

Walking within growing domains: recurrence versus transience*

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Abstract

For normally reflected Brownian motion and for simple random walk on independently growing in time d -dimensional domains, $d \geq 3$, we establish a sharp criterion for recurrence versus transience in terms of the growth rate.

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1 Introduction.

There has been much interest in studies of random walks in random environment (see [13]). Of particular challenge are problems in which the walker affects its environment, as in reinforced random walks. In this context even the most fundamental question of recurrence versus transience is often open. For example, in case of linearly edge-reinforced random walks (LRRW), the existence of phase transition between a.s. recurrence for large enough reinforcement strength and a.s. transience for small enough strength has just been recently shown (see [2, 24] for the recurrence under large reinforcement of LRRW on graphs of bounded degrees, [2] for its transience under small reinforcement for non-amenable graphs, and [8] for such transience in case of the LRRW on \mathbb{Z}^d , $d \geq 3$). The question of M. Keane whether once edge-reinforced random walk (ORRW) on \mathbb{Z}^2 is recurrent, remains widely open (for any reinforcement strength), as does the conjecture by the last author that the recurrence of ORRW on \mathbb{Z}^d , $d \geq 3$, exhibits a phase transition with respect to its reinforcement strength. In contrast, the motion of walker excited towards the origin on the boundary of its range is recurrent in any dimension regardless of the strength of the excitation, see [17, Section 2] and [18], while as shown in [3],

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excitation by means of a drift in \vec{e}_1 direction results in transience for any strength of the drift in any dimension $d \geq 2$ (Cf. [16] for results in one dimension, related excitation models, and open problems).

The case where the walk does not affect the time evolution of its environment is better understood. For example, time homogeneous, translation invariant Markovian evolution of the environment is considered in [7] and the references therein. The quenched CLT for the walk is proved there for stationary initial conditions subject to suitable locality, ellipticity, spatial and temporal mixing of the environment. Our focus is on the recurrence/transience properties of certain time-varying, highly non-reversible evolution. Specifically, we consider the discrete-time simple random walk (SRW) $\{Y_t\}$ on non-decreasing connected graphs \mathbb{G}_t of common vertex set. Namely, having $Y_t = y$, one chooses Y_{t+1} uniformly among all neighbors of y within \mathbb{G}_{t+1} . In this article we propose three natural general conjectures about the recurrence/transience of such processes and prove partial results in this direction, for such SRW on subgraphs of \mathbb{Z}^d , $d \geq 3$, which satisfy the following bounded-shape condition.

Assumption 1.1. *The connected, non-decreasing $t \mapsto \mathbb{D}_t \subseteq \mathbb{Z}^d$, $d \geq 3$ are such that $f(t)\mathbb{B}_1 \cap \mathbb{Z}^d \subseteq \mathbb{D}_t \subseteq f(t)\mathbb{B}_c \cap \mathbb{Z}^d$, for some finite c and non-decreasing, unbounded, strictly positive $f(t)$, $t \geq 0$ (and $\mathbb{B}_c \subset \mathbb{R}^d$ denotes an Euclidean ball of radius c , centered at the origin $0 \in \mathbb{Z}^d$).*

In this context, we propose the following *universality conjecture* (namely, that only the asymptotic growth rate of $t \mapsto f(t)$ matters for transience/recurrence of such SRW).

Conjecture 1.2. *Almost surely, the SRW $\{Y_t\}$ on $\{\mathbb{D}_t\}$ satisfying Assumption 1.1 and starting at $Y_0 = 0$, returns to the origin finitely often iff*

$$J_f := \int_0^\infty \frac{dt}{f(t)^d} < \infty. \tag{1.1}$$

Indeed, we show in Theorem 1.4 that under Assumption 1.1, having $J_f < \infty$ implies that $\mathbb{P}(A) = 0$ for $A := \{\sum_t \mathbb{1}_{\{y\}}(Y_t) = \infty\}$ and any $y \in \mathbb{Z}^d$. For the more challenging part, namely

$$J_f = \infty \Rightarrow \mathbb{P}(A) = 1, \tag{1.2}$$

we resort to connecting the SRW $\{Y_t\}$ with a normally reflected Brownian motion (in short RBM), via an invariance principle (see Lemma 3.2). Thus, our approach yields sample-path recurrence results for reflected Brownian motion on growing domains in \mathbb{R}^d (in short RBMG, see Definition 1.13 and Theorem 1.15), which are of independent interest. This strategy comes however at a cost of imposing certain additional restrictions on $t \mapsto \mathbb{D}_t$. Specifically, when proving in part (b) of Theorem 1.4 the recurrence of the SRW on growing domains \mathbb{D}_t in \mathbb{Z}^d , $d \geq 3$, we further assume that $\mathbb{D}_t = f(t)\mathbb{K} \cap \mathbb{Z}^d$ for some \mathbb{K} regular enough, to which end we recall the following definition.

Definition 1.3. *An open connected $\mathbb{K} \subseteq \mathbb{R}^d$ is called a uniform domain if there exists a constant $C < \infty$ such that for every $x, y \in \mathbb{K}$ there exists a rectifiable curve $\gamma \subseteq \mathbb{K}$ joining x and y , with $length(\gamma) \leq C|x - y|$ and $\min\{|x - z|, |z - y|\} \leq C \text{dist}(z, \partial\mathbb{K})$ for all $z \in \gamma$.*

Dealing with a *discrete time* SRW, we may consider without loss of generality only $t \mapsto f(t)$ piecewise constant, that is, from the collection

$$\mathcal{F} := \{f(\cdot) : f(t) = \sum_{l=1}^\infty a_l \mathbb{1}_{[t_l, t_{l+1})}(t), \text{ for } t_1 = 0, \{t_l\} \uparrow \infty, 0 < a_l \uparrow \infty\}. \tag{1.3}$$

However, as seen in our main result below, for our proof of (1.2) we further require the

following separation of scales

$$\mathcal{F}_* := \{f \in \mathcal{F} : (a_l - a_{l-1}) \uparrow \infty, \sum_{l=1}^{\infty} a_l^{2-d} \log(1 + a_l) < \infty\}. \quad (1.4)$$

Theorem 1.4. Consider a SRW $\{Y_t\}$ on $\{\mathbb{D}_t\}$ satisfying Assumption 1.1, with $Y_0 = 0$.

(a). Whenever $J_f < \infty$, the SRW $\{Y_t\}$ a.s. visits every $y \in \mathbb{Z}^d$ finitely often.

(b). Such SRW $\{Y_t\}$ a.s. visits every $y \in \mathbb{Z}^d$ infinitely often, in case $\mathbb{D}_t = f(t)\mathbb{K} \cap \mathbb{Z}^d$ with $f \in \mathcal{F}_*$ such that $J_f = \infty$ and \mathbb{K} in

$$\mathcal{K} := \{\text{bounded uniform domain } \mathbb{K} \subset \mathbb{R}^d : x \in \mathbb{K} \Rightarrow \lambda x \in \mathbb{K} \quad \forall \lambda \in [0, 1]\}. \quad (1.5)$$

Remark 1.5. Requiring $(a_l - a_{l-1}) \uparrow \infty$ results in $l \mapsto a_l$ super-linear, and hence in the series $\sum_l a_l^{2-d} \log(1 + a_l)$ converging whenever $d \geq 4$ (so the latter restriction on $f \in \mathcal{F}_*$ is relevant only for $d = 3$). We need \mathbb{K} to be a uniform domain only for the invariance principle of Lemma 3.2, and impose on \mathbb{K} the star-shape condition of (1.5) merely to guarantee that the corresponding sub-graphs $t \mapsto \mathbb{D}_t$ are non-decreasing.

One motivating example for our study is the SRW $\{Y_t\}$ on the independently growing Internal Diffusion Limited Aggregation (IDLA) cluster \mathbb{D}_t , formed by particles injected at the origin according to a Poisson process of bounded away from zero intensity $\lambda(t)$, and independently performing SRW with jump-rate v . While the microscopic boundary of such IDLA cluster \mathbb{D}_t is rather involved, it is well known (see [20]), that $M_t^{-1/d} \mathbb{D}_t \rightarrow \mathbb{B}_{\kappa}$, where M_t denotes the number of particles reaching the IDLA cluster boundary by time t , and the value of $\kappa = \kappa_d$ is chosen such that \mathbb{B}_{κ} has volume one. Consequently, from part (a) of Theorem 1.4 we have that

Corollary 1.6. The SRW on such IDLA clusters is a.s. transient when the random variable $J := \int_1^{\infty} M_t^{-1} dt$ is finite.

Further, our analysis (i.e. part (b) of Theorem 1.4), suggests the a.s. recurrence of the SRW on such IDLA clusters, whenever $J = \infty$ (this is also a special case of Conjecture 1.2).

Remark 1.7. In our IDLA clusters example, let $g(t) := \int_0^t \lambda(s) ds$ denote the mean of the Poisson number of particles N_t injected at the origin by time t . Then, a.s. $J < \infty$ iff

$$\hat{J} := \int_1^{\infty} g(t)^{-1} dt < \infty. \quad (1.6)$$

Indeed, clearly $M_t \leq N_t$ and for large t the Poisson variable N_t is concentrated around $g(t)$. Our claim thus follows immediately when $v \rightarrow \infty$, for then one has further that $M_t \uparrow N_t \sim g(t)$. More generally, for v finite and $t \gg 1$, the variable M_t is still concentrated, say around some non-random $u(t)$, which is roughly comparable to $N_{t-cu(t)^{2/d}}$ for some $c = c(v)$, and thereby also to $g(t - cu(t)^{2/d})$. Solving for

$$u(t) := \sup\{u : g(t - cu^{2/d}) \geq u\}$$

it is easy to check that $u(2t) \geq g(t) \wedge (t/c)^{d/2}$, hence for $d \geq 3$

$$\int_1^{\infty} u(t)^{-1} dt < \infty \quad \text{iff} \quad \hat{J} < \infty.$$

Next, considering part (b) of Theorem 1.4 for $\mathbb{K} = \mathbb{B}_1$ we see that (at least subject to our conditions about $\{a_l\}$), Conjecture 1.2 is a consequence of the more general *monotonicity conjecture*:

Conjecture 1.8. *Suppose non-decreasing in t graphs G_t, G'_t of uniformly bounded degrees are such that $G_t \subseteq G'_t$ for all t , and the SRW $\{Y_t\}$ on $\{G_t\}$ is transient, i.e. its sample-path a.s. returns to $Y_0 = y_0$ finitely often. Then, the same holds for the sample path of the SRW $\{Y'_t\}$ on $\{G'_t\}$, starting at $Y'_0 = y_0$.*

Remark 1.9. By Rayleigh monotonicity principle, Conjecture 1.8 trivially holds whenever G_t and G'_t do not depend on t . However, beware that it may fail when the graphs depend on t and unbounded degrees are allowed. For example, on $G_t = \mathbb{Z}^3$ the SRW is transient, but we can force having a.s. infinitely many returns to 0 by adding to the edges of \mathbb{Z}^3 , at times $t_k \uparrow \infty$ fast enough, edges in $G'_t, t \geq t_k$, between 0 and each vertex in a wide enough annulus $A_k := \{x \in \mathbb{Z}^3 : \|x\|_2 \in [r_k, R_k)\}$ (specifically, with $r_k \ll R_k$ suitably chosen to make sure the SRW on G'_t is at times t_k in A_k and thereby force at least one return to zero before exiting A_k , while $t_{k-1} \ll t_k$ gives separation of scales).

As shown for example in Theorem 1.4, when the SRW on the limiting graph G_∞ is transient, one may still get recurrence by imposing slow enough growth on G_t . In contrast, whenever the SRW on G_∞ is recurrent, we have the following consequence of the Conjecture 1.8.

