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Tightness of the recentered maximum of log-correlated Gaussian fields

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Dedicated to the memory of Gloria Ana Valenzuela Aranguiz, beloved mother.

Abstract

We consider a family of centered Gaussian fields on the d-dimensional unit box, whose covariance decreases logarithmically in the distance between points. We prove tightness of the recentered maximum of the Gaussian fields and provide exponentially decaying bounds on the right and left tails. We then apply this result to a version of the two-dimensional continuous Gaussian free field.

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1 Introduction

Main result

Let $\left\{\left(Y_\epsilon^x : x \in [0,1]^d\right)\right\}_{\epsilon>0}$ be a family of centered Gaussian fields indexed by the d-dimensional unit box $[0,1]^d$, where d is any positive integer. Suppose that the family satisfies, for some constant $0 < C_Y < \infty$ and all $x, y \in [0,1]^d$, $\epsilon > 0$,

$$|Cov\left(Y_\epsilon^x, Y_\epsilon^y\right) + \log\left(\max\{\epsilon, \|x-y\|\}\right)| \le C_Y \tag{1.1}$$

and

$$\mathbb{E}\left[\left(Y_\epsilon^x - Y_\epsilon^y\right)^2\right] \le C_Y \epsilon^{-1} \|x-y\| \text{ if } \|x-y\| \le \epsilon, \tag{1.2}$$

where $\|\cdot\|$ is Euclidean distance. Display (1.1) implies that the covariance is logarithmic for distant points and that the variance is nearly constant. The second condition is imposed so that the field does not vary too much for close points. Display (1.2), basic relations between the moments of Gaussian random variables and Kolmogorov's continuity criterion (see [1, Theorem 1.4.17]) imply that the fields have continuous modifications.

When $d = 2$, an example of a field satisfying (1.1) and (1.2) is the bulk of the mollified continuous Gaussian free field (MGFF), which will be defined in Section 3.1, and will be the object of our attention in Section 3.

Set $m_\epsilon = m_{\epsilon,d} = \sqrt{2d}\log(1/\epsilon) - \frac{3/2}{\sqrt{2d}}\log\log(1/\epsilon)$. The main result of this paper is:

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Theorem 1.1. *There exist constants* $0 < c, C < \infty$ *(depending on* C_Y *and* d*) and a small* $\epsilon_0 > 0$ *(depending on* C_Y *and* d*) such that, for all* $\epsilon \in (0, \epsilon_0]$ *and all* $\lambda \geq 0$*,*

$$\mathbb{P}\left(\left|\max_{x\in[0,1]^d} Y_\epsilon^x - m_\epsilon\right| \geq \lambda\right) \leq Ce^{-c\lambda} \tag{1.3}$$

Theorem 1.1 implies, in particular, that $\left\{\max_{x\in[0,1]^d} Y_\epsilon^x - m_\epsilon : \epsilon \in (0, \epsilon_0]\right\}$ is tight and that, for all $\epsilon \in (0, \epsilon_0]$,

$$\left|\mathbb{E}\left[\max_{x\in[0,1]^d} Y_\epsilon^x\right] - m_\epsilon\right| \leq C$$

for some constant C depending on C_Y and d.

The main idea of the proof of Theorem 1.1 is to use Slepian's Lemma (see [2, Theorem 2.2.1]) to compare the maximum of the field Y_ϵ with the maximum of the modified branching Brownian motion (MBBM), which is a continuous time version of the modified branching random walk (MBRW), a field introduced by Bramson and Zeitouni in [3]. Since Slepian's Lemma only allows comparison of fields with the same index set, we will add an appropriately chosen independent continuous field to the MBBM. Adding an independent continuous field to the MBBM does not change the maximum much, provided the continuous field is small and smooth enough. These fields are defined in detail in Section 2.1. After defining the fields, we compare the right and left tails in Sections 2.2 and 2.3, respectively. We then show, in Section 3, that Theorem 1.1 implies tightness of the recentered maximum of the MGFF.

A comment on constants: c will always denote a small positive constant and C will always denote a large positive constant. Both constants are allowed to change from line to line. The dependence of the constants will be explicit or will be clear from the context. The phrase "absolute constant" will refer to fixed numbers that are independent of everything.

Related work

Our approach is motivated by recent advances in the study of the two dimensional discrete Gaussian free field (DGFF). In [3], Bramson and Zeitouni computed the expected maximum of the DGFF up to an order 1 error and concluded tightness of the recentered maximum. In [4], Ding obtained bounds on the right and left tail of the recentered maximum of the DGFF. Later on, in [5], Bramson, Ding and Zeitouni proved convergence in distribution of the recentered maximum. The approach of this line of research is to use first and second moment methods, together with decomposition properties of the DGFF, to obtain good estimates on tail events. Previous work on the DGFF includes [6], where Bolthausen, Deuschel and Giacomin obtained asympotics for the maximum of the DGFF, and [7], where Daviaud studied the extreme points of the DGFF. On the other hand, previous work on the continuous Gaussian free field (CGFF) includes [8], where Hu, Miller, and Peres studied the Hausdorff dimension of the "thick points" of the MGFF, which are closely related to the work of Daviaud. We also mention [9] for a nice discussion of Gaussian fields induced by Markov processes, and [10] for a survey on the CGFF.

Our main result implies, in particular, an analog of [3, Theorem 1.1] for the MGFF. Our approach consists on extending the MBBM by Brownian sheet, so that it is possible to compare the extended field with scaled log-correlated continuous fields. Log-correlated Gaussian fields are subject of current interest (see [11], [12], [13]). In particular, in [12], Madaule proved convergence for stationary centered Gaussian fields

$\left(Z_\epsilon(x) : x \in [0,1]^d\right)$ whose covariance satisfies

$$Cov(Z_\epsilon(0), Z_\epsilon(x)) = \int_0^{\log(1/\epsilon)} k(e^r x)dr,$$

where the fixed kernel $k : \mathbb{R}^d \to \mathbb{R}$ is of class C^1, vanishes outside $[-1,1]^d$, and satisfies $k(0) = 1$. Theorem 1.1 has weaker conditions on the covariance structure, and consequently, only tightness is achieved.

In [13], the authors proved the so called "Freezing Theorem for GFF in planar domains" for a sequence of Gaussian fields approximating the continuous GFF by cutting-off white noise, so that the covariance kernel is proportional to the function $G_t : [0,1]^2 \times [0,1]^2 \to \mathbb{R}$ given by

$$G_t(x,y) = \int_{e^{-t}}^{\infty} p_{\partial[0,1]^2}(r,x,y)dr,$$

where $p_{\partial[0,1]^2}(r,x,y)$ is the transition probability density of a Brownian motion killed at $\partial[0,1]^2$. In the present paper, we consider a sequence of fields approximating the GFF by mollifying the Green function (see (3.5)), and we prove tightness. Convergence for the MGFF is expected to follow by adapting of the arguments given in [5].

2 Comparison to the MBBM

2.1 Auxiliary fields

In this subsection, we rigorously introduce the fields we mentioned in Section 1. A few properties of these fields will be stated; the proofs of these properties will be given in the Appendix.

In order to define these fields, it will be notationally more convenient to use $[0,1)^d$ instead of $[0,1]^d$ as the index set. This will not affect the main result because the supremum of Y_ϵ over $[0,1)^d$ is the same, due to continuity, as the maximum over $[0,1]^d$.

Modified branching Brownian motion

We first divide $[0,1)^d$ into boxes of side length $\epsilon > 0$. Let $V_\epsilon = \left(\epsilon\mathbb{Z}^d\right) \cap [0,1)^d$ and, for $v = (v_i)_{1\le i\le d} \in V_\epsilon$, let $\square_\epsilon^v = (\prod_{1\le i\le d}[v_i, v_i+\epsilon)) \cap [0,1)^d$. Moreover, if $x \in \square_\epsilon^v$, let $[x] := v$. The set V_ϵ is, of course, a discretized version of $[0,1)^d$.

We now define the *modified branching Brownian motion* (MBBM) as the centered Gaussian field $\{\xi_\epsilon^v(t) : v \in V_\epsilon, 0 \le t \le \log(1/\epsilon)\}$ with covariance structure

$$Cov(\xi_\epsilon^v(t), \xi_\epsilon^u(s)) = \int_0^{\min\{t,s\}} \prod_{1\le i\le d} (1 - e^r\,|v_i - u_i|)_+ dr \tag{2.1}$$

for all $0 \le t, s \le \log(1/\epsilon)$ and $v, u \in V_\epsilon$, where $(\cdot)_+ = \max\{\cdot, 0\}$. For simplicity, write $\xi_\epsilon^v = \xi_\epsilon^v(\log(1/\epsilon))$.

Note that, for each point $v \in V_\epsilon$, the process $\left(\xi_\epsilon^v(t)\right)_t$ is a standard Brownian motion. Moreover, for each pair $v, u \in V_\epsilon$, the Brownian motions are correlated until $t = -\log\|v-u\|_\infty$, at which time their increments become independent. The end time is $t = \log(1/\epsilon)$, because, for the "usual" d-ary branching Brownian motion, it takes $\log(1/\epsilon)$ units of time to generate $|V_\epsilon|$ particles (see the proof of Proposition 4.3 for a definition of the usual d-ary branching Brownian motion).

It will be proved in the Appendix (see Proposition 4.1) that the MBBM exists and that it satisfies

$$Var(\xi_\epsilon^v) = \log(1/\epsilon) \tag{2.2}$$

and, for $v \neq u$ (so that $\|v-u\|_\infty \geq \epsilon$),

$$-\log \|v-u\|_\infty - C \leq Cov(\xi_\epsilon^v, \xi_\epsilon^u) \leq -\log \|v-u\|_\infty \tag{2.3}$$

for some constant C depending on d. The MBBM also satisfies (see Proposition 4.2)

$$\mathbb{P}\left(\max_{v\in V_\epsilon} \xi_\epsilon^v \geq m_\epsilon\right) \geq c > 0, \tag{2.4}$$

where c is a constant depending only on d. It will also be proved in the Appendix (see Proposition 4.3) that there exist constants $0 < c, C < \infty$ (depending on d) such that

$$\mathbb{P}\left(\max_{v\in A} \xi_\epsilon^v \geq m_\epsilon + z\right) \leq C\left(\epsilon^d |A|\right)^{1/2} e^{-cz} \tag{2.5}$$

for all $A \subset V_\epsilon$, $z \in \mathbb{R}$ and $\epsilon > 0$ small enough, where $|A|$ is the cardinality of A.

Brownian sheet

As mentioned before, we will need an additional continuous Gaussian field. For $x = (x_i)_{1\leq i\leq d} \in \mathbb{R}_+^d$, let ψ^x denote the centered standard Brownian sheet. Recall that it satisfies

$$\mathbb{E}\left[\psi^x\psi^y\right] = \prod_{1\leq i\leq d} \min\left\{x_i, y_i\right\}.$$

Define a new field $\left(\psi_\epsilon^x : x \in [0,1)^d\right)$, depending on a parameter $p \geq 1$, as follows: for $v \in V_\epsilon$, let l be the linear map from $\Box_\epsilon^v$ onto $[p,2p)^d$ sending v to $(p)_{1\leq i\leq d} = (p,p,\ldots,p)$. Set

$$\left(\psi_\epsilon^x : x \in \Box_\epsilon^v\right) :\stackrel{d}{=} \left(\psi^{l(x)} : x \in \Box_\epsilon^v\right) = \left(\psi^x : x \in [p,2p)^d\right) \tag{2.6}$$

for each $v \in V_\epsilon$, and choose ψ_ϵ^x and ψ_ϵ^y to be independent if $[x] \neq [y]$. Note that the collection of fields $\left\{(\psi_\epsilon^x : x \in \Box_\epsilon^v)\right\}_{v\in V_\epsilon}$ consists of i.i.d. copies of Brownian sheet on $[p,2p)^d$. Using the covariance structure of the Brownian sheet, it is not hard to see that

$$p^d \leq Var\left(\psi_\epsilon^x\right) \leq (2p)^d \tag{2.7}$$

for all $x \in [0,1)^d$, and that (see Proposition 4.5)

$$p^d\epsilon^{-1}\|x-y\|_1 \leq \mathbb{E}\left[\left(\psi_\epsilon^x - \psi_\epsilon^y\right)^2\right] \leq (2p)^d\epsilon^{-1}\|x-y\|_1 \tag{2.8}$$

for all $[x] = [y]$. Note that p can be chosen as large as desired.

