# Central limit theorem for eigenvectors of heavy tailed matrices

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## Abstract

We consider the eigenvectors of symmetric matrices with independent heavy tailed entries, such as matrices with entries in the domain of attraction of $\alpha$-stable laws, or adjacency matrices of Erdös-Rényi graphs. We denote by $U = [u_{ij}]$ the eigenvectors matrix (corresponding to increasing eigenvalues) and prove that the bivariate process

$$B^n_{s,t} := \frac{1}{\sqrt{n}} \sum_{\substack{1 \le i \le ns \\ 1 \le j \le nt}} \left( |u_{ij}|^2 - \frac{1}{n} \right) \qquad (0 \le s, t \le 1),$$

converges in law to a non trivial Gaussian process. An interesting part of this result is the $\frac{1}{\sqrt{n}}$ rescaling, proving that from this point of view, the eigenvectors matrix $U$ behaves more like a permutation matrix (as it was proved in [17] that for $U$ a permutation matrix, $\frac{1}{\sqrt{n}}$ is the right scaling) than like a Haar-distributed orthogonal or unitary matrix (as it was proved in [18, 5] that for $U$ such a matrix, the right scaling is 1).

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## 1 Introduction

During the last decade, many breakthroughs were achieved in the study of random matrices belonging to the GUE universality-class, that is Hermitian matrices with independent and equidistributed entries (modulo the symmetry constraint) with enough finite moments. The first key result about such matrices is due to Wigner [39] in the fifties who showed that the macroscopic behavior of their eigenvalues is universal and asymptotically described by the semi-circle distribution. However, it took a long time to

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get more precise information on the local behavior of the eigenvalues, and for instance about the asymptotic distribution of their spacings. Even though local results were conjectured, for instance by Dyson and Mehta [30], it is only in the nineties that rigorous results were derived, such as the convergence of the joint probability distribution of eigenvalues in an interval of size of order $N^{-1}$ or the fluctuations of the largest eigenvalues, see [37]. Yet these results were restricted to Gaussian ensembles for which the joint law of the eigenvalues is known. Recently, these results were shown to be universal, that is to hold also for matrices with independent non Gaussian entries, provided they have enough finite moments [20, 23, 25, 26, 35]. Such a simple question as the convergence of the law of a single spacing was open, even in the GUE case, until recently when it was solved by Tao [36]. Once considering non Gaussian matrices, it is natural to wonder about the behavior of the eigenvectors and whether they are delocalized (that is go to zero in $L^\infty$ norm) as for GUE matrices. This was indeed shown by Erdös, Schlein and Yau [21].

Despite the numerous breakthroughs concerning random matrices belonging to the GUE universality-class, not much is yet known about other matrices. A famous example of such a matrix is given by the adjacency matrix of an Erdös-Rényi graph. Its entries are independent (modulo the symmetry hypothesis) and equal to one with probability $p = p(N)$, zero otherwise. If $pN$ goes to infinity fast enough, the matrix belongs to the GUE universality class [19]. However if $pN$ converges to a finite non zero constant, the matrix behaves quite differently, more like a "heavy tailed random matrix", *i.e.* a matrix filled with independent entries which have no finite second moment. Also in this case, it is known that, once properly normalized, the empirical measure of the eigenvalues converges weakly but the limit differs from the semi-circle distribution [38, 7, 6, 9, 15, 10]. Moreover, the fluctuations of the empirical measure could be studied [10, 31, 27, 28]. It turns out that it fluctuates much more than in the case of matrices from the GUE universality-class, as fluctuations are square root of the dimension bigger. However, there is no result about the local fluctuations of the eigenvalues except in the case of matrices with entries in the domain of attraction of an $\alpha$-stable law in which case it was shown [2, 34] that the largest eigenvalues are much bigger than the others, converge to a Poisson distribution and have localized eigenvectors. About localization and delocalization of the eigenvectors, some models are conjectured [16, 33] to exhibit a phase transition; eigenvalues in a compact would have more delocalized eigenvectors than outside this compact. Unfortunately, very little could be proved so far in this direction. Only the case where the entries are $\alpha$-stable random variables could be tackled [15]; it was shown that for $\alpha > 1$ the eigenvectors are delocalized whereas for $\alpha < 1$ and large eigenvalues, a weak form of localization holds.