Conjecture 1.10. *If SRW $\{Y_t\}$ on a fixed graph G_∞ of uniformly bounded degrees is recurrent, then the same applies to SRW on non-decreasing $G_t \subseteq G_\infty$, for any choice of $G_t \uparrow G_\infty$.*

In particular, Conjecture 1.10 implies that the SRW on any non-decreasing $D_t \subseteq \mathbb{Z}^2$ is recurrent. We note in passing that monotonicity of $t \mapsto D_t$ is necessary for the latter statement (hence for Conjectures 1.8 and 1.10). Indeed, with D_t being \mathbb{Z}^2 without edges (x, y) for $\|x\|_1 = t$ and $\|y\|_1 = t - 1$, we have $D_t \rightarrow \mathbb{Z}^2$ as $t \rightarrow \infty$, while $\|Y_t\|_1 = t$ for all t .

Remark 1.11. Conjecture 1.10 was proposed to us by J. Ding and upon completing this manuscript we found a more general version of it in [1]. Specifically, [1] conjecture that a random walk $\{Y_t\}$ on graph G_∞ with non-decreasing edge conductances $\{c_t(e)\}$ is recurrent as soon as the walk on $(G_\infty, \{c_\infty(e)\})$ is recurrent (Conjecture 1.10 is just its restriction to $\{0, 1\}$ -valued conductances). This is proved for G_∞ a tree (by potential theory, see [1, Theorem 5.1]). A weaker version of Conjecture 1.8 is also proposed there (and confirmed in [1, Theorem 4.2] for $G_\infty = \mathbb{Z}_+$), whereby the transience of the walk on $(G_\infty, \{c_t(e)\})$ is conjectured to hold whenever the walk on $(G_\infty, \{c_0(e)\})$ and the walk on $(G_\infty, \{c_\infty(e)\})$ are both transient. Finally, we note in passing that the zero-one law $\mathbb{P}(A) \in \{0, 1\}$ in Conjecture 1.2 is not at all obvious given [1, Example 4.5], where $0 < \mathbb{P}(A) < 1$ for some random walk on \mathbb{Z} with certain non-random, non-increasing $c_t(e) \in (0, 1]$.

Remark 1.12. Recall [11] that the SRW on the infinite cluster D_0 of Bernoulli bond percolation on \mathbb{Z}^d is a.s. recurrent for $d = 2$ and transient for any $d \geq 3$. Hence, by Conjectures 1.8 and 1.10 the same should apply to the SRW on any independently growing domains $D_t \supseteq D_0$. Whereas the latter is an open problem, by [15, Theorem 1.1] such conclusion trivially holds when D_t is the set of vertices connected to the origin by time t in First-Passage Percolation with finite, non-negative i.i.d. passage times on \mathbb{Z}^d , subject only to the mild moment condition [15, (1.6)].

We consider also Brownian motions on growing domains, as defined next.

Definition 1.13. *We call (W_t, D_t) reflected Brownian motion on growing domains (RBMG), if the non-random, monotone non-decreasing $D_t \subseteq \mathbb{R}^d$ are such that the normally reflected Brownian motion W on the time-space domain $\mathcal{D} := \{(t, x) \in \mathbb{R}^{d+1} : x \in D_t\}$ is a well-defined strong Markov process solving the corresponding deterministic Skorohod problem. That is, for any $(s, x) \in \overline{\mathcal{D}}$ there is a unique pair of continuous processes (W, L) adapted to the minimal admissible filtration of Brownian motion $\{U_t\}_{t \geq 0}$, with L*

non-decreasing, such that for any $t \geq s$, both $(t, W_t) \in \overline{\mathcal{D}}$ and

$$\begin{aligned} W_t &= x + U_t - U_s + \int_s^t \mathbf{n}(u, W_u) dL_u, \\ L_t &= \int_s^t \mathbb{I}_{\partial\mathcal{D}}(u, W_u) dL_u, \end{aligned} \tag{1.7}$$

where $\mathbf{n}(u, y)$ denotes the inward normal unit vector at $y \in \partial\mathbb{D}_u$.

As shown in [5, Theorem 2.1 and 2.5], Definition 1.13 applies when $\partial\mathcal{D}$ is C^3 -smooth with $\gamma(t, x) \cdot (0, \mathbf{n}(t, x))$ bounded away from zero uniformly on compact time intervals, where $\gamma(t, x)$ denotes the inward normal unit vector at $(t, x) \in \partial\mathcal{D}$. Focusing on $\mathbb{D}_t = f(t)\mathbb{K}$ this condition holds whenever both $f(t)$ and $\partial\mathbb{K}$ are C^3 -smooth. Further, this construction easily extends to handle isolated jumps in $t \mapsto f(t)$.

In the context of \mathbb{R}^d -valued stochastic processes, we define recurrence as follows:

Definition 1.14. *The sample path x_t of a stochastic process $t \mapsto x_t \in \mathbb{D}_t$ with $x_0 = 0$, is called recurrent, if it makes infinitely many excursions to \mathbb{B}_ϵ for any $\epsilon > 0$, and is called transient otherwise. That is, recurrence amounts to the event $A := \bigcap_{\epsilon > 0} A_\epsilon$, where*

$$\begin{aligned} \sigma_\epsilon^{(0)} &:= \inf\{t \geq 0 : \mathbb{D}_t \supseteq \mathbb{B}_\epsilon\}, \\ \tau_\epsilon^{(i)} &:= \inf\{t \geq \sigma_\epsilon^{(i-1)} : |x_t| < \epsilon\}, \quad i \geq 1 \\ \sigma_\epsilon^{(i)} &:= \inf\{t \geq \tau_\epsilon^{(i)} : |x_t| > 1/2\}, \\ A_\epsilon &:= \{\tau_\epsilon^{(i)} < \infty, \forall i\}. \end{aligned}$$

Theorem 1.15. *Suppose $\mathbb{B}_{f(t)} \subseteq \mathbb{D}_t \subseteq \mathbb{R}^d$, $d \geq 3$, and $t \mapsto f(t)$ is positive, non-decreasing.*

- (a). *The sample path of the RBMG (W_t, \mathbb{D}_t) is a.s. transient whenever $J_f < \infty$.*
- (b). *The sample path of the RBMG (W_t, \mathbb{D}_t) is a.s. recurrent whenever $J_f = \infty$, provided $\mathbb{D}_t = f(t)\mathbb{K}$ for C^3 -smooth up to isolated jump points $t \mapsto f(t)$ such that $\int_0^\infty f'(s)^2 ds$ is finite and $\mathbb{K} \in \mathcal{K}$ of C^3 -smooth boundary $\partial\mathbb{K}$.*

Remark 1.16. In part (a) of Theorem 1.15 we implicitly assume that the RBMG (W_t, \mathbb{D}_t) is well defined, in the sense of Definition 1.13. Since $J_f = \infty$ whenever $f(t)$ is bounded, in which case part (b) trivially holds, we assume throughout that $f(t)$ is unbounded. The condition $\int_0^\infty f'(s)^2 ds < \infty$ is needed in part (b) only for $\mathbb{K} \neq \mathbb{B}_r$, and it holds for example whenever $f(\cdot)$ is piecewise constant, or in case $f(s) = (c + s)^\alpha$ for some $c > 0$ and $\alpha \in [0, 1/2)$.

We prove Theorem 1.15 in Section 2 and Theorem 1.4 in Section 3, whereas in Section 4 we show that in the context of Conjecture 1.2, if recurrence/transience occurs a.s. with respect to the origin, then the same applies at any other point.

2 Proof of Theorem 1.15

Since the events A_ϵ are non-decreasing in ϵ , it suffices for Theorem 1.15 to show that $q_\epsilon := \mathbb{P}(W \in A_\epsilon) = \mathbb{I}_{\{J_f = \infty\}}$ for each fixed $\epsilon > 0$. To this end we require the following three lemmas.

- Lemma 2.1.** *Suppose $|x_1| \leq |x_2|$ and for some positive $g \uparrow \infty$, and constant $c > 1$, one has RBMG-S $(W_t^{(1)}, \mathbb{D}_t^{(1)})$, $(W_t^{(2)}, \mathbb{D}_t^{(2)})$, such that $W_0^{(i)} = x_i$, $i = 1, 2$, $\mathbb{D}_t^{(1)} = \mathbb{B}_{g(t)}$, $\mathbb{D}_t^{(2)} \supseteq \mathbb{B}_{cg(t)}$.*
- (a). *Then, there exists a coupling $(W_t^{(1)}, W_t^{(2)})$ with non-negative $\psi_t := |W_t^{(2)}| - |W_t^{(1)}|$.*
 - (b). *Such coupling also exists in case $\mathbb{D}_t^{(2)} = \mathbb{D}_t^{(1)} = \mathbb{B}_{g(t)}$.*

Proof. (a). Given $x, y \in \mathbb{R}^d$ with $|x| = |y|$, let $\mathcal{V} := \text{span}\{x, y\}$ and $\mathbf{O}(x, y)$ denote the unique d -dimensional orthogonal matrix acting as the identity on \mathcal{V}^\perp and as the rotation such that $\mathbf{O}x = y$ on \mathcal{V} . By assumption $\psi_0 \geq 0$. We run the RBMG-s independently, until $\eta_1 := \inf\{t \geq 0 : \psi_t = 0\}$, noting that by continuity of $t \mapsto \psi_t$, the function ψ_t is non-negative on $(0, \eta_1)$. It thus suffices to consider only $\eta_1 < \infty$. In this case, let $\{W_t^{(1)}, t \geq \eta_1\}$, be the solution of (1.7) driven by Brownian motion $\{U_t, t \geq \eta_1\}$ starting at $W_{\eta_1}^{(1)} = U_{\eta_1}$. Setting $\tilde{U}_t := \mathbf{O}(W_{\eta_1}^{(1)}, W_{\eta_1}^{(2)})U_t$ let

$$\tau_1 := \inf\{t \geq \eta_1 : \tilde{U}_t \in \partial\mathbb{D}_t^{(2)}\}.$$

Since $|W_{\eta_1}^{(2)}| = |W_{\eta_1}^{(1)}| \leq g(\eta_1) < cg(\eta_1)$, it follows from the definition of RBMG $(W_t^{(2)}, \mathbb{D}_t^{(2)})$ that $\{\tilde{U}_t\}$ has for $t \in [\eta_1, \tau_1]$ the same law as $\{W_t^{(2)}\}$. In particular, a normal reflection at $\partial\mathbb{B}_{g(t)}$ reduces the norm, hence $|W_t^{(1)}| \leq |U_t| = |\tilde{U}_t| = |W_t^{(2)}|$ for such t . That is $\psi_t \geq 0$ on $t \in [\eta_1, \tau_1]$. With $\psi_{\tau_1} \geq (c-1)g(\tau_1) > 0$, clearly $\eta_2 := \inf\{t \geq \tau_1 : \psi_t = 0\} > \tau_1$. In case $\eta_2 < \infty$, with $W_{\eta_2}^{(2)} \in \mathbb{D}_{\eta_2}^{(1)}$, we repeat the above argument for $[\eta_2, \tau_2]$, then for $[\eta_3, \tau_3]$, etc. By construction, $\eta_n < \tau_n < \eta_{n+1}$ for all n . Moreover, a.s. $\tau_n \rightarrow \infty$ when $n \rightarrow \infty$. Indeed, assuming without loss of generality that $\eta_k < \infty$, we have the stopping times

$$\begin{aligned} \theta_k &:= \inf\{t \geq \eta_k : \tilde{U}_t \in \partial\mathbb{B}_{g(t)}\}, \\ \zeta_k &:= \inf\{t \geq \theta_k : \tilde{U}_t \in \partial\mathbb{B}_{cg(\theta_k)}\}, \end{aligned}$$

such that $\theta_k < \zeta_k \leq \tau_k$ and conditional on the relevant stopped σ -algebra at θ_k , the random variable $\zeta_k - \theta_k$ has the law of the time it takes an independent Brownian motion to get from $\partial\mathbb{B}_{g(\theta_k)}$ to $\partial\mathbb{B}_{cg(\theta_k)}$. With $g(\theta_k) \geq g(0)$, by Brownian scaling it follows that the sequence $\{\tau_k - \eta_k\}$ stochastically dominates the i.i.d. $\{\xi_k\}$, each distributed as $\xi := g(0)^2 \inf\{t \geq 0 : |U_t| = c, |U_0| = 1\}$. This induces stochastic domination of the corresponding partial sums,

$$\sum_{k=1}^n (\tau_k - \eta_k) \succeq \sum_{k=1}^n \xi_k.$$

As $n \rightarrow \infty$ the right-hand-side grows a.s. to infinity and so does the left-hand-side.