To understand the motivation behind the previous definitions, we invite the reader to compare the bounds (1.1) and (1.2) with (2.3) and (2.8), respectively. These bounds will be used in the next sections.

We now proceed to the comparison of the right and left tail of the maximum of the field Y_ϵ (which was defined in Section 1 and satisfies (1.1) and (1.2)) and the maximum of an appropriate combination of the fields ξ_ϵ and ψ_ϵ (which will be specified in the next section). Note that we will only use Brownian sheet when comparing the right tail; for the left tail, we will compare directly the MBBM with the field Y_ϵ on a discrete index set.

2.2 The right tail

Recall from Section 1 that the field Y_ϵ satisfies (1.1) and (1.2), by definition.

Proposition 2.1. *For $\epsilon > 0$, let $(\xi_\epsilon^v : v \in V_\epsilon)$ and $\left(\psi_\epsilon^x : x \in [0,1)^d\right)$ be independent fields, defined as in* (2.1) *and* (2.6)*, respectively. Then, there exist $\delta > 0$ small enough and p large enough (depending on C_Y and d) such that, for all $\epsilon > 0$ small enough (depending on C_Y and d),*

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} Y_{\delta\epsilon}^{\delta x} \geq \lambda\right) \leq \mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x \geq \lambda\right)$$

for all $\lambda \in \mathbb{R}$, where $a(x) := \sqrt{\left(Var(Y_{\delta\epsilon}^{\delta x}) - Var(\psi_\epsilon^x)\right)/Var(\xi_\epsilon^{[x]})}$.

Proof. We first make sure that $Var(Y_{\delta\epsilon}^{\delta x}) - Var(\psi_\epsilon^x) \geq 0$, so that $a(x)$ is well defined. Note that (1.1) and (2.7) imply

$$Var(Y_{\delta\epsilon}^{\delta x}) - Var(\psi_\epsilon^x) \geq \log(1/\epsilon) + \log(1/\delta) - C_Y - (2p)^d \geq 0$$

for all $\epsilon > 0$ small enough (depending on C_Y, d and p). As we will see in this proof, p depends only on C_Y and d, so $a(x)$ is well defined for all $\epsilon > 0$ small enough, depending only on C_Y and d.

We now check the hypotheses of Slepian's Lemma (see [2, Theorem 2.2.1]). The variances of the fields $Y_{\delta\epsilon}^{\delta x}$ and $a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x$ are equal by the definition of $a(x)$. We first choose p so that $a(x) \leq 1$. Note that (1.1) and (2.7) imply

$$a(x)^2 = \frac{Var(Y_{\delta\epsilon}^{\delta x}) - Var(\psi_\epsilon^x)}{Var(\xi_\epsilon^{[x]})} \leq \frac{\log(1/\epsilon) + \log(1/\delta) + C_Y - p^d}{\log(1/\epsilon)},$$

so, by choosing p large enough (depending on C_Y, d and δ), we obtain $a(x) \leq 1$, for all x.

We now compare the covariance for points $x \neq y$, for which we distinguish two cases:

1. $[x] = [y]$ (that is, $\square_\epsilon^{[x]} = \square_\epsilon^{[y]}$). In this case, (1.2) and (2.8) imply

$$\mathbb{E}\left[\left(Y_{\delta\epsilon}^{\delta x} - Y_{\delta\epsilon}^{\delta y}\right)^2\right] \leq C_Y(\delta\epsilon)^{-1}\left\|\delta x - \delta y\right\| \leq p^d\epsilon^{-1}\left\|x - y\right\|_1 \leq \mathbb{E}\left[(\psi_\epsilon^x - \psi_\epsilon^y)^2\right]$$

$$\leq \mathbb{E}\left[\left(a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x - a(y)\xi_\epsilon^{[y]} - \psi_\epsilon^y\right)^2\right]$$

for p large enough (depending on C_Y and d). The last inequality is due to the independence between ξ_ϵ and ψ_ϵ.

2. $[x] \neq [y]$. In this case, we can apply (2.3) and the independence between ξ_ϵ, $\psi_\epsilon^{[x]}$ and $\psi_\epsilon^{[y]}$ to obtain

$$Cov(a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x, a(y)\xi_\epsilon^{[y]} + \psi_\epsilon^y) \leq a(x)a(y)Cov(\xi_\epsilon^{[x]}, \xi_\epsilon^{[y]}) \leq a(x)a(y)\left(-\log\|[x] - [y]\| + C\right).$$

But $a(x)a(y) \leq 1$, so

$$Cov(a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x, a(y)\xi_\epsilon^{[y]} + \psi_\epsilon^y) \leq -\log\|[x] - [y]\| + C.$$

Note that $-\log\|[x] - [y]\| \leq -\log\left(\max\{\epsilon, \|x - y\|\}\right) + C$. Applying (1.1), we obtain

$$-\log\left(\max\{\epsilon, \|x - y\|\}\right) + C \leq -\log\left(\max\{\delta\epsilon, \|\delta x - \delta y\|\}\right) - C_Y \leq Cov(Y_{\delta\epsilon}^{\delta x}, Y_{\delta\epsilon}^{\delta y})$$

for some $\delta > 0$ small enough (depending on C_Y). Proposition 2.1 follows now from Slepian's Lemma. □

Proposition 2.1 provides an upper bound for the right tail of the supremum of $Y_{\delta\epsilon}$ taken over the δ-box $\delta[0,1)^d$. The same proof works for any δ-box. Therefore, a union bound implies

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} Y_{\delta\epsilon}^x \geq \lambda\right) \leq \left(\frac{1}{\delta}\right)^d \mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x \geq \lambda\right) \tag{2.9}$$

for all $\lambda \in \mathbb{R}$.

We now provide an upper bound for the probability on the right hand side of the previous display. We first prove an upper bound on the supremum of the Brownian sheet.

Lemma 2.2. *There exist constants $0 < c, C < \infty$ (depending on p and d) such that*

$$\sup_{v\in V_\epsilon} \mathbb{P}\left(\sup_{x\in\square_\epsilon^v} \psi_\epsilon^x \geq \lambda\right) \leq Ce^{-c\lambda^2}$$

for all $\lambda \geq 0, \epsilon > 0$.

Proof. Let $v \in V_\epsilon$. Fernique's Majorizing Criterion (see [14, Theorem 4.1]) implies that

$$\mathbb{E}\left[\sup_{x\in\square_\epsilon^v} \psi_\epsilon^x\right] \leq C \sup_{x\in\square_\epsilon^v} \int_0^\infty \sqrt{-\log\left(\mu(B(x,r))\right)}dr$$

for some absolute constant C, where μ is the normalized d-dimensional Lebesgue measure on $\square_\epsilon^v$ and $B(x,r) = \left\{y \in \square_\epsilon^v : \mathbb{E}\left[(\psi_\epsilon^x - \psi_\epsilon^y)^2\right] \leq r^2\right\}$. But (2.8) implies

$$B(x,r) \supset \left\{y \in \square_\epsilon^v : (2p)^d\epsilon^{-1}\left\|y-x\right\|_1 \leq r^2\right\}.$$

Therefore, $\mu\left(B(x,r)\right) \geq cr^{2d}$ for some constant $c > 0$ depending on p and d. Applying the previous display and Fernique's Majorizing Criterion, we obtain

$$\mathbb{E}\left[\sup_{x\in\square_\epsilon^v} \psi_\epsilon^x\right] \leq C\int_0^\infty \sqrt{-\log\left(cr^{2d}\right)}dr \leq C < \infty,$$

where C depends on p and d. Borell's Inequality (see [2, Theorem 2.1.1]) and (2.7) imply

$$\mathbb{P}\left(\sup_{x\in\square_\epsilon^v} \psi_\epsilon^x \geq C + \lambda\right) \leq e^{-\lambda^2/(2(2p)^d)},$$

where C is the constant obtained in the previous display. Lemma 2.2 now follows from a change of variables. □

Proposition 2.3. *Let p and δ be as in Proposition 2.1. There exist constants $0 < c, C < \infty$ (depending on C_Y and d) such that*

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x \geq \lambda + m_\epsilon\right) \leq Ce^{-c\lambda}$$

for all $\lambda \geq 0$ and all $\epsilon > 0$ small enough (depending on C_Y and d).

Proof. By letting $\psi_\epsilon^{*,[x]} = \sup_{y\in\square_\epsilon^{[x]}} \psi_\epsilon^y$, we have

$$\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x \leq \max_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^{*,[x]}.$$

The previous display implies

$$\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x \geq m_\epsilon + \lambda \implies \sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^{*,[x]} \geq m_\epsilon + \lambda.$$

We now compute an upper bound for the right hand side of the previous display. Define the random sets $\Gamma_y = \{v \in V_\epsilon : \psi_\epsilon^{*,v} \in [y-1, y)\}$ for $y \geq 1$, and $\Gamma_0 = \{v \in V_\epsilon : \psi_\epsilon^{*,v} \leq 0\}$. Note that

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^{*,[x]} \geq m_\epsilon + \lambda\right) \leq \sum_{y\geq 0} \mathbb{P}\left(\sup_{x:[x]\in\Gamma_y} a(x)\xi_\epsilon^{[x]} \geq m_\epsilon + \lambda - y\right).$$

By the definition of $a(x)$ and the choice of p and δ in Proposition 2.1,

$$a(x)^2 = \frac{Var(Y_{\delta\epsilon}^{\delta x}) - Var(\psi_\epsilon^x)}{Var(\xi_\epsilon^{[x]})} \leq 1,$$

and by (1.1) and (2.7),

$$a(x)^2 \geq \frac{\log(1/\epsilon) + \log(1/\delta) - C_Y - (2p)^d}{\log(1/\epsilon)} \geq \frac{1}{2}$$

for $\epsilon > 0$ small enough (depending on δ, p, C_Y and d, all of which ultimately depend on C_Y and d). Therefore, the last three displays imply

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^{*,[x]} \geq m_\epsilon + \lambda\right) \leq \sum_{y\geq 0} \mathbb{P}\left(\max_{v\in\Gamma_y} \xi_\epsilon^v \geq m_\epsilon + \lambda - 2y\right). \tag{2.10}$$

But $\mathbb{P}\left(\max_{v\in\Gamma_y} \xi_\epsilon^v \geq m_\epsilon + \lambda - 2y\right) = \mathbb{E}\left[\mathbb{P}\left(\max_{v\in\Gamma_y} \xi_\epsilon^v \geq m_\epsilon + \lambda - 2y \mid \Gamma_y\right)\right]$. Since ψ_ϵ and ξ_ϵ are independent, from (2.5) we obtain

$$\mathbb{P}\left(\max_{v\in\Gamma_y} \xi_\epsilon^v \geq m_\epsilon + \lambda - 2y \mid \Gamma_y\right) \leq C\left(\epsilon^d\,|\Gamma_y|\right)^{1/2} e^{-c(\lambda-2y)}.$$

