inequality and (2.10) yields (2.4). The proof is completed.

3 Proof of Theorem 1.1

Given $g \in C_0^\infty(Q_T)$. Let φ be a solution of (2.1)–(2.3). Then

$$\begin{aligned}
&\iint_{Q_T} (u_1 - u_2) g dx dt \\
=&\iint_{Q_T} (u_1 - u_2)\Big(\frac{\partial \varphi}{\partial t} - \lambda \frac{\partial}{\partial t}(\frac{\partial^2 \varphi}{\partial x^2}) \\
&\quad + (\eta + \tilde{A}_\varepsilon)\frac{\partial^2 \varphi}{\partial x^2} - \mu \frac{\partial^2 \varphi}{\partial x^2} - \tilde{B}_{\eta,\varepsilon}\frac{\partial \varphi}{\partial x}\Big) dx dt.
\end{aligned}$$

As indicated above, from the definition of generalized solutions, we have

$$\iint_{Q_T} (u_1 - u_2)\Big(\frac{\partial \varphi}{\partial t} - \lambda \frac{\partial}{\partial t}(\frac{\partial^2 \varphi}{\partial x^2}) + \tilde{A}\frac{\partial^2 \varphi}{\partial x^2} - \mu \frac{\partial^2 \varphi}{\partial x^2} - \tilde{B}\frac{\partial \varphi}{\partial x}\Big) dx dt = 0.$$

Thus

$$\begin{aligned}
&\iint_{Q_T} (u_1 - u_2) g dx dt \\
=&\iint_{Q_T} \eta(u_1 - u_2)\frac{\partial^2 \varphi}{\partial x^2} dx dt + \iint_{Q_T} (u_1 - u_2)(\tilde{A}_\varepsilon - \tilde{A})\frac{\partial^2 \varphi}{\partial x^2} dx dt \\
&- \iint_{Q_T} (u_1 - u_2)(\tilde{B}_{\eta,\varepsilon} - \tilde{B})\frac{\partial \varphi}{\partial x} dx dt. \qquad (3.1)
\end{aligned}$$

Now we are ready to estimate all terms on the right side of (3.1).

First, from Lemma 2.1

$$\begin{aligned}
&\Big|\iint_{Q_T} (u_1 - u_2)(\tilde{A}_\varepsilon - \tilde{A})\frac{\partial^2 \varphi}{\partial x^2} dx dt\Big| \\
\le& C\left(\iint_{Q_T} (\tilde{A}_\varepsilon - \tilde{A})^2 dx dt\right)^{1/2} \left(\iint_{Q_T} (\frac{\partial^2 \varphi}{\partial x^2})^2 dx dt\right)^{1/2} \\
\le& C\eta^{-1}\left(\iint_{Q_T} (\tilde{A}_\varepsilon - \tilde{A})^2 dx dt\right)^{1/2}.
\end{aligned}$$

Hence

$$\lim_{\varepsilon \to 0} \iint_{Q_T} (u_1 - u_2)(\tilde{A}_\varepsilon - \tilde{A})\frac{\partial^2 \varphi}{\partial x^2} dx dt = 0. \qquad (3.2)$$

We also have

$$\begin{aligned}
&\Big|\iint_{Q_T} (u_1 - u_2)(\tilde{B}_{\eta,\varepsilon} - \tilde{B})\frac{\partial \varphi}{\partial x} dx dt\Big| \\
\le& C\Big(\iint_{Q_T} (\tilde{B}_{\eta,\varepsilon} - \tilde{B})^2 dx dt\Big)^{1/2}\Big(\iint_{Q_T} (\frac{\partial \varphi}{\partial x})^2 dx dt\Big)^{1/2} \\
\le& C\eta^{-1/2}\Big(\iint_{Q_T} (\tilde{B}_{\eta,\varepsilon} - \tilde{B})^2 dx dt\Big)^{1/2}.
\end{aligned}$$

Since $\lim_{\varepsilon\to 0} \lambda_{\eta,\varepsilon} = \lambda_\eta = (\eta + \tilde{A})^{-1/2}\tilde{B}$ a.e. in Q_T. Thus

$$\lim_{\varepsilon\to 0} \iint_{Q_T} (u_1 - u_2)(\tilde{B}_{\eta,\varepsilon} - \tilde{B})\frac{\partial\varphi}{\partial x} dxdt = 0. \tag{3.3}$$

Using Lemma 2.1 again, we have

$$\begin{aligned} &\Big|\iint_{Q_T} (u_1 - u_2)\frac{\partial^2\varphi}{\partial x^2} dxdt\Big| \\ &\le C\Big(\iint_{Q_T} (\frac{\partial^2\varphi}{\partial x^2})^2 dxdt\Big)^{1/2} \\ &\le C\eta^{-1/2}. \end{aligned}$$

So

$$\Big|\iint_{Q_T} \eta(u_1 - u_2)\frac{\partial^2\varphi}{\partial x^2} dxdt\Big| \le C\eta^{1/2}. \tag{3.4}$$

Combining (3.1)–(3.4) we finally obtain

$$\Big|\iint_{Q_T} (u_1 - u_2) g dxdt\Big| \le C\eta^{1/2},$$

which implies that

$$\iint_{Q_T} (u_1 - u_2) g dxdt = 0$$

by letting $\eta \to 0$. So the uniqueness of solutions to the problem (1.1)–(1.3) follows from the arbitrariness of g. The proof is completed.

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