(b). We follow the construction and reasoning of part (a), up to time η_1 , setting now $\tilde{U}_t := \mathbf{O}(W_{\eta_1}^{(1)}, W_{\eta_1}^{(2)})W_t^{(1)}$ for all $t \geq \eta_1$. Then, by the invariance to rotations of $\mathbb{B}_{g(t)}$ and the fact that only normal reflections are used, we have that $t \mapsto W_t^{(2)}1_{\{t < \eta_1\}} + \tilde{U}_t1_{\{t \geq \eta_1\}}$ is a realization of the RBMG $(W_t^{(2)}, \mathbb{B}_{g(t)})$, for which ψ_t is non-negative. \square

Lemma 2.2. *Let \mathbb{P}_x denote the law of the RBM Z_t on \mathbb{B}_a , starting at $Z_0 = x$. Consider the stopping times $\tau(a) := \inf\{s \geq 0 : Z_s \in \partial\mathbb{B}_a\}$ and $\sigma(a, r) := \inf\{s \geq 0 : Z_s \in \mathbb{B}_r\}$. Then, there exists $C = C_d(\delta) > 0$ such that for any $t, \delta > 0, \frac{r}{a} \in [\delta, 1), d \geq 3$,*

$$\sup_{x \in \mathbb{B}_a} \mathbb{P}_x(\tau(a) > ta^2) < C^{-1}e^{-Ct}, \tag{2.1}$$

$$\sup_{x \in \mathbb{B}_a \setminus \mathbb{B}_r} \mathbb{P}_x(\sigma(a, r) > ta(a-r)) < C^{-1}e^{-Ct}, \tag{2.2}$$

$$\inf_{x \in \mathbb{B}_{a/2}} \mathbb{P}_x(\tau(a) > a^2) > C. \tag{2.3}$$

Proof. In case the process starts at $z \in \partial\mathbb{B}_r$ we use $\mathbb{P}_{r\bar{e}_1}$ to indicate probabilities of events which are invariant under any rotation of the sample path. Then, with U_t denoting a standard Brownian motion, by Brownian scaling the left-hand side of (2.3) does not

depend on a and is merely the positive probability $\mathbb{P}_{0.5\bar{e}_1}(|U_s| < 1, \forall s \leq 1)$. Further, by the Markov property, invariance to rotations and Brownian scaling, for $x \in \mathbb{B}_a$,

$$\begin{aligned} \mathbb{P}_x(\tau(a) > ta^2) &= \mathbb{P}_x(|Z_s| < a, \forall s \leq ta^2) \leq \left[\sup_{0 \leq |z| < a} \mathbb{P}_z(|Z_s| < a, \forall s \leq a^2) \right]^{[t]} \\ &= [\mathbb{P}_0(|Z_s| < a, \forall s \leq a^2)]^{[t]} = \mathbb{P}_0(|U_s| < 1, \forall s \leq 1)^{[t]} := (1 - \eta)^{[t]}, \end{aligned}$$

with $\eta = \eta_d > 0$, out of which we get (2.1). Proceeding similarly, we have for (2.2) that

$$\begin{aligned} \mathbb{P}_x(\sigma(a, r) > ta(a - r)) &= \mathbb{P}_x(|Z_s| > r, \forall s \leq ta(a - r)) \\ &\leq \left[\sup_{r < |z| \leq a} \mathbb{P}_z(|Z_s| > r, \forall s \leq a(a - r)) \right]^{[t]} = [\mathbb{P}_{a\bar{e}_1}(|Z_s| > r, \forall s \leq a(a - r))]^{[t]} \\ &\leq \sup_{\delta \leq \rho < 1} \mathbb{P}_{\bar{e}_1}(|\widehat{Z}_s| > \rho, \forall s \leq 1 - \rho)^{[t]} := (1 - \zeta(\delta))^{[t]} \end{aligned}$$

where \widehat{Z} denotes the RBM on \mathbb{B}_1 . Further,

$$\zeta \geq \inf_{\delta \leq \rho < 1} \mathbb{P}_{\bar{e}_1}(|U_{1-\rho}| \leq \rho) > 0$$

(by the stochastic domination $|U_s| \succeq |\widehat{Z}_s|$, for example, due to part (a) of Lemma 2.1). \square

Lemma 2.3. *Let \mathbb{P}_x denote the law of the RBM Z_t on \mathbb{B}_a , starting at $Z_0 = x$. Fixing $\epsilon, \delta \in (0, 1/2)$, there exist finite $M_d(\epsilon, \delta)$ and $C = C_d(\epsilon, \delta)$ such that for all M, T, a and r with $M \geq M_d(\epsilon, \delta)$, $T \geq Ma^2 \log a$ and $a - M \geq r \geq a\delta$,*

$$\inf_{x \in \mathbb{B}_r} \mathbb{P}_x(\exists s \leq T : |Z_s| < \epsilon) \geq C^{-1} \left[\frac{T}{a^d} \wedge 1 \right], \tag{2.4}$$

$$\sup_{x \in \partial \mathbb{B}_r} \mathbb{P}_x(\exists s \leq T : |Z_s| < \epsilon) \leq C \left[\frac{T}{a^d} \wedge 1 \right]. \tag{2.5}$$

Proof. Starting at $Z_0 = x \in \partial \mathbb{B}_r$, we set $\sigma_0 := 0$,

$$\begin{aligned} \tau_k &:= \inf\{t \geq \sigma_{k-1} : Z_t \in \partial \mathbb{B}_a\}, \quad k \geq 1 \\ \sigma_k &:= \inf\{t \geq \tau_k : Z_t \in \mathbb{B}_{a/2}\} \end{aligned} \tag{2.6}$$

and call Z . restricted to $[\sigma_k, \sigma_{k+1}]$ the k -th excursion of Z , with $L_k := \sigma_{k+1} - \sigma_k$ denoting its length. Obviously, for $\sigma(a, \epsilon) := \inf\{t \geq 0 : |Z_t| \leq \epsilon\}$ and any $k \in \mathbb{N}$

$$\mathbb{P}(\sigma(a, \epsilon) \leq \sigma_k) - \mathbb{P}(\sigma_k \geq T) \leq \mathbb{P}(\sigma(a, \epsilon) \leq T) \leq \mathbb{P}(\sigma(a, \epsilon) \leq \sigma_k) + \mathbb{P}(\sigma_k \leq T). \tag{2.7}$$

Recall that conditional on their starting and ending positions, these excursions of the RBM Z_t on \mathbb{B}_a are mutually independent. Consequently,

$$\mathbb{P}(\sigma(a, \epsilon) > \sigma_k) = \mathbb{E} \left[\prod_{i=1}^k (1 - b_\epsilon(Z_{\sigma_{i-1}}, Z_{\sigma_i})) \right] \tag{2.8}$$

where $b_\epsilon(x, w) := \mathbb{P}_x(\inf_{t \leq \tau_1} |Z_t| \leq \epsilon \mid Z_{\sigma_1} = w)$ is the probability of entering \mathbb{B}_ϵ in one such excursion. Elementary potential theory (e.g. see [21, Theorem 3.18]), yields the formula

$$b_\epsilon(x) = \frac{|x|^{2-d} - a^{2-d}}{\epsilon^{2-d} - a^{2-d}} \tag{2.9}$$

for the unconditional probability $b_\epsilon(x) := \mathbb{E}[b_\epsilon(x, Z_{\sigma_1})]$. Hence, applying the strong Markov property of Z . at the stopping time σ_i where $|Z_{\sigma_i}| = a/2$, going from $i = k - 1$ backwards to $i = 1$ we deduce that

$$q_k := \mathbb{P}(\sigma(a, \epsilon) \leq \sigma_k) = 1 - (1 - b_\epsilon(r\vec{e}_1))(1 - b_\epsilon(\frac{a}{2}\vec{e}_1))^{k-1}.$$

It is easy to check that $b_\epsilon(\frac{a}{2}\vec{e}_1) = c_0 a^{2-d}(1 + o(1/M))$ for some finite, positive $c_0 = c_0(d, \epsilon)$ and all $a \geq M \geq 1/\delta$, whereas $b_\epsilon(r\vec{e}_1) \leq c' b_\epsilon(\frac{a}{2}\vec{e}_1)$ for some finite $c' = c'(d, \delta)$, and all $r \geq a\delta \geq 1$. Thus, setting $k = \lceil T a^{-2} \kappa^{\mp 1} \rceil$ for some universal κ yet to be determined, we see that

$$C^{-1} \left[\frac{T}{a^d} \wedge 1 \right] \leq q_k \leq C \left[\frac{T}{a^d} \wedge 1 \right], \tag{2.10}$$

for some finite $C = C(d, \epsilon, \delta, \kappa)$ and all $M \geq M_d(\epsilon, \delta)$ large enough. Hence, it suffices to show that for some universal $c = c(d, \epsilon, \delta) > 0$, $\kappa = \kappa(d, \epsilon, \delta)$ finite and all $a \geq 2M_d$, $k \geq 1$, $x \in \partial B_r$

$$\mathbb{P}_x(k^{-1} a^{-2} \sigma_k \geq \kappa) \leq e^{-ck}, \quad \mathbb{P}_x(k^{-1} a^{-2} \sigma_k \leq \kappa^{-1}) \leq e^{-ck}. \tag{2.11}$$

Indeed, our assumption that $T \geq M a^2 \log a$ translates to $ck \geq c\kappa^{\mp 1} M \log a$, so that for all large enough $M \geq M_0(\kappa, c, d, C)$ we have that

$$e^{-ck} \leq \frac{1}{2C} \left[\kappa^{\pm 1} a^{2-d} \wedge 1 \right] \leq \frac{1}{2} q_k,$$

resulting by (2.7) and (2.10) in the claimed bounds.

The universal exponential tail bounds of (2.11), are a direct consequence of having control on the log-moment generating functions $\Lambda_k(\theta) := \log \mathbb{E}[e^{\theta \sigma_k}]$ for large k and small $\hat{\theta} := \theta a^2$. Specifically, by Markov's exponential inequality (also known as Chernoff's bound), we get (2.11) as soon as we show that

$$\kappa_+ := \liminf_{\hat{\theta} \downarrow 0} \limsup_{k \rightarrow \infty} \hat{\theta}^{-1} k^{-1} \sup_{a \geq M} \{ \Lambda_k(\hat{\theta} a^{-2}) \} < \infty, \tag{2.12}$$

$$-\kappa_-^{-1} := \liminf_{\hat{\theta} \downarrow 0} \limsup_{k \rightarrow \infty} \hat{\theta}^{-1} k^{-1} \sup_{a \geq M} \{ \Lambda_k(-\hat{\theta} a^{-2}) \} < 0, \tag{2.13}$$

(provided $\kappa > \kappa_+ \vee \kappa_-$ and $c = \hat{\theta}(\kappa - \kappa_+) \wedge (\kappa_-^{-1} - \kappa^{-1})$ for $\hat{\theta} > 0$ small enough). Turning to control $\Lambda_k(\cdot)$, recall that $\sigma_k = \sum_{i=0}^{k-1} L_i$, with $\{L_i\}$ mutually independent conditional on the values of $\{Z_{\sigma_i}\}$. Thus, proceeding in the same manner as done in (2.8), we have that for any $\theta \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\Lambda_k(\theta) = \log \mathbb{E} \left[\prod_{i=1}^k m(\theta, Z_{\sigma_{i-1}}, Z_{\sigma_i}) \right],$$

where $m(\theta, x, w) := \mathbb{E}_x[e^{\theta L_0} | Z_{\sigma_1} = w]$. By invariance of the joint law of $\{\sigma_k\}$ with respect to rotations of the RBM sample path $t \mapsto Z_t$, the function $m(\theta, x) = \mathbb{E}[m(\theta, x, Z_{\sigma_1})]$ depends only on $|x|$ and is thus denoted hereafter by $m(\theta, |x|)$. Using this and exploiting once more the strong Markov property at the stopping times σ_i where $|Z_{\sigma_i}| = a/2$ (first for $i = k - 1$, then backwards to $i = 1$), we find that

$$\Lambda_k(\theta) = (k - 1) \log m(\theta, a/2) + \log m(\theta, r). \tag{2.14}$$