Then,

$$\mathbb{P}\left(\max_{v\in\Gamma_y} \xi_\epsilon^v \geq m_\epsilon + \lambda - 2y\right) \leq Ce^{-c(\lambda-2y)}\left(\mathbb{E}\left[\epsilon^d\,|\Gamma_y|\right]\right)^{1/2}. \tag{2.11}$$

But, by Lemma 2.2, $\mathbb{E}\left[|\Gamma_y|\right] = \sum_{v\in V_\epsilon} \mathbb{P}\left(\psi_\epsilon^{*,v} \in [y-1,y)\right) \leq C\epsilon^{-d}e^{-cy^2}$. For $y = 0$, we simply use $|\Gamma_0| \leq \epsilon^{-d}$. Therefore, from displays (2.10) and (2.11), we obtain

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^{*,[x]} \geq m_\epsilon + \lambda\right) \leq Ce^{-c\lambda}$$

for some constants $0 < c, C < \infty$ (depending on C_Y and d). □

Proof of Theorem 1.1, (1.3)*, the right tail.* Display (2.9) and Proposition 2.3 imply that there exist constants $0 < c, C < \infty$ (depending on C_Y and d) such that, for all $\epsilon > 0$ small enough (depending on C_Y and d),

$$\mathbb{P}\left(\max_{x\in[0,1)^d} Y_{\delta\epsilon}^x \geq m_\epsilon + \lambda\right) \leq \left(\frac{1}{\delta}\right)^2 \mathbb{P}\left(\max_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x \geq \lambda + m_\epsilon\right) \leq Ce^{-c\lambda}.$$

It is easy to see from the definition that $m_{\delta\epsilon} \leq m_\epsilon + C'$ for some C' depending on δ and d. Therefore,

$$\mathbb{P}\left(\max_{x\in[0,1)^d} Y_{\delta\epsilon}^x \geq m_{\delta\epsilon} + \lambda - C'\right) \leq Ce^{-c\lambda}.$$

The upper bound (1.3) for the right tail follows by adjusting the constants. □

2.3 The left tail

In this subsection we prove the upper bound (1.3) for the left tail. As previously mentioned, we can reduce the set under maximization to a discrete set. More precisely, if $\{D_\epsilon : \epsilon > 0\}$ is any collection of subsets of $[0,1)^d$, then

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} Y_\epsilon^x \le m_\epsilon - \lambda\right) \le \mathbb{P}\left(\sup_{x\in D_\epsilon} Y_\epsilon^x \le m_\epsilon - \lambda\right). \tag{2.12}$$

If we select D_ϵ appropriately, we can perform a comparison with the MBBM using Slepian's Lemma.

Proposition 2.4. *There exist $\delta, \rho > 0$ small enough (depending on C_Y and d) such that*

$$\mathbb{P}\left(\max_{u\in V_{\epsilon/\rho}} Y_{\delta\epsilon}^u \le \lambda\right) \le \mathbb{P}\left(\max_{u\in V_\epsilon\cap\rho[0,1)^d} b(u)\xi_\epsilon^u \le \lambda\right)$$

for all $\epsilon > 0$ and all $\lambda \in \mathbb{R}$, where $b(u) := \sqrt{Var(Y_{\delta\epsilon}^u)/Var(\xi_\epsilon^u)}$ for $u \in V_\epsilon \cap \rho[0,1)^d$.

Proof. Note that (1.1) and (2.3) imply that $b(u) \ge \frac{\log(1/\epsilon)+\log(1/\delta)-C_Y}{\log(1/\epsilon)}$, which is greater than 1 for $\delta > 0$ small enough (depending on C_Y).

Let $u, v \in V_{\epsilon/\rho}$, with $u \neq v$. Then, for $0 < \delta, \rho \le 1$, we have $\|u - v\| \ge \epsilon/\rho \ge \delta\epsilon$. Display (1.1) therefore implies

$$Cov(Y_{\delta\epsilon}^u, Y_{\delta\epsilon}^v) \le -\log\|u - v\| + C_Y.$$

Choose $\rho > 0$ small enough (depending on C_Y and d) so that

$$-\log\|u-v\| + C_Y \le -\log\|\rho u - \rho v\| - C \le Cov(\xi_\epsilon^{\rho u}, \xi_\epsilon^{\rho v}) \le Cov(b(\rho u)\xi_\epsilon^{\rho u}, b(\rho v)\xi_\epsilon^{\rho v}),$$

where the second to last bound follows from (2.3). All the hypotheses of Slepian's Lemma are satisfied, so

$$\mathbb{P}\left(\max_{u\in V_{\epsilon/\rho}} Y_{\delta\epsilon}^u \le \lambda\right) \le \mathbb{P}\left(\max_{u\in V_{\epsilon/\rho}} b(\rho u)\xi_\epsilon^{\rho u} \le \lambda\right)$$

for all $\lambda \in \mathbb{R}$. Proposition 2.4 follows by observing that $\rho V_{\epsilon/\rho} = V_\epsilon \cap \rho[0,1]^d$. □

Proposition 2.5. *Let $\rho > 0$ and $\left\{b(u) : u \in V_\epsilon \cap \rho[0,1]^d\right\}$ be as in Proposition 2.4. Then,*

$$\mathbb{P}\left(\max_{u\in V_\epsilon\cap\rho[0,1]^d} b(u)\xi_\epsilon^u \le m_\epsilon - \lambda\right) \le \mathbb{P}\left(\max_{u\in V_\epsilon\cap\rho[0,1]^d} \xi_\epsilon^u \le m_\epsilon - \lambda/2\right)$$

for all $\lambda \ge 0$ and all $\epsilon > 0$ small enough (depending on C_Y).

Proof. It follows from the definition of $b(u)$ and the choices made in Proposition 2.4 that, for all u,

$$1 \le b(u) = \sqrt{Var(Y_{\delta\epsilon}^u)/Var(\xi_\epsilon^u)} \le \sqrt{\frac{\log(1/\epsilon)+\log(1/\delta)+C_Y}{\log(1/\epsilon)}} \le 2$$

for small enough $\epsilon > 0$ (depending on C_Y and δ, which itself depends on C_Y). Let ν be the (a.s. well-defined) point that maximizes ξ_ϵ^u, for $u \in V_\epsilon \cap \rho[0,1]^d$. Then, the last display implies

$$b(\nu)\xi_\epsilon^\nu \le m_\epsilon - \lambda \implies \xi_\epsilon^\nu \le m_\epsilon/b(\nu) - \lambda/b(\nu) \le m_\epsilon - \lambda/2.$$

□

Our task is now to find an upper bound for the probability on the right hand side of Proposition 2.5.

Proposition 2.6. *There exist constants* $0 < c, C < \infty$ *(depending on* ρ *and* d*) such that*

$$\mathbb{P}\left(\max_{v \in V_\epsilon \cap \rho[0,1]^d} \xi_\epsilon^v \le m_\epsilon - \lambda\right) \le Ce^{-c\lambda}$$

for all $\lambda \ge 0$ *and all* $\epsilon > 0$ *small enough.*

Proof. We distinguish three cases for λ:

1) $\lambda \in [0, 2/\rho]$. In this case, the proposition is trivially true by simply adjusting the constants c and C so that $Ce^{-c\lambda} \ge 1$.

2) $\lambda \in (2/\rho, \sqrt{1/\epsilon}]$. Let $n := \left\lfloor \frac{\rho\lambda}{2} \right\rfloor$ and let $\left\{B^i : i = 1, \ldots, n\right\}$ be a collection of boxes of side length λ^{-1} inside $\rho[0,1)^d$, such that the distance between any pair of boxes is at least λ^{-1}. Set $B_\epsilon^i = B^i \cap V_\epsilon$. We claim that the field

$$\left(\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda) : v \in B_\epsilon^i\right)$$

is a copy of $(\xi_{\lambda\epsilon}^v : v \in V_{\lambda\epsilon})$, and that the fields $\left\{\left(\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda) : v \in B_\epsilon^i\right)\right\}_{1 \le i \le n}$ are independent. Indeed, if $v, u \in B_\epsilon^i$, then (2.1) implies

$$Cov(\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda), \xi_\epsilon^u - \xi_\epsilon^u(\log\lambda)) = \int_{\log(\lambda)}^{\log(1/\epsilon)} \prod_{1 \le j \le d} \left(1 - e^r |v_j - u_j|\right)_+ dr \tag{2.13}$$

$$= \int_0^{-\log(\lambda\epsilon)} \prod_{1 \le j \le d} \left(1 - e^r |\lambda v_j - \lambda u_j|\right)_+ dr,$$

and the set $\lambda B_\epsilon^i = \left\{\lambda v : v \in B_\epsilon^i\right\}$ coincides with $V_{\lambda\epsilon}$ after a translation. This shows that $\left(\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda) : v \in B_\epsilon^i\right) \overset{d}{=} (\xi_{\lambda\epsilon}^v : v \in V_{\lambda\epsilon})$. Moreover, from (2.13), it is easy to see that $\|v - u\| \ge \lambda^{-1}$ (which is true for points v, u in different boxes B_ϵ^i, by construction) implies

$$Cov(\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda), \xi_\epsilon^u - \xi_\epsilon^u(\log\lambda)) = 0,$$

as desired.

Therefore, independence of the fields $\left\{\left(\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda) : v \in B_\epsilon^i\right)\right\}_{1 \le i \le n}$ and (2.4) imply

$$\mathbb{P}\left(\max_{v \in \cup_i B_\epsilon^i} (\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda)) \le m_{\lambda\epsilon}\right) \le e^{-cn}$$

for some constant $c > 0$ depending on d. But $n \ge c\lambda$ for some constant $c > 0$ depending on ρ. Therefore,

$$\mathbb{P}\left(\max_{v \in \cup_i B_\epsilon^i} (\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda)) \le m_{\lambda\epsilon}\right) \le e^{-c\lambda}$$

for some constant $c > 0$ depending on both d and ρ. By letting $\nu = \arg\max\left\{\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda) : v \in \bigcup_i B_\epsilon^i\right\}$, the previous display implies

$$\mathbb{P}\left(\max_{v \in V_\epsilon \cap \rho[0,1)^d} \xi_\epsilon^v \le m_\epsilon - \lambda\right) \le \mathbb{P}\left(\xi_\epsilon^\nu \le m_\epsilon - \lambda\right) \le \mathbb{P}\left(\xi_\epsilon^\nu(\log\lambda) \le m_\epsilon - m_{\lambda\epsilon} - \lambda\right) + \mathbb{P}\left(\xi_\epsilon^\nu - \xi_\epsilon^\nu(\log\lambda) \le m_{\lambda\epsilon}\right)$$

$$\le \mathbb{P}\left(\xi_\epsilon^\nu(\log\lambda) \le m_\epsilon - m_{\lambda\epsilon} - \lambda\right) + e^{-c\lambda}.$$

Moreover, it is clear from (2.1) that the fields $(\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda) : v \in V_\epsilon)$ and $(\xi_\epsilon^v(\log\lambda) : v \in V_\epsilon)$ are independent. Hence, ν is independent from $\xi_\epsilon^{(\cdot)}(\log\lambda)$, and $\xi_\epsilon^\nu(\log\lambda)$ is therefore a Gaussian random variable with mean zero and variance $\log\lambda$. But

$$m_\epsilon - m_{\lambda\epsilon} \le \sqrt{2d}\log\lambda.$$

Therefore, the last two displays imply

$$\mathbb{P}\left(\max_{v\in V_\epsilon\cap\rho[0,1)^d}\xi_\epsilon^v\le m_\epsilon-\lambda\right)\le Ce^{-c\frac{(\lambda-\sqrt{2d}\log\lambda)^2}{\log\lambda}}+e^{-c\lambda}\le Ce^{-c\lambda},$$

proving Proposition 2.6 in the case $\lambda\in[2/\rho,\sqrt{1/\epsilon}]$. 3) $\lambda\in(\sqrt{1/\epsilon},\infty)$. In this case, we have

$$\mathbb{P}\left(\max_{v\in V_\epsilon\cap\rho[0,1)^d}\xi_\epsilon^v\le m_\epsilon-\lambda\right)\le\mathbb{P}\left(\xi_\epsilon^v\le m_\epsilon-\lambda\right)\le Ce^{-c\frac{(\lambda-m_\epsilon)^2}{\log(1/\epsilon)}}\le Ce^{-c\lambda}$$

(where v is any point), which implies Proposition 2.6 in this case. □

Using Propositions 2.4, 2.5 and 2.6, we are now ready to finish the proof of Theorem 1.1.