In this article, we study another type of properties of the eigenvectors of a random matrix. Namely we consider the bivariate process

$$G_{s,t}^n := \sum_{\substack{1 \le i \le ns \\ 1 \le j \le nt}} \left(|u_{ij}|^2 - \frac{1}{n}\right) \qquad (0 \le s \le 1, \quad 0 \le t \le 1),$$

where $U = [u_{ij}]$ is an orthogonal matrix whose columns are the eigenvectors of an Hermitian random matrix $A = [a_{k\ell}]$. In the case where $A$ is a GUE matrix [18], and then a more general matrix in the GUE universality-class [8], it was shown that this process converges in law towards a bivariate Brownian bridge (see also the closely related issues considered in [4]). Here, we investigate the same process in the case where $A$ is a heavy tailed random matrix and show that it fluctuates much more, namely it is $\frac{1}{\sqrt{n}}G^n$ which converges in law. The limit is a Gaussian process whose covariance

depends on the model through the function

$$\Phi(\lambda) = \lim_{n \to \infty} n \, \mathbb{E}[e^{-i\lambda a_{k\ell}} - 1].$$

Of course if one considers heavy tailed variables in the domain of attraction of the same $\alpha$-stable law, and they suitably renormalize them, the function $\Phi$ and therefore the covariance will be the same. However, the covariance may vary when $\Phi$ does and it is not trivial in the sense that it does not vanish uniformly if $\Phi$ is not linear (a case which corresponds to light tails). In the related setting when the $u_{ij}$'s are the entries of a uniformly distributed random permutation, *i.e.* of a somehow sparse matrix, the process $\frac{1}{\sqrt{n}} G^n$ also converges in law, towards the bivariate Brownian bridge [17].

More precisely, we consider a real symmetric random $n \times n$ matrix $A$ that can be either a Wigner matrix with exploding moments (which includes the adjacency matrix for Erdös-Rényi graphs) or a matrix with i.i.d. entries in the domain of attraction of a stable law (or more generally a matrix satisfying the hypotheses detailed in Hypothesis 2.1). We then introduce an orthogonal matrix $U = [u_{ij}]$ whose columns are the eigenvectors of $A$ so that we have $A = U \operatorname{diag}(\lambda_1, \ldots, \lambda_n) U^*$. We then define the bivariate processes

$$B_{s,t}^n := \frac{1}{\sqrt{n}} \sum_{\substack{1 \leq i \leq ns \\ 1 \leq j \leq nt}} \left( |u_{ij}|^2 - \frac{1}{n} \right) \qquad (0 \leq s \leq 1, \quad 0 \leq t \leq 1)$$

and

$$C_{s,\lambda}^n := \frac{1}{\sqrt{n}} \sum_{\substack{1 \leq i \leq ns \\ 1 \leq j \leq n \, ; \, \lambda_j \leq \lambda}} \left( |u_{ij}|^2 - \frac{1}{n} \right) \qquad (0 \leq s \leq 1, \quad \lambda \in \mathbb{R})$$

and prove, in Theorem 2.5, that both of these processes (with a little technical restriction on the domain of $B$) converge in law to (non trivial) Gaussian processes linked by the relation

$$B_{s, F_{\mu_\Phi}(\lambda)} = C_{s,\lambda},$$

where $F_{\mu_\Phi}(\lambda) = \mu_\Phi((-\infty, \lambda])$ denotes the cumulative distribution function of the limit spectral law $\mu_\Phi$ of $A$, *i.e.*

$$F_{\mu_\Phi}(\lambda) = \lim_{n \to \infty} F_n(\lambda), \quad \text{with} \quad F_n(\lambda) := \frac{1}{n} |\{i \, ; \, \lambda_i \leq \lambda\}|. \tag{1.1}$$

The idea of the proof is the following one. We first notice that for any $s \in [0, 1]$, the function $\lambda \mapsto C_{s,\lambda}^n$ is the cumulative distribution function of the random signed measure $\nu_{s,n}$ on $\mathbb{R}$ defined by

$$\nu_{s,n} := \frac{1}{\sqrt{n}} \sum_{1 \leq i \leq ns} \sum_{j=1}^{n} \left( |u_{ij}|^2 - \frac{1}{n} \right) \delta_{\lambda_j} \tag{1.2}$$

(*i.e.* that for any $\lambda \in \mathbb{R}$, $C_{s,\lambda}^n = \nu_{s,n}((-\infty, \lambda])$). Then, we introduce the Cauchy transform $X^n(s, z) := \int_{\lambda \in \mathbb{R}} \frac{\mathrm{d}\nu_{s,n}(\lambda)}{z - \lambda}$ of $\nu_{s,n}$ and prove (Proposition 2.9) that the process $(X^n(s, z))_{s,z}$ converges in law to a limit Gaussian process $(H_{s,z})$. This convergence is proved thanks to the classical CLT for martingales (Theorem 6.4 of the Appendix) together with the Schur complement formula and fixed points characterizations like the ones of the papers [7, 6, 10]. Then to deduce the convergence in law of the process $(C_{s,\lambda}^n)_{s,\lambda}$, we use the idea that the cumulative distribution function of a signed measure is entirely determined by its Cauchy transform. In fact, as the measures $\nu_{s,n}$ of (1.2) are random, things are slightly more complicated, and we need to prove a tightness lemma for the process $(C_{s,\lambda}^n)_{s,\lambda}$ (specifically Lemma 6.1 of the Appendix, first applied to