Further, L_0 is the sum of two independent variables, having the laws of $\tau(a)$ and $\sigma(a, a/2)$ of Lemma 2.2. Thus, the universal upper bounds (2.1) and (2.2) imply that for any $0 \leq \hat{\theta} < C_d(\delta)$ (and $C_d(\delta) > 0$ as in Lemma 2.2),

$$\sup_{r \geq a\delta} \{ m(\hat{\theta} a^{-2}, r) \} \leq \left[1 + \frac{\hat{\theta}}{C_d(C_d - \hat{\theta})} \right]^2. \tag{2.15}$$

Combining this with (2.14) we get (2.12) (with $\kappa_+ = 2C_d^{-2}$ finite). Recall that for any $Y, y \geq 0$,

$$\log \mathbb{E}[e^{-Y}] \leq -(1 - \mathbb{E}[e^{-Y}]) \leq -(1 - e^{-y})\mathbb{P}(Y \geq y),$$

hence $\log m(-\hat{\theta}a^{-2}, r) \leq 0$ and

$$\log m(-\hat{\theta}a^{-2}, a/2) \leq -(1 - e^{-\hat{\theta}})\mathbb{P}(L_1 \geq a^2).$$

It thus follows from (2.3) and the stochastic domination $L_1 \succeq \tau(a)$ starting at some position $x \in \partial\mathbb{B}_{a/2}$ that $\log m(-\hat{\theta}a^{-2}, a/2) \leq -C_d\hat{\theta}/e$ for all $a > 0$ and $\hat{\theta} \leq 1$, thereby establishing (2.13) with $\kappa_-^{-1} = C_d/e$ positive, and completing the proof of the lemma. \square

Proof of Theorem 1.15. This proof consists of six steps. First, for $\mathbb{D}_t = \mathbb{B}_{f(t)}$ and $f \in \mathcal{F}_*$ of (1.4), we prove in Step I the a.s. recurrence of the RBMG when $J_f = \infty$, and in Step II its a.s. transience when $J_f < \infty$. Relaxing these conditions, in Step III we prove part (a), and in Step IV get part (b) for $\mathbb{K} = \mathbb{B}_1$. The a.s. sample-path recurrence when $J_f = \infty$ is then established for $\mathbb{K} \in \mathcal{K}$ of (1.5), when both $\partial\mathbb{K}$ and $t \mapsto f(t)$ are C^3 -smooth (see Step V), and further extended to $f(\cdot)$ having isolated jump points (see Step VI).

Step I. For $f \in \mathcal{F}_*$ we set $\Delta T_l := t_{l+1} - t_l$ and $p_l := a_l^{2-d} \log(1 + a_l)$, so that $\sum_l p_l < \infty$ and

$$J_f = \sum_{l=1}^{\infty} a_l^{-d} \Delta T_l. \tag{2.16}$$

Considering here $\mathbb{D}_t = \mathbb{B}_{f(t)}$ for $f \in \mathcal{F}_*$, we proceed to prove the a.s. recurrence of the RBMG sample path in case $J_f = \infty$. To this end, consider the events $\Gamma_l := \{\exists t \in [t_{l-1}, t_l] : |W_t| < \epsilon\}$, adapted to the filtration $\mathcal{G}_l := \sigma\{W_s, s \leq t_l\}$. Fixing $\delta \in (0, 1/2)$ we set $r_l := (a_{l-1} + 1) \vee \delta a_l$ and further assume that

$$\Delta T_l \geq 2M_d a_l^2 \log(1 + a_l). \tag{2.17}$$

Then, since

$$W_{t_l} \in \overline{\mathbb{B}_{a_{l-1}}} \quad \text{and} \quad \mathbb{D}_t = \mathbb{B}_{a_l}, \quad \forall t \in [t_l, t_{l+1}), \tag{2.18}$$

we have by (2.4) that

$$\zeta_l := \mathbb{P}(\Gamma_{l+1} | \mathcal{G}_l) = \mathbb{P}(\Gamma_{l+1} | W_{t_l}) \geq \inf_{x \in \overline{\mathbb{B}_{a_{l-1}}}} \mathbb{P}_x(\exists s \leq \Delta T_l : |Z_s| < \epsilon) \geq C^{-1} \left[\frac{\Delta T_l}{a_l^d} \wedge 1 \right].$$

Recall that J_f of (2.16) is infinite, hence a.s. $\{\sum_{l=1}^{\infty} \zeta_l = \infty\}$, which implies that Γ_l occurs infinitely often (by the conditional version of Borel-Cantelli II, see [9, Theorem 5.3.2]). That is,

$$\exists l_k \uparrow \infty \quad \& \quad s_k \in [t_{l_k}, t_{l_k+1}) \quad \text{such that} \quad |W_{s_k}| < \epsilon. \tag{2.19}$$

By transience of the $d \geq 3$ dimensional Brownian motion we can set $k_1 = 1$ and recursively pick $u_j := \inf\{t > s_{k_j} : |W_t| > 1/2\}$, $k_{j+1} := \inf\{k : s_k > u_j\}$, for $j = 1, 2, \dots$, thus yielding the event A_ϵ . To remove the spurious condition (2.17) set $\psi_l := \Delta T_l / (a_l^2 \log(1 + a_l))$, so $\sum_l \psi_l p_l = \sum_l a_l^{-d} \Delta T_l$ diverges by (2.16) whereas $\sum_l p_l$ is finite. Hence, $\sum_l \psi_l p_l \mathbb{1}_{\{\psi_l \geq 2M_d\}} = \infty$, and the preceding argument is applicable even when restricted to $\{l_k\} \uparrow \infty$ such that $\psi_{l_k} \geq 2M_d$.

Step II. Still considering $\mathbb{D}_t = \mathbb{B}_{f(t)}$ for $f \in \mathcal{F}_*$, we show next that $\mathbb{P}(A_\epsilon) = 0$ whenever J_f of (2.16) is finite. To this end, we note that

$$\tau_l := \inf\{s \geq 0 : W_s \notin \overline{\mathbb{B}_{r_l}}\},$$

for $l = 1, 2, \dots$, are a.s. finite and proceed to show that

$$\sum_l \mathbb{P}(\tilde{\Gamma}_l) < \infty, \tag{2.20}$$

where $\tilde{\Gamma}_l := \{\exists t \in [\tau_l, \tau_{l+1}) : |W_t| < \epsilon\}$. Indeed, in this case by Borel-Cantelli I, a.s. the RBMG does not re-enter \mathbb{B}_ϵ during $[\tau_l, \infty)$, for some l finite. In any finite time, even the RBM on \mathbb{B}_1 a.s. makes only finitely many excursions between \mathbb{B}_ϵ and $\mathbb{B}_{1/2}^c$, hence $\mathbb{P}(A_\epsilon) = 0$. Turning to prove (2.20), recall that $t_l \leq \tau_l$ and $t_{l+1} \leq \tau_{l+1}$, so the interval $[\tau_l, \tau_{l+1})$ splits into $[\tau_l, \xi_{l+1})$ and $[\xi_{l+1}, \tau_{l+1})$, where

$$\xi_{l+1} := \inf\{s \geq t_{l+1} : W_s \notin \overline{\mathbb{B}_{r_l}}\}.$$

Restricted to $t \in [\tau_l, \xi_{l+1})$, the process $\{W_t\}$ has the law of a RBM on \mathbb{B}_{a_l} , and the length of $[\tau_l, \xi_{l+1})$ is at most ΔT_l plus the length of $[t_{l+1}, \xi_{l+1})$. By (2.1), for some constant $C = C_d(\delta) > 0$, any l and all t ,

$$\mathbb{P}(\xi_{l+1} - t_{l+1} > ta_l^2) < C^{-1}e^{-Ct}. \tag{2.21}$$

Combining (2.5) with (2.21) for $t = M \log a_l$, $M = M_d \vee \frac{2}{C}$, we have that

$$\mathbb{P}(\exists s \in [\tau_l, \xi_{l+1}) : |W_s| < \epsilon) \leq C[a_l^{-d}\Delta T_l + Mp_l] + C^{-1}a_l^{-2}, \tag{2.22}$$

with the first term on the right-hand-side summable in l iff $J_f < \infty$ (the other two terms are summable for any $f \in \mathcal{F}_*$). Further, restricted to $t \in [\xi_{l+1}, \tau_{l+1})$, the process $\{W_t\}$ has the law of Brownian motion $\{U_t\}$ (since $r_{l+1} < a_{l+1}$), hence

$$\mathbb{P}(\exists s \in [\xi_{l+1}, \tau_{l+1}) : |W_s| < \epsilon) = \frac{r_l^{2-d} - r_{l+1}^{2-d}}{\epsilon^{2-d} - r_{l+1}^{2-d}} \leq 2\epsilon^{d-2}(r_l^{2-d} - r_{l+1}^{2-d}). \tag{2.23}$$

Bounding $\mathbb{P}(\tilde{\Gamma}_l)$ by the sum of the left-hand-sides of (2.22) and (2.23), we thus conclude that $\sum_l \mathbb{P}(\tilde{\Gamma}_l) < \infty$ whenever $J_f < \infty$.

Step III. Given non-decreasing, unbounded, positive $t \mapsto f(t)$ (which without loss of generality we assume hereafter to be also right-continuous), let $g \in \mathcal{F}_*$ with $a_l = 2^{l-1}f(0)$ and $t_l := \inf\{t \geq 0 : f(t) \geq 2^{l-1}f(0)\}$. Since $g(t) \leq f(t) \leq 2g(t)$ for all $t \geq 0$, we have by part (a) of Lemma 2.1, the coupling $|\widetilde{W}_t| \leq |W'_t|$ for RBMG $(\widetilde{W}_t, \mathbb{B}_{g(t)/2})$ and (W'_t, \mathbb{D}_t) such that $\mathbb{D}_t \supseteq \mathbb{B}_{f(t)}$. Further, as $J_{4g} \leq J_f \leq J_{\frac{1}{2}g}$, if $J_f < \infty$ then $J_{\frac{1}{2}g} = 8^d J_{4g} < \infty$ and in view of Step II, a.s. $\{\widetilde{W}_t\}$ enters \mathbb{B}_ϵ finitely often. Hence, $\mathbb{P}(A_\epsilon) = 0$, yielding the stated a.s. transience of the sample path for any such RBMG (W'_t, \mathbb{D}_t) , thereby completing the proof of part (a).

Step IV. Returning to $\mathbb{D}_t = \mathbb{B}_{f(t)} = f(t)\mathbb{B}_1$, now for $t \mapsto f(t)$ which is further C^3 -smooth up to isolated jump points, we have by yet another application of part (a) of Lemma 2.1 that $|W'_t| \leq |W''_t|$ for the RBMG $(W''_t, \mathbb{B}_{4g(t)})$. Assuming that $J_f = \infty$, or equivalently that $J_{4g} = \infty$ (with $g \in \mathcal{F}_*$ chosen as in Step III), we know from Step I that for any u fixed, $\{W''_t, t \geq u\}$ a.s. makes infinitely many excursions from \mathbb{B}_ϵ to $\mathbb{B}_{1/2}^c$. With $|W'_t| \leq |W''_t|$ we consequently get (2.19) (for any unbounded t_l), which as we have already seen in Step I of the proof, implies that $\{W'_t\}$ a.s. makes infinitely many excursions from \mathbb{B}_ϵ to $\mathbb{B}_{1/2}^c$.