Proof of 1.1, (1.3)*, the left tail.* Propositions 2.4, 2.5 and 2.6 imply the existence of constants $0<\delta,\rho,c,C<\infty$, depending on C_Y and d, such that

$$\mathbb{P}\left(\max_{u\in V_{\epsilon/\rho}}Y_{\delta\epsilon}^u\le m_\epsilon-\lambda\right)\le Ce^{-c\lambda}$$

for all $\lambda\ge 0$ and all $\epsilon>0$ small enough (depending on C_Y). But $m_{\delta\epsilon}\le m_\epsilon+C'$, where C' depends on δ and d. Therefore,

$$\mathbb{P}\left(\max_{u\in V_{\epsilon/\rho}}Y_{\delta\epsilon}^u\le m_{\delta\epsilon}-\lambda-C'\right)\le Ce^{-c\lambda}.$$

The bound (1.3) for the left tail follows by adjusting the constants. □

3 Example: a mollified Gaussian free field in $d=2$

The Gaussian free field in two dimensions provides an important example of a log-correlated field. Intuitively speaking, the reason for the log-correlation is simply that, in $d=2$, the Green function for the Laplacian is logarithmic.

We begin by recalling in Section 3.1 the definitions of the Dirichlet product and the Hilbert space induced by it. We then use this Hilbert space to define the continuous Gaussian free field and the mollified Gaussian free field. After that, we prove some useful properties of these fields, which will be used to check the hypotheses of Theorem 1.1. Finally, in Section 3.2, we use Theorem 1.1 to prove tightness of the recentered maximum of the family of mollified Gaussian free fields.

3.1 Continuous and mollified Gaussian free fields

Dirichlet product

We begin by recalling the definition of the Dirichlet product. Let $C_c^\infty\left((0,1)^2\right)$ denote the set of real valued C^∞ functions with compact support in $(0,1)^2$. For $\phi,\psi\in C_c^\infty\left((0,1)^2\right)$, let

$$\langle\phi,\psi\rangle_\nabla=\int\nabla\phi(x)\nabla\psi(x)dx$$

denote the Dirichlet product, where ∇ is the gradient and dx is two-dimensional Lebesgue measure. Note that the Dirichlet product satisfies

$$\langle\phi,\psi\rangle_\nabla=\int\phi(x)(-\Delta\psi)(x)dx, \tag{3.1}$$

where Δ is the standard Laplacian. The Dirichlet product induces a norm on $C_c^\infty\left((0,1)^2\right)$ by

$$\|\phi\|_\nabla = \sqrt{\langle \phi, \phi \rangle_\nabla},$$

called the Dirichlet norm. Denote by $W = W\left((0,1)^2\right)$ the completion of $C_c^\infty\left((0,1)^2\right)$ with respect to the Dirichlet norm. The set W, together with the Dirichlet product on W, defines a Hilbert space.

The Dirichlet norm satisfies Poincare's Inequality: there exists a constant C (which depends only on the domain $(0,1)^2$) such that

$$\|\phi\|_{L^2} \le C\,\|\nabla \phi\|_{L^2}$$

for all $\phi \in C_c^\infty$. Poincare's Inequality implies that the Dirichlet norm is equivalent to the norm

$$\|\phi\|_{L^2} + \left\|\frac{\partial}{\partial x_1}\phi\right\|_{L^2} + \left\|\frac{\partial}{\partial x_2}\phi\right\|_{L^2}.$$

Recall that the completion of $C_c^\infty\left((0,1)^2\right)$ with respect to the latter norm is called a $(1,2)$-Sobolev space (i.e., measurable functions such that their weak derivatives up to order 1 exist and belong to $L^2\left((0,1)^2\right)$). Since the norms are equivalent, the space $(W, \|\cdot\|_\nabla)$ is also a Sobolev space. Therefore, for any $g \in W$ and any measurable set $E \subset [0,1]^2$, the integral $\int_E g(x)dx$ is well-defined.

For a given open set $U \subset (0,1)^2$, Poincare's Inequality implies that the linear mapping $W \to \mathbb{R}$ given by

$$g \mapsto \int_U g(x)dx$$

is $\|\cdot\|_\nabla$-continuous. Note that, since W is a Hilbert space, the Riesz representation theorem implies the existence of a function $f = f_U \in W$ such that

$$\langle g, f_U \rangle_\nabla = \int_U g(x)dx \tag{3.2}$$

for all $g \in W$.

Gaussian free fields

The continuous Gaussian free field is defined as follows: since $\langle \cdot, \cdot \rangle_\nabla$ is positive definite, there exists a family $\left\{X^f : f \in W\right\}$ of centered Gaussian variables, defined on some probability space $(\Omega, \mathbb{P})$, such that

$$Cov(X^f, X^g) = \langle f, g \rangle_\nabla$$

for all $f \in W$. The family $\left\{X^f : f \in W\right\}$ is called the *continuous Gaussian free field*.

We next define a field indexed by the set $[0,1]^2$. Fix $\epsilon > 0$, and let $x \in [0,1]^2$. By (3.2), there exists a function $f_{x,\epsilon} \in W$ such that

$$\langle f_{x,\epsilon}, g \rangle_\nabla = \frac{1}{\pi \epsilon^2} \int_{D(x,\epsilon)\cap(0,1)^2} g(u)du \tag{3.3}$$

for all $g \in W$, where $D(x,\epsilon)$ is the disk of radius ϵ centered at x. Using (3.1) and (3.3), it is not hard to show that

$$f_{x,\epsilon}(u) = \frac{1}{\pi \epsilon^2} \int_{D(x,\epsilon)\cap(0,1)^2} G(u,v)dv, \tag{3.4}$$

where $G = G_{(0,1)^2}$ is the Green function of $(0,1)^2$ for the operator $-\Delta$, with Dirichlet boundary conditions on $\partial(0,1)^2$. For the domain $(0,1)^2$, the Green function can be explicitly stated as:

$$G(u,v) = \frac{4}{\pi^2} \sum_{n,m\geq 1} \frac{1}{n^2+m^2} \sin(n\pi u_1)\sin(m\pi u_2)\sin(n\pi v_1)\sin(m\pi v_2),$$

where $u = (u_1, u_2) \in [0,1]^2$. The field $\left(X^{f_{x,\epsilon}} : x \in [0,1]^2\right)$ will be called *ϵ-mollified Gaussian free field* (MGFF). To simplify notation, set $X_\epsilon^x = X^{f_{x,\epsilon}}$. Note that, by definition,

$$Cov(X_\epsilon^x, X_\epsilon^y) = \langle f_{x,\epsilon}, f_{y,\epsilon}\rangle_\nabla = \frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)\cap(0,1)^2} f_{y,\epsilon}(u)du$$

and, from (3.4), we obtain

$$Cov(X_\epsilon^x, X_\epsilon^y) = \frac{1}{(\pi\epsilon^2)^2}\int_{D(x,\epsilon)\cap(0,1)^2\times D(y,\epsilon)\cap(0,1)^2} G(u,v)\,dudv, \tag{3.5}$$

for all $x, y \in [0,1]^2$.

Orthogonal decomposition

The next proposition shows that the MGFF satisfies a tree-like decomposition property.

Proposition 3.1. *Let $Q = \frac{1}{2}(0,1)^2 \subset (0,1)^2$ be a sub-square of side length $1/2$. Then, X_ϵ^x can be decomposed as*

$$X_\epsilon^x = \hat{X}_\epsilon^x + \phi^x,$$

where $\left(\hat{X}_\epsilon^x : x \in \overline{Q}\right)$ is a copy of $\left(X_{2\epsilon}^x : x \in [0,1]^2\right)$, $\left(\phi^x : x \in [0,1]^2\right)$ is a centered Gaussian field, and $\left(\hat{X}_\epsilon^x : x \in \overline{Q}\right)$ is independent of $\left(\phi^x : x \in [0,1]^2\right)$.

Proof. Denote by $C_c^\infty(Q)$ the set of real valued C^∞ functions with compact support in Q, and let $W(Q)$ be the corresponding Hilbert space induced by the Dirichlet product in $C_c^\infty(Q)$. Note that $C_c^\infty(Q) \subset C_c^\infty\left((0,1)^2\right)$ and

$$\langle f, g\rangle_{\nabla,Q} := \int_Q \nabla f(u)\cdot\nabla g(u)\,du = \int_{(0,1)^2} \nabla f(u)\cdot\nabla g(u)\,du \tag{3.6}$$

for all $f, g \in C_c^\infty(Q)$. By taking the completion of $C_c^\infty(Q)$ with respect to the Dirichlet product, we see that $W(Q)$ is a Hilbert subspace of $W\left((0,1)^2\right)$ and that (3.6) holds for all $f, g \in W(Q)$.

Let $f_{x,\epsilon}$ be as in (3.3) and decompose it as

$$f_{x,\epsilon} = g_{x,\epsilon} + h_{x,\epsilon},$$

where $g_{x,\epsilon} \in W(Q)$ and $h_{x,\epsilon} \in W(Q)^\perp$ (the orthogonal space). Set

$$\hat{X}_\epsilon^x = X^{g_{x,\epsilon}}$$

and

$$\phi^x = X^{h_{x,\epsilon}}.$$

Since $g_{x,\epsilon} \perp h_{y,\epsilon}$ for all $x, y \in [0,1]^2$, the families $\left(\hat{X}_\epsilon^x : x \in [0,1]^2\right)$ and $\left(\phi^x : x \in [0,1]^2\right)$ are independent. Also, since $f \mapsto X^f$ is a linear embedding of W into $L^2(\Omega, \mathbb{P})$,

$$X_\epsilon^x = \hat{X}_\epsilon^x + \phi^x \text{ a.s.}$$

for every $x \in [0,1]^2$.

We show now that $\left(\hat{X}_\epsilon^x : x \in \overline{Q}\right)$ is a copy of $\left(X_{2\epsilon}^x : x \in [0,1]^2\right)$.