the process $(B^n_{s,t})$ and then transferred to $(C^n_{s,\lambda})$ by Formula (1.3) below). This lemma reduces the problem to the proof of the unicity of the possible limits for $(C^n_{s,\lambda})$. Then, we use the formula

$$\int_{\lambda \in \mathbb{R}} \frac{C^n_{s,\lambda}}{(z-\lambda)^2} \mathrm{d}\lambda = -X^n(s,z)$$

and Lemma 6.2 of the appendix to be able to claim that $(C^n_{s,\lambda})_{s,\lambda}$ has a unique limit point. The result proved for $(C^n_{s,\lambda})$ can then be transferred to $(B^n_{s,t})$ thanks to Formula (1.3) below, where $F_n(\lambda)$, defined at (1.1) above, converges to the deterministic limit $F_{\mu_\Phi}(\lambda)$:

$$C^n_{s,\lambda} = B^n_{s,F_n(\lambda)}. \tag{1.3}$$

**Remark 1.1.** *The objects introduced in the previous paragraph for the sketch of the proof enlighten the reason of the presence of the factor $\frac{1}{\sqrt{n}}$ in the definitions of the processes $B^n_{s,t}$ and $C^n_{s,t}$ (recall that this factor does not appear in the corresponding formulas when Wigner instead of heavy-tailed matrices are concerned). Let $\mu_n$ denote the empirical spectral law of $A$ and let, for $i = 1,\ldots,n$, $\mu_{n,\mathbf{e}_i}$ denote the empirical spectral law of $A$ according to the $i$th vector $\mathbf{e}_i$ for the canonical basis: that is, for any test function $f$,*

$$\int f(x)\mathrm{d}\mu_n(x) = \frac{1}{n}\operatorname{Tr}(f(A)) \quad ; \quad \int f(x)\mathrm{d}\mu_{n,\mathbf{e}_i}(x) = \sum_{j=1}^n |u_{i,j}|^2 f(\lambda_j) = (f(A))_{ii}. \tag{1.4}$$

*Then*

$$C^n_{s,\lambda} = \frac{1}{\sqrt{n}} \sum_{i=1}^{ns} (\mu_{n,\mathbf{e}_i} - \mu_n)((-\infty,\lambda]),$$

*so that $C^n_{s,\lambda}$ is the centered version of a sum of random variables $\mu_{n,\mathbf{e}_i}((-\infty,\lambda])$ $(1 \le i \le ns)$. It has been proved in [14] that in the Lévy case, the random probability measures $\mu_{n,\mathbf{e}_i}$ converge to i.i.d. copies of a nontrivial limiting random probability measure (the spectral measure at the root of a suitable random weighted tree). This contrasts with case of Wigner matrices, where concentration implies that the limiting measure of $\mu_{n,\mathbf{e}_i}$ is deterministic, and give a heuristic explanation, in the Lévy case, of why one has to renormalize by $\sqrt{n}$ in $C^n_{s,\lambda}$. Note however that this explanation is not enough to prove that the variance of $C^n_{s,\lambda}$ does not explode nor vanish because $C^n_{s,\lambda}$ is a sum of a large number of $\mu_{n,\mathbf{e}_i}((-\infty,\lambda])$'s, that are correlated at finite $n$ (for example because the process vanishes on the boundary).*

**Organization of the paper.** The main results are stated in Section 2. In Section 3, we give a proof of Theorem 2.5, based on Proposition 2.9, which is proved in Section 4. Proposition 2.7 is proved in Section 5. At last, some technical results are proved or recalled in the Appendix.

**Notations.** For $u$, $v$ depending implicitly on $n$, we write $u \ll v$ when $u/v \longrightarrow 0$ as $n \to \infty$. For $x$ a random variable, $\operatorname{Var}(x)$ denotes the *variance* of $x$, *i.e.* $\mathbb{E}[|x|^2] - |\mathbb{E}x|^2$. Power functions are defined on $\mathbb{C}\backslash\mathbb{R}_-$ via the standard determination of the argument on this set taking values in $(-\pi,\pi)$. The set $\mathbb{C}^+$ (resp. $\mathbb{C}^-$) denotes the open upper (resp. lower) half plane and for any $z \in \mathbb{C}$, $\operatorname{sgn}_z := \operatorname{sign}(\Im z)$. At last, bor any variable $x$, $\partial_x$ denotes $\frac{\partial}{\partial x}$.