Step V. We next extend the a.s. recurrence of the RBMG (W_t, \mathbb{D}_t) sample path to $\mathbb{D}_t = f(t)\mathbb{K}$ with $J_f = \infty$, \mathbb{K} from \mathcal{K} of (1.5), such that $\partial\mathbb{K}$ and $f(t)$ are both C^3 -smooth, and $\int_0^\infty f'(s)^2 ds < \infty$. To this end, we assume without loss of generality that $\mathbb{B}_1 \subseteq \mathbb{K} \subseteq \mathbb{B}_\epsilon$ and note that $t \mapsto \int_0^t \frac{1}{f(u)} dL_u =: \tilde{L}_t$ increases only when $X_t := \frac{1}{f(t)}W_t$ is at $\partial\mathbb{K}$. Hence,

applying Ito's formula to the $C^{1,2}$ -function $v(t, x) = \frac{1}{f(t)}x$ (with $v_{xx} = 0$), and the semimartingale $\{W_t\}$ of (1.7), we get that (X, \tilde{L}) is the strong Markov process solving the deterministic Skorohod problem corresponding for $(s, x) \in \mathbb{R}_+ \times \mathbb{K}$ to

$$X_t = x + \int_s^t \frac{1}{f(u)} dB_u + \int_s^t \mathbf{n}(X_u) d\tilde{L}_u, \tag{2.24}$$

$$\tilde{L}_t = \int_s^t \mathbb{I}_{\partial\mathbb{K}}(X_u) d\tilde{L}_u, \tag{2.25}$$

where $\mathbf{n}(x)$ denotes the inward unit normal vector at $x \in \partial\mathbb{K}$ and

$$B_t = U_t - \int_0^t f'(s)X_s ds, \quad B_0 = 0. \tag{2.26}$$

Further, with $X_t \in \mathbb{K} \subseteq \mathbb{B}_c$ and $\int_0^\infty f'(s)^2 ds < \infty$, the quadratic variation $\langle M \rangle_t = \int_0^t |f'(s)X_s|^2 ds$ of the continuous (local) martingale

$$M_t = \int_0^t f'(s)X_s dU_s,$$

has uniformly in t bounded exponential moments. That is, for any $\kappa > 1$,

$$\mathbb{E} \left[\exp \{ \kappa \langle M \rangle_\infty \} \right] \leq \exp \left\{ c^2 \kappa \int_0^\infty f'(s)^2 ds \right\} < \infty.$$

By Novikov's criterion, $Z_t = \exp(M_t - \frac{1}{2} \langle M \rangle_t)$ is a uniformly integrable continuous martingale (see [23, Proposition VIII.1.15]). The same applies for $Z_t^{-1} = \exp(\widehat{M}_t - \frac{1}{2} \langle \widehat{M} \rangle_t)$ and the martingale $\widehat{M}_t = - \int_0^t f'(s)X_s dB_s$ under the measure \mathbb{Q} such that $\{B_t, t \in [0, \infty)\}$ is a standard Brownian motion in \mathbb{R}^d . Hence, by Girsanov's theorem, restricted to the completion of the canonical Brownian filtration, the measure \mathbb{Q} is equivalent to \mathbb{P} (see [23, Proposition VIII.1.1]). Moreover, under \mathbb{Q} the process $\{X_t\}$ is a normally reflected time changed Brownian motion (in short τ_{CRBM}), on \mathbb{K} for the deterministic time change

$$\tau(t) := \int_0^t f(s)^{-2} ds. \tag{2.27}$$

Applying the same procedure for the RBMG $(W'_t, f(t)\mathbb{B}_c)$, such that $W'_0 = W_0$, yields another probability measure \mathbb{H} , likewise equivalent to \mathbb{P} , under which $Y_t := \frac{1}{f(t)}W'_t$ is a τ_{CRBM} on \mathbb{B}_c for the same time change $\tau(\cdot)$ as in (2.27). Further, $\{W' \in A\}$ iff $\{X \in A^{(f)}\}$, and $\{W' \in A\}$ iff $\{Y \in A^{(f)}\}$, where $A^{(f)} := \bigcap_{\epsilon > 0} A_\epsilon^{(f)}$ and similarly to Definition 1.14 we have that

$$\begin{aligned} \sigma_\epsilon^{(0,f)} &:= 0 \\ \tau_\epsilon^{(i,f)} &:= \inf \{ t \geq \sigma_\epsilon^{(i-1,f)} : |x_t| < \epsilon / f(t) \}, \quad i \geq 1 \\ \sigma_\epsilon^{(i,f)} &:= \inf \{ t \geq \tau_\epsilon^{(i,f)} : |x_t| > 1 / (2f(t)) \}, \\ A_\epsilon^{(f)} &:= \{ \tau_\epsilon^{(i,f)} < \infty, \forall i \} \end{aligned}$$

(as $\epsilon < f(0)$ with no loss of generality). For $J_f = \infty$ recall that $\mathbb{P}(W' \in A) = 1$ as shown in Step IV. Hence,

$$\mathbb{P}(Y \in A^{(f)}) = 1 \Leftrightarrow \mathbb{H}(Y \in A^{(f)}) = 1 \stackrel{(a)}{\Leftrightarrow} \mathbb{Q}(X \in A^{(f)}) = 1 \Leftrightarrow \mathbb{P}(X \in A^{(f)}) = 1$$

out of which we deduce that $\mathbb{P}(W \in A) = 1$ as well. The key implication, marked by (a), is a consequence of the proof of [22, Theorem 5.4]. The latter theorem is a

comparison result about Neumann heat kernels over domains \mathbb{D}_i , $i = 1, 2$, such that $\mathbb{D}_2 \subseteq \mathbb{B} \subseteq \mathbb{D}_1 \subseteq \mathbb{R}^d$, $d \geq 2$, for bounded domains $\mathbb{D}_1, \mathbb{D}_2$ of C^2 -smooth boundary, and some ball \mathbb{B} centered at 0, such that for any $x \in \mathbb{D}_2$, the line segment from 0 to x is in \mathbb{D}_2 . Its proof in [22] is by constructing a (mirror) coupling between the RBM \widehat{X} on \mathbb{D}_2 and the RBM \widehat{Y} on \mathbb{D}_1 , such that $|\widehat{X}_s| \leq |\widehat{Y}_s|$ for all $s \geq 0$ and any common starting point $x \in \mathbb{D}_2$. We use it here for $\mathbb{D}_2 = \mathbb{K} \subseteq \mathbb{B}_c = \mathbb{D}_1$ and note that the monotonicity of the radial component under this coupling extends to the τ CRBM-s X_s (under \mathbb{Q}), and Y_s (under \mathbb{H}), thereby assuring that $Y \in A^{(f)}$, \mathbb{H} -a.s. implies $X \in A^{(f)}$, \mathbb{Q} -a.s.

Step VI. We proceed to show that the conclusion of Step V holds in case $t \mapsto f(t)$ has jumps $\Delta_j > 0$ at isolated jump points $t_1 < \dots < t_j < \dots$. That is, $f(t) = f_c(t) + f_d(t)$ with a C^3 -smooth function $f_c(\cdot)$ and piecewise constant $f_d(t) = \sum_j \Delta_j \mathbb{I}_{t \geq t_j}$. Setting $t_0 = 0$ and re-using the notations of Step V, upon applying Ito's formula we get that X_t (and Y_t) solve the corresponding deterministic Skorohod problem (2.24)-(2.25) within each interval $[t_{i-1}, t_i)$, and B_t is again defined via (2.26) except for $f'_c(t)$ replacing $f'(t)$. In addition, $X_{t_i} = \eta_i X_{t_i^-}$ and $Y_{t_i} = \eta_i Y_{t_i^-}$ for $i = 1, 2, \dots$, where $\eta_i = f(t_i^-)/f(t_i) < 1$. Since $\int_0^\infty f'_c(s)^2 ds$ is finite, as in Step V we have measures \mathbb{Q} and \mathbb{H} , both equivalent to \mathbb{P} , under which within each interval $[t_{i-1}, t_i)$ the processes X_t and Y_t are τ CRBM-s on \mathbb{K} and \mathbb{B}_c , respectively, for the same time change $\tau(\cdot)$. With $J_f = \infty$, we already saw in Step IV that $\mathbb{P}(W' \in A) = 1$. Following the argument of Step V this would yield that $\mathbb{P}(W \in A) = 1$, provided we suitably extend the scope of the implication (a). That is, suffices to show the existence of coupling between RBM-s \widehat{X} on \mathbb{K} and \widehat{Y} on \mathbb{B}_c , such that $|\widehat{X}_s| \leq |\widehat{Y}_s|$ for all $s \geq 0$, in the setting where at a sequence of isolated times $s_i = \tau(t_i)$ one applies the common shrinkage by $\eta_i \in (0, 1)$ to both \widehat{X} and \widehat{Y} . To achieve this, starting at $\widehat{Y}_0 = \widehat{Y}'_0 = \widehat{X}_0 = x$, we produce inductively for $i = 0, 1, \dots$ another copy $\{\widehat{Y}'_s : s \in [s_i, s_{i+1})\}$ of the RBM on \mathbb{B}_c , with jumps from $\widehat{Y}'_{s_i^-}$ to $\widehat{Y}'_{s_i} = \widehat{X}_{s_i}$ and a coupling such that $|\widehat{X}_s| \leq |\widehat{Y}'_s| \leq |\widehat{Y}_s|$ for all s . Indeed, as explained in Step V, employing [22, Theorem 5.4] separately within each interval $[s_i, s_{i+1})$ yields a (mirror) coupling of \widehat{Y}' and \widehat{X} that maintains the stated relation $|\widehat{X}_s| \leq |\widehat{Y}'_s|$. Further, applying part (b) of Lemma 2.1 inductively in $i \geq 0$, we couple \widehat{Y}'_s and \widehat{Y}_s within each interval $[s_i, s_{i+1})$, such that $|\widehat{Y}'_s| \leq |\widehat{Y}_s|$ for all $s \geq 0$, provided $|\widehat{Y}'_{s_i}| \leq |\widehat{Y}_{s_i}|$ for all $i \geq 0$. Starting at $\widehat{Y}_0 = \widehat{Y}'_0$, we have the latter inequality at $i = 0$. Then, for $i \geq 1$ we have by induction, upon utilizing our coupling on $[s_{i-1}, s_i)$ that $|\widehat{X}_{s_i^-}| \leq |\widehat{Y}'_{s_i^-}| \leq |\widehat{Y}_{s_i^-}|$. Hence $|\widehat{Y}'_{s_i}| = |\widehat{X}_{s_i}| \leq |\widehat{Y}_{s_i}|$ (after the common shrinkage by factor η_i), as needed for concluding the proof. \square

3 Proof of Theorem 1.4

Hereafter we denote the inner boundary of a discrete set \mathbb{G} by $\partial\mathbb{G}$ and fix \mathbb{K} from the collection \mathcal{K} of (1.5), scaled by a constant factor so as to have $\mathbb{K} \supseteq \mathbb{B}_2$ and hence $(\mathbb{B}_a \cap \mathbb{Z}^d) \cap \partial(a\mathbb{K} \cap \mathbb{Z}^d) = \emptyset$ for all $a \geq a_d$ large enough. We then have the following SRW analog of Lemma 2.2.

Lemma 3.1. *Let \mathbb{P}_x denote the law of SRW $\{Z_t, t \geq 0\}$ on $a\mathbb{K} \cap \mathbb{Z}^d$, $d \geq 3$, starting at $Z_0 = x \in \mathbb{Z}^d$. Considering the stopping times $\tau(a) := \inf\{s \geq 0 : Z_s \in \mathbb{B}_a^c\}$ and $\sigma(a, r) := \inf\{s \geq 0 : Z_s \in \overline{\mathbb{B}}_r\}$, there exists $C = C_d(\delta) > 0$ and $a_d = a_d(\delta) < \infty$ such that for any $t, \delta > 0$, $a \geq a_d$, $\frac{r}{a} \in [\delta, 1)$,*

$$\sup_{x \in \mathbb{B}_a} \mathbb{P}_x(\tau(a) > ta^2) < C^{-1}e^{-Ct}, \tag{3.1}$$

$$\sup_{x \in a\mathbb{K} \setminus \mathbb{B}_r} \mathbb{P}_x(\sigma(a, r) > ta^2) < C^{-1}e^{-Ct}, \tag{3.2}$$

$$\inf_{x \in \overline{\mathbb{B}}_{a/2}} \mathbb{P}_x(\tau(a) > a^2) > C. \tag{3.3}$$

In proving Lemma 3.1 we rely on the following invariance principle in bounded uniform domains, which allows us to transform hitting probabilities of SRW to the corresponding probabilities for an RBM.