Claim 3.2. *For every $k \in W(Q)$,*

$$\langle g_{x,\epsilon}, k\rangle_{\nabla,Q} = \frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)\cap Q} k(u)\,du.$$

Proof of Claim 3.2. By (3.6),

$$\langle g_{x,\epsilon}, k\rangle_{\nabla,Q} = \langle g_{x,\epsilon}, k\rangle_{\nabla}$$

and, since $g_{x,\epsilon} = f_{x,\epsilon} - h_{x,\epsilon}$,

$$\langle g_{x,\epsilon}, k\rangle_{\nabla} = \langle f_{x,\epsilon}, k\rangle_{\nabla} - \langle h_{x,\epsilon}, k\rangle_{\nabla}.$$

But $h_{x,\epsilon} \perp k$, so the second term on the right hand side of the previous display vanishes. Using (3.3) and the two previous displays, we obtain

$$\langle g_{x,\epsilon}, k\rangle_{\nabla,Q} = \frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)\cap(0,1)^2} k(u)\,du.$$

Since $k \in W(Q)$, the function k vanishes outside of Q. Therefore,

$$\langle g_{x,\epsilon}, k\rangle_{\nabla,Q} = \frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)\cap Q} k(u)\,du,$$

as desired. □

Claim 3.2 implies, in analogy with (3.5), that the following is true for all $x, y \in \overline{Q}$:

$$Cov\left(\hat{X}^x_\epsilon, \hat{X}^y_\epsilon\right) = \langle g_{x,\epsilon}, g_{y,\epsilon}\rangle_\nabla = \langle g_{x,\epsilon}, g_{y,\epsilon}\rangle_{\nabla,Q} = \frac{1}{(\pi\epsilon^2)^2}\int_{D(x,\epsilon)\cap Q\times D(y,\epsilon)\cap Q} G_Q(u,v)\,dudv,$$

where G_Q is the Green function of Q for the operator $-\Delta$, with Dirichlet boundary conditions on ∂Q.

Claim 3.3. *For every $u, v \in [0,1]^2$,*

$$G_Q\left(u/2, v/2\right) = G(u,v).$$

Proof of Claim 3.3. Let $\phi \in C_c^\infty\left((0,1)^2\right)$ and note that $(\Delta\phi)(2u) = \frac{1}{4}\Delta\left(\phi(2u)\right)$. By the change of variables $u' = u/2$,

$$\int_{(0,1)^2} G_Q\left(u/2, v/2\right)(\Delta\phi)(u)\,du = \int_Q G_Q\left(u', v/2\right)\Delta\left(\phi\left(2u'\right)\right)\,du' = -\phi(2v/2),$$

where the last equality holds by definition of G_Q. On the other hand,

$$\int_{(0,1)^2} G(u,v)\,(\Delta\phi)(u)\,du = -\phi(v),$$

by definition of G. Since

$$\int_{(0,1)^2} G_Q\left(u/2, v/2\right)(\Delta\phi)(u)\,du = \int_{(0,1)^2} G(u,v)\,(\Delta\phi)(u)\,du$$

for every $\phi \in C_c^\infty\left((0,1)^2\right)$, the functions $G_Q(u/2, v/2)$ and $G(u,v)$ are identical (Lebesgue-a.e.). □

The change of variables $u' = 2u, v' = 2v$ implies

$$Cov\left(\hat{X}_\epsilon^x, \hat{X}_\epsilon^y\right) = \frac{1}{(\pi\epsilon^2)^2}\int_{D(x,\epsilon)\cap Q\times D(y,\epsilon)\cap Q} G_Q(u,v)\,dudv$$

$$= \frac{1}{(\pi(2\epsilon)^2)^2}\int_{D(2x,2\epsilon)\cap(0,1)^2\times D(2y,2\epsilon)\cap(0,1)^2} G_Q(u'/2,v'/2)\,du'dv',$$

and Claim 3.3 implies that the previous display is

$$= \frac{1}{(\pi(2\epsilon)^2)^2}\int_{D(2x,2\epsilon)\cap(0,1)^2\times D(2y,2\epsilon)\cap(0,1)^2} G(u',v')\,du'dv' = Cov\left(X_{2\epsilon}^{2x}, X_{2\epsilon}^{2y}\right).$$

For Gaussian fields, equality of the covariance structure implies that the fields have the same distribution. Therefore,

$$\left(\hat{X}_\epsilon^x : x \in \overline{Q}\right) \overset{d}{=} \left(X_{2\epsilon}^{2x} : x \in \overline{Q}\right),$$

and the right hand side is clearly equal to $\left(X_{2\epsilon}^x : x \in [0,1]^2\right)$, which finishes the proof of Proposition 3.1. ☐

Proposition 3.1 is true for any sub-square $Q \subset (0,1)^2$ of side length $1/2$, because Green functions are translation invariant (i.e., $G_{Q+z}(u+z, v+z) = G_Q(u,v)$ for any $z \in \mathbb{R}^2, u, v \in Q$, where G_{Q+z} is the Green function of $Q+z$ for the operator $-\Delta$, with Dirichlet boundary conditions on $\partial Q + z$).

Estimates on the covariance

In this subsection, we prove that the "bulk" of the field $\left\{\sqrt{2\pi}X_\epsilon^x : x \in [0,1]^2\right\}$ satisfies both (1.1) and (1.2). Recall that $\Gamma(\cdot,\cdot) = \Gamma(\|\cdot - \cdot\|) = \frac{1}{2\pi}\log(1/\|\cdot - \cdot\|)$ is the Green function of $\mathbb{R}^2$ for the operator $-\Delta$.

Proposition 3.4. *Let $K \subset (0,1)^2$ be such that $k = dist(\partial(0,1)^2, K) > 0$, and let $0 < \epsilon < k/2$. Then, there exists a constant $C < \infty$, depending on k only, such that, for all $x \in K, y \in [0,1]^2$,*

$$\left|\frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)} G(u,y)du - \frac{1}{2\pi}\log(1/\epsilon)\right| \leq C$$

if $\|y - x\| < \epsilon$, and

$$\left|\frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)} G(u,y)du - \frac{1}{2\pi}\log(1/\|x-y\|)\right| \leq C$$

if $\|y - x\| \geq \epsilon$.

Proof. The function $(G-\Gamma)(x,y)$ is symmetric, harmonic in each variable, and continuous. Hence,

$$\left|\frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)} (G-\Gamma)(u,y)\,du\right| \leq \sup_{\{u:dist(u,\partial(0,1)^2)\geq k/2\}}\ \sup_{\{y\in[0,1]^2\}} |(G-\Gamma)(u,y)|$$

$$\leq \sup_{u:dist(u,\partial(0,1)^2)\geq k/2} |\Gamma(dist(u,\partial(0,1)^2))| = \Gamma(k/2),$$

where the second bound is obtained by applying the maximum principle to $(G-\Gamma)(u,\cdot)$, noting that $G(u,\cdot)$ vanishes at the boundary of $(0,1)^2$, and using that Γ is decreasing. Therefore, it is enough to prove Proposition 3.4 with G replaced by Γ.

Suppose that $\|x-y\| < \epsilon$. Then,

$$\left|\frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)}\Gamma(u,y)-\Gamma(\epsilon)du\right| = \left|\frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)}\Gamma\left(\frac{\|u-y\|}{\epsilon}\right)du\right|.$$

The change of variables $u'=(u-y)/\epsilon$ implies that the previous display is

$$= \left|\frac{1}{\pi}\int_{D((x-y)/\epsilon,1)}\Gamma(u')du'\right| \le \sup_{z\in\overline{D(0,1)}}\left|\frac{1}{\pi}\int_{D(z,1)}\Gamma(u')du'\right| \le C,$$

by continuity in z and compactness of $\overline{D(0,1)}$, where C is an absolute constant.

Suppose now that $\|x-y\| \ge \epsilon$. Then,

$$\left|\frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)}\Gamma(u,y)du-\Gamma(\|x-y\|)\,du\right| = \left|\frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)}\Gamma\left(\left\|\frac{u-y}{\|x-y\|}\right\|\right)du\right|.$$

The change of variables $u'=(u-y)/\|x-y\|$ implies that the previous line is

$$= \left|\frac{1}{\pi(\epsilon/\|x-y\|)^2}\int_{D\left(\frac{x-y}{\|x-y\|},\frac{\epsilon}{\|x-y\|}\right)}\Gamma(u')\,du'\right| \le \sup_{0\le r\le 1}\sup_{\|z\|=1}\left|\frac{1}{\pi r^2}\int_{D(z,r)}\Gamma(u')\,du'\right| < C,$$

by continuity in r,z and compactness of $\{0\le r\le 1\}\times\{\|z\|=1\}$, where C is an absolute constant. □

Note that the fact that we are integrating over disks is not essential. We could define similar MGFF for other mollifiers.

A trivial corollary (which follows from elementary properties of $\log$) of the previous proposition is

Corollary 3.5. *Let K,k,ϵ be as in Proposition 3.4 and let $c_0>0$. Then, there exists a constant C (depending on k and c_0) such that, for all $x\in K, y\in[0,1]^2$,*

$$\left|\frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)}G(u,y)du-\frac{1}{2\pi}\log(1/\epsilon)\right| \le C$$

whenever $\|x-y\| < c_0\epsilon$, and

$$\left|\frac{1}{\pi\epsilon^2}\int_{D(x,\epsilon)}G(u,y)du-\frac{1}{2\pi}\log(1/\|x-y\|)\right| \le C$$

whenever$\|x-y\| \ge c_0\epsilon$.

Now we prove an important corollary of Proposition 3.4.

Corollary 3.6. *Let K,k be as in Proposition 3.4. Then, there exists a constant C (depending only on k) such that, for all $x,y\in K,\epsilon>0$,*

$$|Cov(X_\epsilon^x,X_\epsilon^y)+\frac{1}{2\pi}\log(\max\{\epsilon,\|x-y\|\})| \le C. \quad (3.7)$$

Moreover, if $\|x-y\|\le\epsilon$, then

$$\mathbb{E}\left(X_\epsilon^x-X_\epsilon^y\right)^2 \le C\epsilon^{-1}\|x-y\|. \quad (3.8)$$

Proof. Let us prove (3.7). If $\|x-y\| \le 2\epsilon$, by Corollary 3.5,

$$\left| \frac{1}{\pi\epsilon^2} \int_{D(y,\epsilon)} G(u,v)\, dv - \Gamma(\epsilon) \right| \le C$$

for every $u \in D(x,\epsilon)$. Integrating the last inequality over $u \in D(x,\epsilon)$ and using (3.5), we obtain that

$$|Cov(X_\epsilon^x, X_\epsilon^y) - \Gamma(\epsilon)| \le C$$

for all $\|x-y\| \le 2\epsilon$ (and in particular, for $\|x-y\| \le \epsilon$).

If $\|x-y\| \ge 2\epsilon$, Corollary 3.5 implies

$$\left| \frac{1}{\pi\epsilon^2} \int_{D(y,\epsilon)} G(u,v) dv - \Gamma(\|y-u\|) \right| \le C$$

for every $u \in D(x,\epsilon)$. But $\Gamma(3/2) \le \Gamma(1 + \frac{\epsilon}{\|x-y\|}) \le \Gamma(\|y-u\|) - \Gamma(\|x-y\|) \le \Gamma(1 - \frac{\epsilon}{\|x-y\|}) \le \Gamma(1/2)$ for all $u \in D(x,\epsilon)$. Therefore,

$$\left| \frac{1}{\pi\epsilon^2} \int_{D(y,\epsilon)} G(u,v) dv - \Gamma(\|x-y\|) \right| \le C.$$

The same (with a different constant) holds for $\|x-y\| \ge \epsilon$, because Γ is logarithmic. Integrating over $u \in D(x,\epsilon)$ finishes the proof of (3.7).