Lemma 3.2. [4, 6] Fix a bounded uniform domain $D \subseteq \mathbb{R}^d$ and let $Y_t^n := n^{-1}Y_{\lfloor n^2 t \rfloor}$ denote the SRW on $D \cap (n^{-1}\mathbb{Z})^d$, induced by the discrete-time SRW $\{Y_t\}$ on $nD \cap \mathbb{Z}^d$. If $Y_0^n = x_n \rightarrow x \in \overline{D}$, then $\{Y_t^n; t \geq 0\}$ converges weakly in $D([0, \infty), \overline{D})$ to $\{W_{\kappa t}; t \geq 0\}$, where W is the RBM on D starting from x , time changed by constant κ .

Proof. Lemma 3.2 merely adapts facts from [6, Theorem 3.17 and Section 4.2] to our context (alternatively, it also follows by strengthening [4, Theorem 3.6] as suggested in [4, Remark 3.7]). The original result presented in [6] is for variable-speed and constant-speed random walks (vSRW, cSRW) on bounded uniform domain with random conductances uniformly bounded up and below. We are in a special case where all edges in $nD \cap \mathbb{Z}^d$ are present and have equal non-random conductance. Hence, here the cSRW is merely a continuous-time SRW Z_t of unit jump rate on $nD \cap \mathbb{Z}^d$ and further the invariance principle holds for $Z_t^n := n^{-1}Z_{n^2 t}$ and any choice of $x \in \overline{D}$. Indeed, while RBM W_t constructed via Dirichlet forms is typically well defined only for a quasi-everywhere starting point in \overline{D} , here this can be refined to every starting point. This is because in a uniform domain, such RBM admits a jointly-continuous transition density $p(t, x, y)$ on $\mathbb{R}_+ \times \overline{D} \times \overline{D}$ of Aronson’s type (see [12, Theorem 3.10]), thereby eliminating the exceptional set in [10, Theorem 4.5.4].

It remains only to infer the invariance principle for the discrete-time SRW $\{Y_t^n\}$ out of the invariance principle for $\{Z_t^n\}$. To this end, recall the representation $Y_t^n = Z_{n^{-2}L(n^2t)}^n$ for $L(t) := \inf\{s \geq 0 : N(s) = \lfloor t \rfloor\}$ and the independent Poisson process $N(t)$ of intensity one. Now, fixing T finite, by the functional strong law of large numbers for Poisson processes,

$$\sup_{t \in [0, T]} |n^{-2}L(n^2t) - t| \xrightarrow{a.s.} 0, \quad \text{for } n \rightarrow \infty.$$

Further, by [6, Proposition 3.10 and Section 4.2], for any $r > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}_{n x_n} \left(\sup_{|s_1 - s_2| \leq \delta, s_i \leq T} |Z_{s_2}^n - Z_{s_1}^n| > r \right) = 0.$$

Hence,

$$\sup_{0 \leq t \leq T} |Y_t^n - Z_t^n| = \sup_{0 \leq t \leq T} |Z_{n^{-2}L(n^2t)}^n - Z_t^n| \xrightarrow{P} 0.$$

and it follows that $(Y_t^n; t \geq 0) \xrightarrow{d} (W_{\kappa t}; t \geq 0)$ as $n \rightarrow \infty$. □

Remark 3.3. Lemma 3.2 generalizes to $Y_t^{a_n} \xrightarrow{d} W_{\kappa t}$, for $Y_t^{a_n} := a_n^{-1}Y_{\lfloor a_n^2 t \rfloor}$ that is induced by the discrete-time SRW on $a_n D \cap \mathbb{Z}^d$ and any fixed $a_n \uparrow \infty$ (just note that the conditions laid out in [6, first paragraph, Page 13] hold with a_n replacing n).

Proof of Lemma 3.1. Consider the RBM W on $\overline{\mathbb{K}} \supseteq \mathbb{B}_2$ and the rescaled discrete time SRW $Z_t^a := a^{-1}Z_{\lfloor a^2 t \rfloor}$. Starting with the proof of (3.1), for $a > 0$ and $y \in \overline{\mathbb{K}}$, let

$$q^{\text{RW}}(a, y) := \mathbb{P}_y(Z_s^a \in \mathbb{B}_1, \forall s \leq 1), \quad m^{\text{RW}}(a) := \sup_{y \in \mathbb{B}_1 \cap (a^{-1}\mathbb{Z})^d} q^{\text{RW}}(a, y),$$

$$q^{\text{BM}}(y) := \mathbb{P}_y(W_{\kappa s} \in \mathbb{B}_1, \forall s \leq 1), \quad m^{\text{BM}} := \sup_{y \in \overline{\mathbb{B}}_1} q^{\text{BM}}(y).$$

Then, by the Markov property of the srw, for any $a, t > 0$ and $x \in \mathbb{B}_a \cap \mathbb{Z}^d$,

$$\begin{aligned} \mathbb{P}_x(\tau(a) > ta^2) &= \mathbb{P}_x(Z_s \in \mathbb{B}_a, \forall s \leq ta^2) = \mathbb{P}_{a^{-1}x}(Z_s^a \in \mathbb{B}_1, \forall s \leq t) \\ &\leq \left[\sup_{y \in \mathbb{B}_1 \cap (a^{-1}\mathbb{Z})^d} q^{\text{rw}}(a, y) \right]^{\lfloor t \rfloor} = m^{\text{rw}}(a)^{\lfloor t \rfloor}. \end{aligned} \tag{3.4}$$

An RBM on uniform domain admits jointly continuous, positive transition density ([12, Theorem 3.10]), and in particular $m^{\text{BM}} = 1 - 2\eta$ for some $\eta \in (0, 1/2)$. As we show in the sequel, setting $\xi := \frac{1-\eta}{1-2\eta} > 1$,

$$a_d := \sup\{a > 0 : m^{\text{rw}}(a) > \xi m^{\text{BM}}\}, \tag{3.5}$$

is finite. It then follows from (3.4) that for some positive C , all $a > a_d$ and $t > 0$,

$$\sup_{x \in \mathbb{B}_a \cap \mathbb{Z}^d} \mathbb{P}_x(\tau(a) > ta^2) \leq m^{\text{rw}}(a)^{\lfloor t \rfloor} \leq (\xi m^{\text{BM}})^{\lfloor t \rfloor} = (1 - \eta)^{\lfloor t \rfloor} \leq C^{-1} e^{-Ct}.$$

To complete the proof of (3.1), suppose to the contrary that $a_d = \infty$ in (3.5), namely $m^{\text{rw}}(a_l) > \xi m^{\text{BM}}$ for some $a_l \uparrow \infty$. Taking the uniformly bounded $y_l \in \mathbb{B}_1 \cap (a_l^{-1}\mathbb{Z})^d$ such that $q^{\text{rw}}(a_l, y_l) = m^{\text{rw}}(a_l)$, we pass to a sub-sequence $\{l_n\}$ along which $y_{l_n} \rightarrow x \in \overline{\mathbb{B}}_1$. Then, considering Remark 3.3 for the sequence a_{l_n} , we deduce that as $n \rightarrow \infty$,

$$m^{\text{rw}}(a_{l_n}) = q^{\text{rw}}(a_{l_n}, y_{l_n}) \rightarrow q^{\text{BM}}(x) \leq m^{\text{BM}},$$

in contradiction with our assumption that $m^{\text{rw}}(a_{l_n}) > \xi m^{\text{BM}}$ for some $\xi > 1$ and all n .

Likewise, whenever $x \in \overline{\mathbb{B}}_{a/2} \cap \mathbb{Z}^d$ we have that

$$\mathbb{P}_x(\tau(a) > a^2) = \mathbb{P}_{a^{-1}x}(Z_s^a \in \mathbb{B}_1, \forall s \leq 1) \geq \inf_{y \in \overline{\mathbb{B}}_{1/2} \cap (a^{-1}\mathbb{Z})^d} q^{\text{rw}}(a, y) := m_{\text{rw}}(a)$$

and by the same reasoning as before,

$$\liminf_{a \rightarrow \infty} m_{\text{rw}}(a) \geq \inf_{z \in \overline{\mathbb{B}}_{1/2}} \{q^{\text{BM}}(z)\} > 0,$$

yielding the bound (3.3). Next, fixing $\delta > 0$ we turn to the stopping time $\sigma(a, r)$ and set

$$\begin{aligned} q_*^{\text{rw}}(a, y) &:= \mathbb{P}_y(Z_s^a \notin \overline{\mathbb{B}}_\delta, \forall s \leq 1), & m_*^{\text{rw}}(a) &:= \sup_{y \in (\mathbb{K} \setminus \overline{\mathbb{B}}_\delta) \cap (a^{-1}\mathbb{Z})^d} q_*^{\text{rw}}(a, y), \\ q_*^{\text{BM}}(y) &:= \mathbb{P}_y(W_{\kappa_s} \notin \overline{\mathbb{B}}_\delta, \forall s \leq 1), & m_*^{\text{BM}} &:= \sup_{y \in \mathbb{K} \setminus \overline{\mathbb{B}}_\delta} q_*^{\text{BM}}(y), \end{aligned}$$

getting by Markov property of the srw that for any $a, t > 0$, $r/a \in [\delta, 1)$ and $x \in (a\mathbb{K} \setminus \mathbb{B}_r) \cap \mathbb{Z}^d$

$$\begin{aligned} \mathbb{P}_x(\sigma(a, r) > ta^2) &= \mathbb{P}_x(Z_s \notin \overline{\mathbb{B}}_r, \forall s \leq ta^2) \leq \mathbb{P}_{a^{-1}x}(Z_s^a \notin \overline{\mathbb{B}}_\delta, \forall s \leq t) \\ &\leq \left[\sup_{y \in (\mathbb{K} \setminus \overline{\mathbb{B}}_\delta) \cap (a^{-1}\mathbb{Z})^d} q_*^{\text{rw}}(a, y) \right]^{\lfloor t \rfloor} = m_*^{\text{rw}}(a)^{\lfloor t \rfloor}. \end{aligned} \tag{3.6}$$

By the same arguments as in case of (3.4), again $m_*^{\text{BM}} = 1 - 2\eta$ for some $\eta \in (0, 1/2)$, and in view of Remark 3.3 the corresponding constant a_d as in (3.5), is finite, with (3.6) thus yielding (3.2). \square

Equipped with Lemma 3.1 we can now establish the following srw analog of Lemma 2.3.

Lemma 3.4. Let \mathbb{P}_x denotes the law of srw $\{Z_t, t \geq 0\}$ on $a\mathbb{K} \cap \mathbb{Z}^d$, with $Z_0 = x \in \mathbb{Z}^d$.

(a). For $\delta \in (0, 1/2)$, there exist $C = C_d(\delta) > 0$ and $M_d = M_d(\delta)$ finite, such that for all $M \geq M_d$, and any $T \geq Ma^2 \log a$, $a - M \geq r \geq a\delta$,

$$\inf_{x \in r\mathbb{K}} \mathbb{P}_x(\exists s \leq T : |Z_s| = 0) \geq C^{-1} \left[\frac{T}{a^d} \wedge 1 \right], \tag{3.7}$$

$$\sup_{x \in \partial(r\mathbb{K} \cap \mathbb{Z}^d)} \mathbb{P}_x(\exists s \leq T : |Z_s| = 0) \leq C \left[\frac{T}{a^d} \wedge 1 \right]. \tag{3.8}$$

(b). The uniform bound (3.8) applies for srw $\{Z_t\}$ on growing domains $\tilde{\mathbb{D}}_t \supseteq \mathbb{B}_{a+1} \cap \mathbb{Z}^d$, starting at arbitrary $Z_0 \in \mathbb{B}_{a\delta}^c$.