We now prove (3.8). Display (3.5) implies

$$Cov(X_\epsilon^x, X_\epsilon^x - X_\epsilon^y) = \frac{1}{\pi^2\epsilon^4} \int_{D(x,\epsilon)} \int_{D(x,\epsilon)} G - \frac{1}{\pi^2\epsilon^4} \int_{D(x,\epsilon)} \int_{D(y,\epsilon)} G$$

$$= \frac{1}{\pi^2\epsilon^4} \left(\int_{D(x,\epsilon)\backslash D(y,\epsilon)} \int_{D(x,\epsilon)} G - \int_{D(y,\epsilon)\backslash D(x,\epsilon)} \int_{D(x,\epsilon)} G \right).$$

We can use Corollary 3.5 to obtain an upper bound of the first term and a lower bound of the second term of the previous display. Then, the previous display is

$$\le \frac{1}{\pi\epsilon^2} \left(\int_{D(x,\epsilon)\backslash D(y,\epsilon)} (\Gamma(\epsilon) + C) - \int_{D(y,\epsilon)\backslash D(x,\epsilon)} (\Gamma(\epsilon) - C) \right) = \frac{C}{\pi\epsilon^2} \left| D(x,\epsilon)\backslash D(y,\epsilon) \right|,$$

where $|D(x,\epsilon)\backslash D(y,\epsilon)|$ is the Lebesgue measure of the set $D(x,\epsilon)\backslash D(y,\epsilon)$. Elementary geometry implies $|D(x,\epsilon)\backslash D(y,\epsilon)| \le C\epsilon \|x-y\|$. Repeating the previous argument for $Cov(X_\epsilon^y, X_\epsilon^y - X_\epsilon^x)$ finishes the proof. ☐

3.2 Tightness for the MGFF

In the next theorem we provide upper bounds on the left and right tail of the MGFF, and we compute the expected maximum up to an order 1 term.

Theorem 3.7. *For $\epsilon > 0$ small enough, let $X_\epsilon^x, x \in [0,1]^2$ be the MGFF. Then, there exist absolute constants $0 < c, C < \infty$ such that*

$$\mathbb{P}\left(\left| \max_{x\in[0,1]^2} X_\epsilon^x - \sqrt{\frac{1}{2\pi}} m_\epsilon \right| \ge +\lambda \right) \le Ce^{-c\lambda} \tag{3.9}$$

for all $\lambda \ge 0$. Moreover,

$$\mathbb{E}\left[\max_{x\in[0,1]^2} X_\epsilon^x \right] = \sqrt{\frac{1}{2\pi}} m_\epsilon + O(1).$$

Proof. Let Q be the open square of side length $1/2$, which is concentric with $(0,1)^2$, and let $q : [0,1]^2 \to \overline{Q}$ be the natural concentric contraction. Consider the field $Y_\epsilon^x := X_{\epsilon/2}^{q(x)}; x \in [0,1]^2$. By Corollary 3.6,

$$Cov(Y_\epsilon^x, Y_\epsilon^y) = Cov(X_{\epsilon/2}^{q(x)}, X_{\epsilon/2}^{q(y)}) = \frac{1}{2\pi}\log\left(\max\left\{\epsilon/2, \|q(x)-q(y)\|\right\}\right) + O(1)$$

$$= \frac{1}{2\pi}\log\left(\max\left\{\epsilon, \|x-y\|\right\}\right) + O(1)$$

for all $x, y \in [0,1]^2$, and

$$\mathbb{E}\left(Y_\epsilon^x - Y_\epsilon^y\right)^2 = \mathbb{E}\left(X_{\epsilon/2}^{q(x)} - X_{\epsilon/2}^{q(y)}\right)^2 \le C\epsilon^{-1}2\left\|q(x)-q(y)\right\| = C\epsilon^{-1}\left\|x-y\right\|$$

for all $x, y \in [0,1]^2$ such that $\|x-y\| \le \epsilon$. An application of Theorem 1.1 yields the existence of absolute constants $0 < c, C < \infty$ such that

$$\mathbb{P}\left(\max_{x\in[0,1]^2} Y_\epsilon^x - \sqrt{\frac{1}{2\pi}}m_\epsilon \ge \lambda\right) = \mathbb{P}\left(\max_{x\in\overline{Q}} X_{\epsilon/2}^x - \sqrt{\frac{1}{2\pi}}m_\epsilon \ge \lambda\right) \le Ce^{-c\lambda} \tag{3.10}$$

and

$$\mathbb{P}\left(\max_{x\in\overline{Q}} X_{\epsilon/2}^x - \sqrt{\frac{1}{2\pi}}m_\epsilon \le \lambda\right) \le Ce^{-c\lambda} \tag{3.11}$$

for all $\lambda \ge 0$. Bound (3.11) easily implies that

$$\mathbb{P}\left(\max_{x\in[0,1]^2} X_{\epsilon/2}^x - \sqrt{\frac{1}{2\pi}}m_\epsilon \le \lambda\right) \le \mathbb{P}\left(\max_{x\in\overline{Q}} X_{\epsilon/2}^x - \sqrt{\frac{1}{2\pi}}m_\epsilon \le \lambda\right) \le Ce^{-c\lambda}$$

for all $\lambda \ge 0$, proving (3.9) for the left tail (after using $m_{\epsilon/2} = m_\epsilon + O(1)$, and adjusting the constants).

In order to prove the bound (3.9) for the right tail, we use Proposition 3.1 and the comment that follows it to decompose

$$X_{\epsilon/2}^x = \hat{X}_{\epsilon/2}^x + \phi^x,$$

where $\left(\hat{X}_{\epsilon/2}^x : x \in \overline{Q}\right) \overset{d}{=} \left(X_\epsilon^x : x \in [0,1]^2\right)$ and the fields $\left(\phi^x : x \in \overline{Q}\right), \left(\hat{X}_{\epsilon/2}^x : x \in \overline{Q}\right)$ are independent. If $\chi = \arg\max\left\{\hat{X}_{\epsilon/2}^x : x \in \overline{Q}\right\}$, then

$$\left\{\phi^\chi \ge 0, \hat{X}_{\epsilon/2}^\chi - \sqrt{\frac{1}{2\pi}}m_\epsilon \ge \lambda\right\} \subset \left\{\max_{x\in\overline{Q}} X_{\epsilon/2}^x - \sqrt{\frac{1}{2\pi}}m_\epsilon \ge \lambda\right\}.$$

But independence of ϕ and χ implies

$$\mathbb{P}\left(\phi^\chi \ge 0, \hat{X}_{\epsilon/2}^\chi - \sqrt{\frac{1}{2\pi}}m_\epsilon \ge \lambda\right) = \frac{1}{2}\mathbb{P}\left(\hat{X}_{\epsilon/2}^\chi - \sqrt{\frac{1}{2\pi}}m_\epsilon \ge \lambda\right)$$

because ϕ is a centered field. By using the last display and (3.10), we obtain

$$\mathbb{P}\left(\hat{X}_{\epsilon/2}^\chi - \sqrt{\frac{1}{2\pi}}m_\epsilon \ge \lambda\right) \le 2\mathbb{P}\left(\max_{x\in\overline{Q}} X_{\epsilon/2}^x - \sqrt{\frac{1}{2\pi}}m_\epsilon \ge \lambda\right) \le Ce^{-c\lambda}$$

for some absolute constants $0 < c, C < \infty$.

The bound (3.9) and $m_{\epsilon/2} = m_\epsilon + O(1)$ implies tightness of the family

$$\left\{ \max_{x\in[0,1]^2} X_\epsilon^x - \sqrt{\frac{1}{2\pi}} m_\epsilon : \epsilon > 0 \right\},$$

and the same bound also implies

$$\mathbb{E}\left[\max_{x\in[0,1]^2} X_\epsilon^x \right] = \sqrt{\frac{1}{2\pi}} m_\epsilon + O(1),$$

finishing the proof. □

4 Appendix

We prove here the claims made in Section 2.1.

Proposition 4.1. *The MBBM, defined by display* (2.1)*, exists and satisfies*

$$Var(\xi_\epsilon^v(t)) = t$$

for all $0 \le t \le \log(1/\epsilon)$ *and all* $v \in V_\epsilon$*, and*

$$t - C \le Cov(\xi_\epsilon^v(t), \xi_\epsilon^w(t)) \le t$$

for all $0 \le t \le -\log \|v - w\|_\infty$ *and all* $v, w \in V_\epsilon$*, where* C *is a constant depending on the dimension.*

Proof. We show that the mapping $(V_\epsilon \times [0, \log(1/\epsilon)])^2 \to \mathbb{R}$ given by

$$((v,t),(u,s)) \mapsto \int_0^{\min\{t,s\}} \prod_{1\le i\le d} (1 - e^r |v_i - u_i|)_+ dr$$

is positive definite. Note first that

$$\prod_{1\le i\le d} (1 - e^r |v_i - u_i|)_+ = \int_{\mathbb{R}^d} 1_{A(v,r)}(z) 1_{A(u,r)}(z) dz,$$

where dz is d-dimensional Lebesgue measure and $A(v,r)$ is the d-dimensional box of side length 1, centered at $e^r v$. Let $\{(v^\alpha, t^\alpha)\}_\alpha$ be any finite subset of $V_\epsilon \times [0, \log(1/\epsilon)]$, and let $\{c_\alpha\}_\alpha$ be arbitrary real numbers. Then, applying the previous display, we obtain

$$\sum_{\alpha,\beta} c_\alpha c_\beta \int_0^{\min\left\{t^\alpha, t^\beta\right\}} \prod_{1\le i\le d} (1 - e^r \left| v_i^\alpha - v_i^\beta \right|)_+ dr$$

$$= \int_0^\infty \int_{\mathbb{R}^d} \sum_{\alpha,\beta} c_\alpha c_\beta 1_{[0,t^\alpha]}(r) 1_{[0,t^\beta]}(r) 1_{A(v^\alpha,r)}(z) 1_{A(v^\beta,r)}(z)\, dz\, dr$$

$$= \int_0^\infty \int_{\mathbb{R}^d} \left(\sum_\alpha c_\alpha 1_{[0,t^\alpha]}(r) 1_{A(v^\alpha,r)}(z) \right)^2 dz\, dr \ge 0,$$

as desired. This shows that the MBBM exists.

For any $v \in V_\epsilon$ and $t \le \log(1/\epsilon)$,

$$Var(\xi_\epsilon^v(t)) = \int_0^t \prod_{1\le i\le d} (1)\, dr = t.$$

Moreover, if $v \neq w$,

$$\prod_{1\le i\le d} (1 - e^r |v_i - w_i|)_+ \begin{cases} > 0 & \text{if } r < -\log \|v-w\|_\infty \\ = 0 & \text{if } r \ge -\log \|v-w\|_\infty \end{cases}.$$

Therefore, if $t < -\log \|v-w\|_\infty$,

$$t \ge Cov(\xi_\epsilon^v(t), \xi_\epsilon^w(t)) \ge \int_0^t \prod_{1\le i\le d} (1 - e^r |v_i - w_i|)\, dr \ge \int_0^t (1 - e^r \|v-w\|_\infty)^d \, dr,$$

where the last inequality follows because $1 - e^r |v_i - w_i| \ge 1 - e^r \|v-w\|_\infty$ for all $i \in \{1, 2, \ldots, d\}$. Expanding and integrating, we obtain that the last display is

$$\ge t + \sum_{k=1}^d \binom{d}{k} (-1)^k \|v-w\|_\infty^k \left(\frac{e^{kt}-1}{k}\right) \ge t - \sum_{k=1}^d \binom{d}{k} \|v-w\|_\infty^k \left(e^{kt}+1\right). \tag{4.1}$$

But since $\|v-w\|_\infty \le 1$ and $t < -\log \|v-w\|_\infty$, we have

$$\|v-w\|_\infty^k \left(e^{kt}+1\right) \le \left(\|v-w\|_\infty e^{-\log \|v-w\|_\infty}\right)^k + 1 \le 2.$$