Proof. (a). We adapt the proof of Lemma 2.3 to the current setting of discrete time srw Z_t on $a\mathbb{K} \cap \mathbb{Z}^d$, by taking throughout $\epsilon = 0$ and re-defining the excursions of length $L_k := \sigma_{k+1} - \sigma_k$, $k \geq 0$, to be determined now by the stopping times $\sigma_0 = 0$ and

$$\begin{aligned} \tau_k &:= \inf\{t \geq \sigma_{k-1} : Z_t \in \mathbb{B}_a^c\}, \quad k \geq 1 \\ \sigma_k &:= \inf\{t \geq \tau_k : Z_t \in \overline{\mathbb{B}}_{a/2}\}. \end{aligned}$$

Since the laws of increments of srw are not invariant to rotations, $x \mapsto m(\theta, x) = \mathbb{E}_x[e^{\theta L_0}]$ is not a radial function. However, replacing Lemma 2.2 (which we used when bounding $m(\theta, x)$ in case of Brownian motion), by the *universal* bounds of Lemma 3.1, yields (2.12) and (2.13) for the srw case considered here. Thereby, applying the discrete analogue of (2.9)

$$b(x) := \mathbb{P}_x(\inf_{t \leq \tau_1} |Z_t| = 0) = c_d(|x|^{2-d} - a^{2-d}) + O(|x|^{1-d}), \tag{3.9}$$

where $0 < c_d < \infty$ is a dimensional constant (see [19, Proposition 1.5.9]), at $x \in \partial(r\mathbb{K} \cap \mathbb{Z}^d)$ and $x \in \partial(\overline{\mathbb{B}}_{a/2} \cap \mathbb{Z}^d)$, yields the srw analog of (2.10), out of which the stated conclusions follow.

(b). Let $I_k := [\sigma_k, \tau_{k+1})$, $k \geq 0$. Our assumptions that $Z_0 \in \mathbb{B}_{a\delta}^c$ and $\tilde{\mathbb{D}}_t \supseteq \mathbb{B}_{a+1} \cap \mathbb{Z}^d$ result in $\{Z_t, t \in I_k\}$ having for each $k \geq 0$ the same conditional law given Z_{σ_k} , as in part (a). Since the event $|Z_t| = 0$ can only occur for $t \in \cup_k I_k$, the derivation leading to the srw analog of (2.10) applies here as well. Further, conditional on $Z_{\sigma_k} = x$, each L_k , $k \geq 1$, stochastically dominates the random variable $\tau(a)$ of Lemma 3.1 starting at same point x . Consequently

$$\mathbb{E}_x[e^{-\hat{\theta}L_k/a^2}] \leq \mathbb{E}_x[e^{-\hat{\theta}\tau(a)/a^2}],$$

and utilizing the uniform in x and a control on the r.h.s. due to (3.1), establishes yet again the analog of (2.13). Examining the proof of (2.5) in Lemma 2.3 we see that this suffices for re-producing the corresponding uniform upper bound (3.8). \square

Proof of Theorem 1.4. (a). Fix $f(t)$ such that $J_f < \infty$ and consider the srw $\{Y_t\}$ on $\mathbb{D}_t \subseteq \mathbb{Z}^d$, $d \geq 3$ for which Assumption 1.1 holds. Similarly to Step II of the proof of Theorem 1.15, for $a_l := (c + 1)^l$, $l \geq 1$, define

$$\begin{aligned} t_l &:= \inf\{s \geq 1 : \mathbb{D}_s \cap \mathbb{B}_{a_{l+1}}^c \neq \emptyset\}, \\ \tau_l &:= \inf\{s \geq 0 : Y_s \in \mathbb{B}_{a_l}^c\}, \\ \tilde{\Gamma}_l &:= \{\exists t \in [\tau_l, \tau_{l+1}) : Y_t = 0\}. \end{aligned}$$

With $f(\cdot)$ unbounded, for any l eventually $\mathbb{D}_s \supseteq \mathbb{B}_{a_l} \cap \mathbb{Z}^d$ and by the transience of the srw on \mathbb{Z}^d , necessarily τ_l are a.s. finite. Thus, by Borel-Cantelli I,

$$\sum_l \mathbb{P}(\tilde{\Gamma}_l) < \infty \quad \Rightarrow \quad \mathbb{P}_0(Y_t = 0 \text{ f.o.}) = 1. \tag{3.10}$$

Turning to bound $\mathbb{P}(\tilde{\Gamma}_l)$, note that $\tau_{l+1} \geq t_l$ and $\mathbb{D}_{t_l} \supseteq \mathbb{B}_{1+a_l} \cap \mathbb{Z}^d$ (by Assumption 1.1 and the choice of a_l). Hence, by (3.1), we have that for some constants $C_d > 0$ and $l_d < \infty$, all $l \geq l_d$ and $t \geq 0$,

$$\mathbb{P}(\tau_{l+1} - t_l > ta_l^2) < C_d^{-1}e^{-C_d t}. \tag{3.11}$$

Let $\Delta T_l := (t_l - t_{l-1})$ and for $\delta = 1/(c + 1) < 1/2$ and $M_d = M_d(\delta)$ of Lemma 3.4, set $T_l^* := M_d \log a_l$ and $T_l = \Delta T_l + T_l^* a_l^2$. Since $\tau_l \geq t_{l-1}$ the length of $[\tau_l, \tau_{l+1})$ is at most ΔT_l plus the length of $[t_l, \tau_{l+1})$, which by (3.11) is with high probability under $T_l^* a_l^2$. Further, $\mathbb{D}_{\tau_l} \supseteq \mathbb{D}_{t_{l-1}} \supseteq \mathbb{B}_{1+a_{l-1}} \cap \mathbb{Z}^d$ and $Y_{\tau_l} \in \mathbb{B}_{a_l}^c$, hence from part (b) of Lemma 3.4 we have that,

$$\begin{aligned} \mathbb{P}(\tilde{\Gamma}_l) &\leq \mathbb{P}(\tau_{l+1} - t_l > T_l^* a_l^2) + \mathbb{P}(\exists s \in [\tau_l, \tau_l + T_l] : Y_s = 0) \\ &\leq C_d^{-1}e^{-C_d T_l^*} + C a_{l-1}^{-d} T_l. \end{aligned} \tag{3.12}$$

With our choice of a_l growing exponentially in l , the terms $e^{-C_d T_l^*}$ and $a_l^{2-d} T_l^*$ in the bound (3.12) are summable over $l \in \mathbb{N}$. Hence, the left-hand-side of (3.10) is finite whenever $\sum_l a_{l-1}^{-d} \Delta T_l$ is finite. Further, Assumption 1.1 and our definition of t_l imply that $f(t_l - 1) \leq 1 + a_{l+1}$. Thus,

$$J_f \geq \sum_{l \geq 2} f(t_l - 1)^{-d} \Delta T_l \geq (1 + c)^{-3d} \sum_{l \geq 2} a_{l-1}^{-d} \Delta T_l.$$

Consequently, finite J_f results in $\mathbb{P}_0(Y_t = 0 \text{ f.o.}) = 1$, which by Proposition 4.2 extends to $\mathbb{P}_0(Y_t = y \text{ f.o.}) = 1$ for all $y \in \mathbb{Z}^d$, as claimed.

(b). Fix $f \in \mathcal{F}_*$ such that $J_f = \infty$ and $\mathbb{K} \in \mathcal{K}$. Since $J_{f/r} = \infty$ for any $r > 0$ and $\mathbb{D}_t = (f(t)/r)(r\mathbb{K}) \cap \mathbb{Z}^d$, taking r large enough we have with no loss of generality that $\mathbb{K} \supseteq \mathbb{B}_2$. Then, considering the srw on \mathbb{D}_t , upon replacing (2.4) by (3.7), the argument we have used in Step I of the proof of Theorem 1.15 applies here as well, apart from the obvious notational changes (of replacing \mathbb{B}_{a_l} and $\bar{\mathbb{B}}_{a_{l-1}}$ in (2.18) by $a_l \mathbb{K} \cap \mathbb{Z}^d$ and the collection of all $x \in \mathbb{Z}^d$ within distance one of $a_{l-1} \mathbb{K}$, respectively). \square

4 On recurrence probability independence of target states

The following, xy -recurrence property, generalizes Definition 1.14 to arbitrary starting and target locations, $x, y \in \mathbb{R}^d$, respectively.

Definition 4.1. Suppose $\mathbb{D}_t \uparrow \mathbb{R}^d$, $x \in \mathbb{D}_0$, $y \in \mathbb{R}^d$. The sample path x_t of a stochastic process $t \mapsto x_t \in \mathbb{D}_t$ is xy -recurrent if $x_0 = x$ and the event $A(y) := \cap_{\epsilon > 0} A_\epsilon(y)$ occurs, where

$$\begin{aligned} \sigma_\epsilon^{(0)} &:= \inf\{t \geq 0 : \mathbb{D}_t \supseteq \mathbb{B}_\epsilon + x\}, \\ \tau_\epsilon^{(i)} &:= \inf\{t \geq \sigma_\epsilon^{(i-1)} : |x_t - y| < \epsilon\}, \quad i \geq 1 \\ \sigma_\epsilon^{(i)} &:= \inf\{t \geq \tau_\epsilon^{(i)} : |x_t - y| > 1/2\}, \\ A_\epsilon(y) &:= \{\tau_\epsilon^{(i)} < \infty, \forall i\}. \end{aligned}$$

Proposition 4.2. Suppose $\{X_t\}$ is a srw on $\mathbb{D}_t \uparrow \mathbb{Z}^d$, or alternatively that (X_t, \mathbb{D}_t) is the RBMG of Definition 1.13 with \mathbb{D}_0 open connected set and $\mathbb{D}_t \uparrow \mathbb{R}^d$. Then, the probability q_{xy} of xy -recurrence does not depend on y . In case of RBMG, if $q_{zy} \in \{0, 1\}$ for some $z \in \mathbb{D}_0$ then $q_{xy} = q_{zy}$ for all $x \in \mathbb{D}_0$, whereas in case of srw, if $q_{zy} = 0$ for some $z \in \mathbb{D}_0$ then $q_{xy} = 0$ whenever $\|x - z\|_1$ is even.

Remark 4.3. Adapting the approach we use for the RBMG, it is not hard to show that for continuous time srw (on growing domains $\mathbb{D}_t \uparrow \mathbb{Z}^d$), having $q_{zz} \in \{0, 1\}$ for some $z \in \mathbb{D}_0$

results in $q_{xy} = q_{zz}$ for all $x \in \mathbb{D}_0$ and $y \in \mathbb{Z}^d$. This approach is based on the equivalence of hitting measures of suitable sets when starting the process at nearby initial states. This however does not apply for discrete time SRW, hence our limited conclusion in that case.

Proof. This proof consists of the following four steps. Starting with the SRW we show in Step I that q_{xy} does not depend on y , then for $x, z \in \mathbb{D}_0$ with $\|x - z\|_1$ even, we prove in Step II that $q_{zz} = 0$ implies $q_{xx} = 0$. In case of the RBMG we have that $q_{xy}^\epsilon := \mathbb{P}_x(A_\epsilon(y)) \downarrow q_{xy}$ and deduce the stated claims upon showing in Step III that if $q_{zz} \in \{0, 1\}$ then $q_{xz} = q_{zz}$ for any $x, z \in \mathbb{D}_0$, then conclude in Step IV that $q_{xy}^\epsilon = q_{xx}^\epsilon$ for any fixed $\epsilon > 0$ and all $y \in \mathbb{R}^d$ (even when $0 < q_{xx} < 1$).