Therefore, display (4.1) is

$$\ge t - 2\sum_{k=1}^d \binom{d}{k} \ge t - C$$

for some constant $C < \infty$ depending on d only. Similarly, if $t \ge -\log \|v-w\|_\infty$,

$$-\log \|v-w\|_\infty \ge Cov(\xi_\epsilon^v(t), \xi_\epsilon^w(t)) \ge -\log \|v-w\|_\infty - C.$$

□

Proposition 4.2. *Let $(\xi_\epsilon^v : v \in V_\epsilon)$ be the MBBM and let m_ϵ be the number defined in the line preceding Theorem 1.1. Then, there exists a constant $c > 0$ (depending on the dimension) such that*

$$\mathbb{P}\left(\max_{v\in V_\epsilon} \xi_\epsilon^v \ge m_\epsilon\right) \ge c.$$

Proof. We use a second moment method. Let $T = T_\epsilon = \log(1/\epsilon)$ and let

$$A_v = \left\{\xi_\epsilon^v \ge m_\epsilon, \xi_\epsilon^v(t) \le \frac{m_\epsilon}{T} t + 1 \text{ for all } 0 \le t \le T\right\},$$

$$Z = \sum_{v\in V_\epsilon} 1_{A_v}.$$

Note that

$$\mathbb{P}\left(\max_{v\in V_\epsilon} \xi_\epsilon^v \ge m_\epsilon\right) \ge \mathbb{P}(Z > 0) \ge \frac{(\mathbb{E}[Z])^2}{\mathbb{E}[Z^2]}, \tag{4.2}$$

where the second inequality follows by Cauchy-Schwarz. We first compute a lower bound for $\mathbb{E}[Z]$. Note that

$$\mathbb{E}[Z] = \epsilon^{-d} \mathbb{P}(A_v).$$

Let $\bar{\xi}_\epsilon^v(t) = \xi_\epsilon^v(t) - \frac{m_\epsilon}{T} t$. Define a probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left(-\frac{m_\epsilon}{T} \bar{\xi}_\epsilon^v(T) - \frac{m_\epsilon^2}{2T}\right).$$

Girsanov's Theorem (see [16, Theorem 5.1]) implies that $\bar{\xi}_\epsilon^v(t)$ is Brownian motion under $\mathbb{Q}$. Note that

$$\mathbb{P}(A_v) = \int_{A_v} \exp\left(-\frac{m_\epsilon}{T}\bar{\xi}_\epsilon^v(T) - \frac{m_\epsilon^2}{2T}\right) d\mathbb{Q} \geq \exp\left(-\frac{m_\epsilon}{T} - \frac{m_\epsilon^2}{2T}\right)\mathbb{Q}(A_v)$$

$$\geq ce^{-\sqrt{2d}}\epsilon^d T^{3/2}\mathbb{Q}(A_v)$$

for some absolute constant $c > 0$. It follows easily from the Reflection Principle (see [16, Proposition 6.19]) that $\mathbb{Q}(A_v) = \mathbb{Q}(\bar{\xi}_\epsilon^v \geq 0, \bar{\xi}_\epsilon^v(t) \leq 1 \text{ for all } 0 \leq t \leq T) \geq cT^{-3/2}$ for some absolute constant $c > 0$. Combining the three previous displays, we obtain

$$\mathbb{E}[Z] \geq c \tag{4.3}$$

for some constant $c > 0$, depending on the dimension d.

We now compute an upper bound for $\mathbb{E}\left[Z^2\right]$. Note that

$$\mathbb{E}\left[Z^2\right] = \sum_{v,w\in V_\epsilon} \mathbb{P}(A_v \cap A_w) = \sum_{v,w\in V_\epsilon} \mathbb{P}\left(\bar{\xi}_\epsilon^v, \bar{\xi}_\epsilon^w \geq 0, \bar{\xi}_\epsilon^v(t), \bar{\xi}_\epsilon^w(t) \leq 1 \text{ for all } 0 \leq t \leq T\right). \tag{4.4}$$

Both $\xi_\epsilon^v(\cdot), \xi_\epsilon^w(\cdot)$ are Brownian motions, which have independent increments starting at time $s = s_{v,w} = -\log\left(\max\left\{\epsilon, \|v-w\|_\infty\right\}\right)$. Therefore,

$$\mathbb{P}(A_v \cap A_w) \leq \sum_{-\infty<x,y\leq 1} p(x)p(y)\mathbb{P}\left(\bar{\xi}_\epsilon^v(t), \bar{\xi}_\epsilon^w(t) \leq 1 \text{ for all } t \in [0,s], \bar{\xi}_\epsilon^v(s) \in [x-1,x], \bar{\xi}_\epsilon^w(s) \in [y-1,y]\right)$$

$$\leq \sum_{-\infty<y\leq x\leq 1} 2p(x)p(y)\mathbb{P}\left(\bar{\xi}_\epsilon^v(t), \bar{\xi}_\epsilon^w(t) \leq 1 \text{ for all } t \in [0,s], \bar{\xi}_\epsilon^v(s) \in [x-1,x], \bar{\xi}_\epsilon^w(s) \in [y-1,y]\right), \tag{4.5}$$

where

$$p(x) = \sup_{z\in[x-1,x]} \mathbb{P}\left(\bar{\xi}_\epsilon^v(t) \leq 1-z \text{ for all } t \in [0,T-s], \bar{\xi}_\epsilon^v(T-s) \geq -z\right).$$

Assume $0 < s < T$. Applying Girsanov's Theorem and the Reflection Principle, we obtain

$$p(x) \leq C\exp\left(\frac{m_\epsilon}{T}x - \frac{m_\epsilon^2}{2T^2}(T-s)\right)\frac{(1-x)}{(T-s)^{3/2}}$$

for some constant C. Therefore, from (4.5) and the last display,

$$\mathbb{P}(A_v \cap A_w) \leq \sum_{-\infty<y\leq x\leq 1} Cp(x)^2\mathbb{P}\left(\bar{\xi}_\epsilon^v(t), \bar{\xi}_\epsilon^w(t) \leq 1 \text{ for all } t \in [0,s], \bar{\xi}_\epsilon^v(s) \in [x-1,x], \bar{\xi}_\epsilon^w(s) \in [y-1,y]\right)$$

$$\leq \sum_{-\infty<x\leq 1} Cp(x)^2\mathbb{P}\left(\bar{\xi}_\epsilon^v(t) \leq 1 \text{ for all } t \in [0,s], \bar{\xi}_\epsilon^v(s) \in [x-1,x]\right).$$

Applying Girsanov's Theorem and the Reflection Principle again,

$$\mathbb{P}(A_v \cap A_w) \leq C\sum_{-\infty<x\leq 1} p(x)^2\exp\left(-\frac{m_\epsilon}{T}x - \frac{m_\epsilon^2}{2T^2}s\right)\frac{(1-x)}{s^{3/2}}$$

$$\leq C\frac{1}{(T-s)^3 s^{3/2}}\exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right) \tag{4.6}$$

for some constant C.

Consider now the case $s=0$. Then, the independence of $\xi_\epsilon^v(\cdot)$ and $\xi_\epsilon^w(\cdot)$ implies

$$\mathbb{P}\left(A_v\cap A_w\right)=\mathbb{P}\left(A_v\right)^2=\mathbb{P}\left(\bar\xi_\epsilon^v(t)\le 1 \text{ for all } t\in[0,T], \bar\xi_\epsilon^v(T)\ge 0\right)^2$$
$$\le C\frac{1}{T^3}\exp\left(-\frac{m_\epsilon^2}{T}\right), \tag{4.7}$$

where the last bound follows from Girsanov's Theorem and the Reflection Principle. In the case $s=T$,

$$\mathbb{P}\left(A_v\cap A_w\right)\le\mathbb{P}\left(A_v\right)\le C\frac{1}{T^{3/2}}\exp\left(-\frac{m_\epsilon^2}{2T}\right). \tag{4.8}$$

In consequence, for any pair $v,w\in V_\epsilon$, displays (4.6), (4.7) and (4.8) imply

$$\mathbb{P}\left(A_v\cap A_w\right)\le C\frac{1}{\left((T-s)\vee 1\right)^3(s\vee 1)^{3/2}}\exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right),$$

where $\cdot\vee\cdot=\max\{\cdot,\cdot\}$. For any fixed $v\in V_\epsilon$, there are $O(e^{(d-1)(T-s)})$ points w such that $-\log\|v-w\|_\infty=s$. Therefore, from (4.4) and the last display, we obtain

$$\mathbb{E}\left[Z^2\right]\le C\sum_{0\le s\le T}|V_\epsilon|\,e^{(d-1)(T-s)}\frac{1}{\left((T-s)\vee 1\right)^3(s\vee 1)^{3/2}}\exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right)$$
$$\le C+C\sum_{0<s<T}|V_\epsilon|\,e^{(d-1)(T-s)}\frac{\exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right)}{(T-s)^3 s^{3/2}}$$
$$=C+C\sum_{0<s<T}e^{dT}e^{(d-1)(T-s)}\frac{\exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right)}{(T-s)^3 s^{3/2}}.$$

But,

$$\sum_{0<s<T}e^{dT}e^{(d-1)(T-s)}\frac{\exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right)}{(T-s)^3 s^{3/2}}\le\sum_{0<s<T}e^{d(2T-s)}\frac{\exp\left(\left(-d+\frac{3\log T}{2T}\right)(2T-s)\right)}{(T-s)^3 s^{3/2}}$$
$$=\sum_{0<s<T}\frac{\exp\left(\frac{3}{2}\frac{\log T}{T}(2T-s)\right)}{(T-s)^3 s^{3/2}}\le C\sum_{0<s<T/2}\frac{1}{s^{3/2}}+\sum_{T/2\le s<T}\frac{\exp\left(\frac{3}{2}\frac{\log T}{T}(T-s)\right)}{(T-s)^3}\frac{T^{3/2}}{s^{3/2}}$$
$$\le C+C\sum_{0<s\le T/2}\frac{T^{3s/2T}}{s^3}\le C<\infty,$$

because the last expression is (eventually) decreasing in T. Proposition 4.2 follows from the last display, (4.2) and (4.3). □

Proposition 4.3. *Let $(\xi_\epsilon^v : v\in V_\epsilon)$ be the MBBM and let m_ϵ be the number defined in the line preceding Theorem 1.1. Then, there exist constants $0<c,C<\infty$ (depending on the dimension d) such that*

$$\mathbb{P}\left(\max_{v\in A}\xi_\epsilon^v\ge m_\epsilon+z\right)\le C\left(\epsilon^d|A|\right)^{1/2}e^{-cz}$$

for all $A\subset V_\epsilon$, $z\in\mathbb{R}$ and $\epsilon>0$ small enough.

Proof. We introduce the d-ary branching Brownian motion (BBM) as follows: let $\epsilon = 2^{-n}$ for some $n \in \mathbb{N}$. At each time $T_k = k \log 2; k = 0, 1, \dots, n$, we partition $[0,1)^d$ into 2^{kd} disjoint boxes of side length 2^{-k}. For a pair $v, w \in V_\epsilon$, denote by $l(v,w)$ the first time that v, w lie in different boxes of the partition. With this notation, define the BBM as the Gaussian field $(\eta_\epsilon^v(t) : v \in V_\epsilon, t \in [0, T_n])$ with

$$Cov(\eta_\epsilon^v(t), \eta_\epsilon^w(s)) = \min\{t, s, l(v,w)\}.$$

For simplicity, let $T = T_n$ and $\eta_\epsilon^v = \eta_\epsilon^v(T)$. It is not hard to show that such a field exists. Note that our BBM can be interpreted as a branching Brownian motion that splits every $\log 2$ units of time into 2^d independent Brownian motions. Following the argument given in [15, Lemma 3.7], one can show that there exists C (depending on the dimension) such that

$$\mathbb{P}\left(\max_{v\in A} \xi_\epsilon^v \geq m_\epsilon + \lambda\right) \leq C\mathbb{P}\left(\max_{v\in A} \eta_{\epsilon/C}^v \geq m_\epsilon + \lambda\right)$$

for all $A \subset V_\epsilon \subset V_{\epsilon/C}$ and all $\lambda \in \mathbb{R}$. Therefore, it is enough to prove Proposition 4.3 for the BBM. We do so by following very closely the proof in [5, Lemma 3.8].