Step I. For the SRW X_t on $\mathbb{D}_t \subseteq \mathbb{Z}^d$ and fixed $s \in \mathbb{N}$ we denote by $\mathbb{P}_x^s(\cdot)$ the law of SRW X_t on the shifted-domains \mathbb{D}_{t+s} starting at $X_0 = x$. Then, for any $x, y \in \mathbb{Z}^d$ and $s \geq 0$,

$$q_{xy}(s) := \mathbb{P}(X_t = y \text{ i.o.} \mid X_s = x) = \mathbb{P}_x^s(X_t = y \text{ i.o.}),$$

with $q_{xy} := q_{xy}(0)$. Since $\mathbb{D}_t \uparrow \mathbb{Z}^d$, clearly any $y, w \in \mathbb{Z}^d$ are also in \mathbb{D}_t provided $t \geq t_0(y, w)$ is large enough, with some non-self-intersecting path in \mathbb{D}_{t_0} connecting y and w . Setting $\mathcal{F}_t^X := \sigma\{X_s, s \leq t\} \uparrow \mathcal{F}_\infty$ and events $\Gamma_{s,t,z,w} := \{X_s = z, X_u = w \text{ some } u > t\}$, we thus have $\eta = \eta(y, w) > 0$ such that for any starting point x , all z, s and $t \geq t_0 \vee s$,

$$\mathbb{P}_x(\Gamma_{s,t,z,w} \mid \mathcal{F}_t^X) \geq \eta \mathbb{I}_{\{X_s=z, X_t=y\}}.$$

Further as $t \rightarrow \infty$ we have that

$$\Gamma_{s,t,z,w} \downarrow \Gamma_{s,z,w} := \{X_s = z \text{ and } X_u = w \text{ i.o. in } u\}.$$

Clearly, $\Gamma_{s,z,w} \in \mathcal{F}_\infty$ so it follows by Lévy's upward theorem (and dominated convergence, see [9, Theorem 5.5.9]), that for any x , a.s.

$$\mathbb{I}_{\Gamma_{s,z,w}} = \mathbb{P}_x(\Gamma_{s,z,w} \mid \mathcal{F}_\infty) = \lim_{t \rightarrow \infty} \mathbb{P}_x(\Gamma_{s,t,z,w} \mid \mathcal{F}_t^X) \geq \eta \limsup_{t \rightarrow \infty} \mathbb{I}_{\{X_s=z, X_t=y\}} = \eta \mathbb{I}_{\Gamma_{s,z,y}}.$$

The same applies with the roles of y and w exchanged and consequently, a.s. $\Gamma_{s,z,y} = \Gamma_{s,z,w}$ for all $z, y, w \in \mathbb{Z}^d$ and $s \geq 0$. In particular, $q_{zy}(s) = \mathbb{P}(\Gamma_{s,z,y} \mid X_s = z)$ is thus independent of y , for any z and $s \geq 0$.

Step II. Assuming now that $q_{zz} = q_{zz}(0) = 0$ for some $z \in \mathbb{D}_0$, we have from Step I that $q_{zx} = 0$. As explained before (in Step I), $s_0 := \inf\{t : \mathbb{P}_z(X_t = x) > 0\}$ is a finite integer and clearly $\mathbb{P}_z(X_{2s+s_0} = x) > 0$ for any $s \geq 0$. By the Markov property at time $2s + s_0$,

$$0 = q_{zx} \geq \mathbb{P}_z(X_{2s+s_0} = x, X_t = x \text{ i.o.}) = \mathbb{P}_z(X_{2s+s_0} = x)q_{xx}(2s + s_0).$$

Consequently, for any $s \geq 0$,

$$\mathbb{P}_x(X_{2s+s_0} = x, X_t = x \text{ i.o.}) = \mathbb{P}_x(X_{2s+s_0} = x)q_{xx}(2s + s_0) = 0.$$

Starting at $X_0 = x$, the event $\{X_t = x\}$ is possible only at t even. Since $\|x - z\|_1$ is even, so is the value of s_0 and from the preceding we know that \mathbb{P}_x -a.s. any visit of x at even integer larger than s_0 results in only finitely many visits to x . Since there can be only finitely many visits of x up to time s_0 , we conclude that $q_{xx} = 0$.

Step III. Dealing hereafter with the RBMG, recall that $A_\epsilon(y) \downarrow A(y)$ for $A_\epsilon(y) = \{\exists s_k, u_k \uparrow \infty : |X_{s_k} - y| < \epsilon, |X_{u_k} - y| > 1/2, u_k \in (s_k, s_{k+1})\}$. Let $\mathbb{P}_x^s(\cdot)$ stand for the law of the RBMG $\{X_t\}$ on shifted-domains \mathbb{D}_{t+s} starting at $X_0 = x$, and $q_{xy}^\epsilon(s) := \mathbb{P}_x^s(A_\epsilon(y))$ with $q_{xy}^\epsilon = q_{xy}^\epsilon(0)$, so that $q_{xy}^\epsilon \downarrow q_{xy}^0 = q_{xy}$ when $\epsilon \downarrow 0$. We first prove that if $q_{zz} \in \{0, 1\}$ for some $z \in \mathbb{D}_0$ then $q_{xz} = q_{zz}$ for any $x \in \mathbb{D}_0$ such that $\frac{x+z}{2} + B_\alpha \subseteq \mathbb{D}_0$ for some

$\alpha > |x - z|/2$. Indeed, with $\mathbb{P}^{x,\alpha}$ denoting the joint law of $(X_{\tau_\alpha}, \tau_\alpha)$ for the first exit time $\tau_\alpha := \inf\{s \geq 0 : X_s \notin \frac{x+z}{2} + \mathbb{B}_\alpha\}$ and $X_0 = x$, we have that

$$q_{xz}^\epsilon = \int q_{x'z}^\epsilon(\gamma) d\mathbb{P}^{x,\alpha}(x', \gamma), \tag{4.1}$$

for any fixed $\epsilon > 0$. By dominated convergence this identity extends to $\epsilon = 0$ and considering it for $x = z$ (and $\epsilon = 0$), we deduce that $q_{x'z}(\gamma) = q_{zz} \in \{0, 1\}$ for $\mathbb{P}^{z,\alpha}$ -a.e. (x', γ) . By our assumption about the points x and z , the measure $\mathbb{P}^{z,\alpha}$ is merely the joint law of exit position and time for $\frac{x+z}{2} + \mathbb{B}_\alpha$ and Brownian motion X_s starting at z and as such it has a continuous Radon-Nikodym density with respect to the product of the uniform surface measure ω_{d-1} on $\partial(\frac{x+z}{2} + \mathbb{B}_\alpha)$ and the Lebesgue measure on $(0, \infty)$ (for example, see [14, Theorem 1 and 3]). Further, the latter density is strictly positive due to the continuity of (killed) Brownian transition kernel. Since the same applies to the corresponding Radon-Nikodym density between $\mathbb{P}^{x,\alpha}$ and $d\omega_{d-1} \times dt$, we conclude that $\mathbb{P}^{z,\alpha}$ and $\mathbb{P}^{x,\alpha}$ are mutually equivalent measures. In particular, $q_{x'z}(\gamma) = q_{zz}$ also for $\mathbb{P}^{x,\alpha}$ -a.e. (x', γ) and hence it follows from (4.1) at $\epsilon = 0$, that $q_{xz} = q_{zz}$. Now, since \mathbb{D}_0 is an open connected subset of \mathbb{R}^d , any $x, z \in \mathbb{D}_0$ are connected by a continuous path $w : [0, 1] \rightarrow \mathbb{D}_0$ such that $\text{dist}(w(\cdot), \mathbb{D}_0^c) > 0$. Consequently, there exists a finite sequence of points $\{w_k\}_{k=0}^K \subseteq \mathbb{D}_0$ with $w_0 = z$, $w_K = x$ and $\frac{w_{k-1} + w_k}{2} + \mathbb{B}_{\alpha_k} \subseteq \mathbb{D}_0$, for $\alpha_k > |w_k - w_{k-1}|/2$ and all $1 \leq k \leq K$. Applying iteratively the preceding argument, we conclude that if $q_{zz} \in \{0, 1\}$ then $q_{zz} = q_{w_1z} = \dots = q_{w_{K-1}z} = q_{xz}$, as claimed.

Step IV. Next, fixing $\epsilon > 0$ and $x \in \mathbb{D}_0$ we proceed to show that $q_{xz}^\epsilon = q_{xy}^\epsilon$ for any $z, y \in \mathbb{R}^d$. To this end, let $t_0 = t_0(y, z)$ be large enough so that \mathbb{D}_{t_0} contains the compact

$$\mathbb{K} := \{w : \inf_{\lambda \in [0,1]} |w - \lambda z - (1 - \lambda)y| \leq 1\}.$$

We set $\mathcal{F}_t^X := \sigma\{X_s, s \leq t\} \uparrow \mathcal{F}_\infty$ and consider the \mathcal{F}^X -stopping times $\theta_{t,z} \geq \tau_{t,z} \geq t$, given by

$$\tau_{t,z} := \inf\{u \geq t : |X_u - z| < \epsilon\}, \quad \theta_{t,z} := \inf\{v \geq \tau_{t,z} : |X_v - z| \geq 1/2\}$$

(with $\theta_{t,y} \geq \tau_{t,y} \geq t$ defined analogously). We claim that \mathbb{P}_x -a.s. for some non-random $\eta = \eta(z, y, \epsilon) > 0$ and any $t \geq t_0$,

$$\mathbb{P}_x(\theta_{t,y} < \infty | \mathcal{F}_{\theta_{t,z}}^X) \geq \eta \mathbb{I}_{\{\theta_{t,z} < \infty\}}. \tag{4.2}$$

Indeed, assuming without loss of generality that $\theta = \theta_{t,z}$ is finite, for any given $w = X_\theta \in z + \partial\mathbb{B}_{1/2}$ let $\psi(\cdot)$ denote the line segment from $\psi(0) = w$ to $\psi(1) = y$. The event $\Gamma_{w,y} := \sup_{s \in [0,1]} |X_{\theta+s} - \psi(s)| < \epsilon$ implies that $\tau_{t,y} \leq \theta + 1$ is finite and thereby also that $\theta_{t,y} < \infty$. Further, since $\psi(\cdot) \subseteq \mathbb{K} \subseteq \mathbb{D}_t$ is of distance $1/2 > \epsilon$ from $\partial\mathbb{K}$, the probability of $\Gamma_{w,y}$ given \mathcal{F}_θ^X is merely $\delta(w) := \mathbb{P}(\sup_{s \in [0,1]} |U_s + \psi(0) - \psi(s)| < \epsilon)$ for a standard d -dimensional Brownian motion $\{U_s\}$. Clearly, $\eta = \inf\{\delta(w) : |w - z| = 1/2\} > 0$, yielding (4.2). Now, considering the conditional expectation of (4.2) given \mathcal{F}_t^X , we find that

$$\mathbb{P}_x(\theta_{t,y} < \infty | \mathcal{F}_t^X) \geq \eta \mathbb{P}_x(\theta_{t,z} < \infty | \mathcal{F}_t^X).$$

Further, the \mathcal{F}_∞ -measurable event $A_\epsilon(y)$ is the limit of $\{\theta_{t,y} < \infty\}$ as $t \rightarrow \infty$ (and the same applies to $A_\epsilon(z)$), so it follows from Lévy's upward theorem (see [9, Theorem 5.5.9]), that \mathbb{P}_x -a.s.

$$\mathbb{I}_{A_\epsilon(y)} = \mathbb{P}_x(A_\epsilon(y) | \mathcal{F}_\infty) = \lim_{t \rightarrow \infty} \mathbb{P}_x(\theta_{t,y} < \infty | \mathcal{F}_t^X) \geq \eta \lim_{t \rightarrow \infty} \mathbb{P}_x(\theta_{t,z} < \infty | \mathcal{F}_t^X) = \eta \mathbb{I}_{A_\epsilon(z)}.$$

The same applies with the roles of y and z exchanged and consequently, \mathbb{P}_x -a.s. $A_\epsilon(y) = A_\epsilon(z)$. In particular, $q_{xy}^\epsilon = \mathbb{P}_x(A_\epsilon(y)) = \mathbb{P}_x(A_\epsilon(z)) = q_{xz}^\epsilon$, as claimed. \square

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