We will use the following estimate, which is proved in [5, Lemma 3.6]: let W_s be standard Brownian motion under $\mathbb{P}$ and fix a large constant C_1. Then, if

$$\mu_{q,r}^*(x) = \mathbb{P}\left(W_q \in dx, W_s \leq r + C_1(\min\{s, q-s\})^{1/20} \text{ for all } 0 \leq s \leq q\right)/dx,$$

we have

$$\mu_{q,r}^*(x) \leq C_2 r(r-x)/q^{3/2} \tag{4.9}$$

for all $x \leq r$, where C_2 depends on C_1.

We next define the event

$$G(\lambda) = \left\{\exists t \leq T, v \in V_\epsilon : \eta_\epsilon^v(t) - \frac{m_\epsilon}{T}t - 10\log\left(\min\{t, T-t\}\right)_+ \geq \lambda\right\}$$

and we prove the following claim:

Claim 4.4. *There exists a constant $C > 0$ (depending on d) such that*

$$\mathbb{P}(G(\lambda)) \leq C\lambda e^{-\sqrt{2d}\lambda}$$

for all $\lambda \geq 1$.

Proof. Following the proof of [5, Lemma 3.7], we define $\psi_t = \lambda + 10\log\left(\min\{t, T-t\}\right)_+$ and $\chi_{T_k}(x) = \mathbb{P}\left(\eta_\epsilon^v(t) - \frac{m_\epsilon}{T}t \leq \psi_t \text{ for all } t \leq T_k, \eta_\epsilon^v(T_k) - \frac{m_\epsilon}{T}T_k \in dx\right)/dx$. Then, by decomposing based on the first time such that $\eta_\epsilon^v(t) - \frac{m_\epsilon}{T}t \geq \psi_t$, we obtain that

$$\mathbb{P}(G(\lambda)) \leq \sum_{k=1}^n 2^{dk} \int_{-\infty}^{\psi_{T_k}} \chi_{T_k}(x)\mathbb{P}\left(\max_{s\leq \log 2} \eta_\epsilon^v(s) \geq \psi_{T_k} - x - C\right) dx,$$

where C is an absolute constant. Display (4.9) and Girsanov's Theorem imply that

$$\chi_{T_k}(x) \leq C2^{-dk}e^{-x\left(\sqrt{2d} - O(\log T/T)\right)}\psi_{T_k}(\psi_{T_k} - x),$$

where C depends on d. On the other hand,

$$\mathbb{P}\left(\max_{s\leq \log 2} \eta_\epsilon^v(t) \geq \psi_{T_k} - x - C\right) \leq Ce^{-\left(\psi_{T_k} - x - C\right)^2/2\log 2}$$

for some absolute constant C. Therefore, by the three previous displays, we obtain

$$\mathbb{P}\left(G(\lambda)\right) \le C \sum_{k=1}^{n} \psi_{T_k} \int_{-\infty}^{\psi_{T_k}} e^{-x\left(\sqrt{2d}-O(\log T/T)\right)} (\psi_{T_k}-x) e^{-\left(\psi_{T_k}-x-C\right)^2/2\log 2} dx.$$

A change of variables $u = \psi_{T_k} - x$ yields

$$\mathbb{P}\left(G(\lambda)\right) \le C \sum_{k=1}^{n} \psi_{T_k} e^{-\sqrt{2d}\psi_{T_k}}$$

$$= C \sum_{k=1}^{n} \left(\lambda + 10\log\left(\min\left\{T_k, T-T_k\right\} \vee 1\right)\right) e^{-\sqrt{2d}\left(\lambda+10\log\left(\min\left\{T_k,T-T_k\right\}\vee 1\right)\right)}$$

$$= C \sum_{k=1}^{n} \frac{\left(\lambda + 10\log\left(\min\left\{T_k, T-T_k\right\} \vee 1\right)\right)}{\left(\min\left\{T_k, T-T_k\right\} \vee 1\right)^{10}} e^{-\sqrt{2d}\lambda} \le C\lambda e^{-\sqrt{2d}\lambda},$$

where $\cdot \vee \cdot = \max\{\cdot,\cdot\}$, and the convergence of the last sum is due the exponent 10 in the denominator (with room to spare). □

We now finish the proof of Proposition 4.3. Fix $A \subset V_\epsilon$ and $z \in \mathbb{R}$. For $z + \left(|V_\epsilon| / |A|\right)^{1/4} \ge 1$, let $\lambda = z + \left(|V_\epsilon| / |A|\right)^{1/4}$, and continuing with the notation of Claim 4.4, we let

$$F_v = \left\{\eta_\epsilon^v(t) \le \frac{m_\epsilon}{T} t + \psi_t \text{ for all } 0 \le t \le T, \eta_\epsilon^v \ge m_\epsilon + z\right\},$$

where $v \in V_\epsilon$. We now compute

$$\mathbb{P}\left(F_v(\lambda)\right) = \int_z^{\psi_T} \frac{d\mathbb{P}}{d\mathbb{Q}}(x+m_\epsilon)\chi_T(x) dx$$

$$\le C \int_z^{\psi_T} 2^{-dn} e^{-x\left(\sqrt{2d}-O(\log T/T)\right)} \psi_T \left(\psi_T - x\right) dx$$

$$\le C 2^{-dn} \psi_T e^{-\sqrt{2d}\psi_T} \int_0^{\psi_T - z} e^u u\, du \le C 2^{-dn} \psi_T e^{-\sqrt{2d}z} \left(\psi_T - z\right).$$

Recalling that $\psi_T = \lambda = z + \left(|V_\epsilon| / |A|\right)^{1/4}$, we obtain

$$\mathbb{P}\left(F_v(\lambda)\right) \le C 2^{-dn} \left(z + \left(|V_\epsilon| / |A|\right)^{1/4}\right) \left(|V_\epsilon| / |A|\right)^{1/4} e^{-\sqrt{2d}z}$$

$$\le C 2^{-dn} \left(|V_\epsilon| / |A|\right)^{1/2} e^{-cz}.$$

Adding the last display for $v \in A$ and using Claim 4.4, we obtain

$$\mathbb{P}\left(\max_{v\in A} \eta_\epsilon^v \ge m_\epsilon + z\right) \le C\left(\epsilon^d |A|\right)^{1/2} e^{-cz} + C\left(z + \left(|V_\epsilon| / |A|\right)^{1/4}\right) e^{-\sqrt{2d}\left(z+\left(|V_\epsilon|/|A|\right)^{1/4}\right)}$$

$$\le C\left(\epsilon^d |A|\right)^{1/2} e^{-cz}$$

for some $0 < c, C < \infty$ (depending on d only), as desired. The previous computation was made under the assumption $z + \left(|V_\epsilon| / |A|\right)^{1/4} \ge 1$. Assume now $\left(|V_\epsilon| / |A|\right)^{1/4} - 1 \le -z$. In this case,

$$\left(\epsilon^d |A|\right)^{1/2} e^{-cz} \ge c\left(\epsilon^d |A|\right)^{1/2} e^{c\left(\epsilon^d|A|\right)^{-1/4}}.$$

But $\inf_{0<x<1} x^{1/2} e^{cx^{-1/4}} \ge c > 0$, where c depends only d. Therefore, in this case, Proposition 4.3 holds trivially by adjusting the constant C. □

Proposition 4.5. *Let* $(\psi_\epsilon^x : x \in \square_\epsilon^v)$ *be the Brownian sheet defined in* (2.6)*. Then, for all* $x, y \in \square_\epsilon^v$*,*

$$p^d \epsilon^{-1} \|x-y\|_1 \leq \mathbb{E}\left[(\psi_\epsilon^x - \psi_\epsilon^y)^2\right] \leq (2p)^d \epsilon^{-1} \|x-y\|_1 .$$

Proof. By (2.6),

$$\mathbb{E}\left[(\psi_\epsilon^x - \psi_\epsilon^y)^2\right] = \mathbb{E}\left[\left(\psi^{l(x)} - \psi^{l(y)}\right)^2\right], \tag{4.10}$$

where l is the linear map from $\square_\epsilon^v$ onto $[p, 2p)^d$ sending v to $(p)_{1\leq i\leq d} = (p, p, \ldots, p)$. Call $l(x) = x'$ and $l(y) = y'$. Note that

$$\mathbb{E}\left[\left(\psi^{x'} - \psi^{y'}\right)^2\right] = \left(\prod_{1\leq i\leq d} x_i' - \prod_{1\leq i\leq d} \min\{x_i', y_i'\}\right) + \left(\prod_{1\leq i\leq d} y_i' - \prod_{1\leq i\leq d} \min\{x_i', y_i'\}\right) =: A+B. \tag{4.11}$$

Consider the first term, A. Adding and subtracting the intermediate terms

$$\left(\prod_{j\leq i\leq d} x_i'\right)\left(\prod_{1\leq i\leq j-1} \min\{x_i', y_i'\}\right)$$

for $j = 2, \ldots, d$, we obtain

$$\begin{aligned} A &= \sum_{1\leq j\leq d}\left(\left(\prod_{j\leq i\leq d} x_i'\right)\left(\prod_{1\leq i\leq j-1} \min\{x_i', y_i'\}\right) - \left(\prod_{j+1\leq i\leq d} x_i'\right)\left(\prod_{1\leq i\leq j} \min\{x_i', y_i'\}\right)\right) \\ &= \sum_{1\leq j\leq d}\left(\prod_{j+1\leq i\leq d} x_i'\right)\left(\prod_{1\leq i\leq j-1} \min\{x_i', y_i'\}\right)\left(x_j' - \min\left\{x_j', y_j'\right\}\right). \end{aligned}$$

Since both x' and y' belong to $[p, 2p)^d$, we obtain

$$p^{d-1} \sum_{1\leq j\leq d}\left(x_j' - \min\left\{x_j', y_j'\right\}\right) \leq A \leq (2p)^{d-1} \sum_{1\leq j\leq d}\left(x_j' - \min\left\{x_j', y_j'\right\}\right).$$

An analogous expression holds for B. Then,

$$p^{d-1} \sum_{1\leq j\leq d}\left(x_j' + y_j' - 2\min\left\{x_j', y_j'\right\}\right) \leq A+B \leq (2p)^{d-1} \sum_{1\leq j\leq d}\left(x_j' + y_j' - 2\min\left\{x_j', y_j'\right\}\right),$$

so, from the last display, (4.10) and (4.11)

$$p^{d-1} \|x'-y'\|_1 \leq \mathbb{E}\left[(\psi_\epsilon^x - \psi_\epsilon^y)^2\right] \leq (2p)^{d-1} \|x'-y'\|_1 .$$

But, from the definition of x' and y', we see that $\|x'-y'\|_1 = \epsilon^{-1} p \|x-y\|_1$. Therefore,

$$p^d \epsilon^{-1} \|x-y\|_1 \leq \mathbb{E}\left[(\psi_\epsilon^x - \psi_\epsilon^y)^2\right] \leq (2p)^d \epsilon^{-1} \|x-y\|_1 ,$$

as desired. □

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