## 2 Main results

Although technical, the model introduced in Hypothesis 2.1 below has the advantage to be general enough to contain several models of interest.

**Hypothesis 2.1.** *Let, for each $n \geq 1$, $A_n = [a_{ij}]$ be an $n \times n$ real symmetric random matrix whose sub-diagonal entries are some i.i.d. copies of a random variable $a$ (depending implicitly on $n$) such that:*
- *The random variable $a$ can be decomposed into $a = b + c$ such that as $n \to \infty$,*

$$\mathbb{P}(c \neq 0) \ll n^{-1} \tag{2.1}$$

$$\mathrm{Var}(b) \ll n^{-1/2} \tag{2.2}$$

*Moreover, if the $b_i$'s are independent copies of $b$,*

$$\lim_{K \nearrow \infty} \lim_{n \nearrow \infty} \mathbb{P}\left( \sum_{i=1}^{n} (b_i - \mathbb{E}(b_i))^2 \geq K \right) = 0. \tag{2.3}$$

- *For any $\varepsilon > 0$ independent of $n$, the random variable $a$ can be decomposed into $a = b_\varepsilon + c_\varepsilon$ such that*

$$\limsup_{n \to \infty} n\, \mathbb{P}(c_\varepsilon \neq 0) \leq \varepsilon \tag{2.4}$$

*for all $k \geq 1$, $n\, \mathbb{E}[(b_\varepsilon - \mathbb{E}\, b_\varepsilon)^{2k}]$ has a finite limit $C_{\varepsilon,k}$ as $n \to \infty$.*
- *For $\phi_n$ the function defined on the closure $\overline{\mathbb{C}^-}$ of $\mathbb{C}^- := \{\lambda \in \mathbb{C}\, ; \Im \lambda < 0\}$ by*

$$\phi_n(\lambda) := \mathbb{E}\left[ \exp(-i\lambda a^2) \right], \tag{2.5}$$

*we have the convergence, uniform on compact subsets of $\overline{\mathbb{C}^-}$,*

$$n(\phi_n(\lambda) - 1) \longrightarrow \Phi(\lambda), \tag{2.6}$$

*for a certain function $\Phi$ defined on $\overline{\mathbb{C}^-}$.*
- *The function $\Phi$ of (2.6) admits the decomposition*

$$\Phi(z) = \int_0^\infty g(y) e^{i \frac{y}{z}} \mathrm{d}y \tag{2.7}$$

*where $g(y)$ is a function such that for some constants $K, \gamma > -1, \kappa \geq 0$, we have*

$$|g(y)| \leq K \mathbb{1}_{y \leq 1} y^\gamma + K \mathbb{1}_{y \geq 1} y^\kappa, \qquad \forall y > 0. \tag{2.8}$$

- *The function $\Phi$ of (2.6) also either has the form*

$$\Phi(x) = -\sigma(ix)^{\alpha/2} \tag{2.9}$$

*or admits the (other) decomposition, for $x, y$ non zero:*

$$\Phi(x+y) = \iint_{(\mathbb{R}^+)^2} e^{i \frac{v}{x} + i \frac{v'}{y}} \mathrm{d}\tau(v,v') + \int_{\mathbb{R}^+} e^{i \frac{v}{x}} \mathrm{d}\mu(v) + \int_{\mathbb{R}^+} e^{i \frac{v'}{y}} \mathrm{d}\mu(v') \tag{2.10}$$

*for some complex measures $\tau, \mu$ on respectively $(\mathbb{R}^+)^2$ and $\mathbb{R}^+$ such that for all $b > 0$, $\int e^{-bv} \mathrm{d}|\mu|(v)$ is finite and for some constants $K > 0$, $-1 < \gamma \leq 0$ and $\kappa \geq 0$, and*

$$\frac{\mathrm{d}|\tau|(v,v')}{\mathrm{d}v \mathrm{d}v'} \leq K\big(v^\gamma \mathbb{1}_{v \in ]0,1]} + v^\kappa \mathbb{1}_{v \in ]1,\infty[}\big)\big(v'^\gamma \mathbb{1}_{v' \in ]0,1]} + v'^\kappa \mathbb{1}_{v' \in ]1,\infty[}\big). \tag{2.11}$$

**Remark 2.2.** *When $\Phi$ satisfies (2.9) (e.g. for Lévy matrices), (2.10) holds as well. Indeed, for all $x, y \in \mathbb{C}^+$ (with a constant $C_\alpha$ that can change at every line),*

$$\Phi(x^{-1} + y^{-1}) = C_\alpha \big(\frac{1}{x} + \frac{1}{y}\big)^{\alpha/2} = C_\alpha \frac{1}{x^{\alpha/2}} \frac{1}{y^{\alpha/2}} (x+y)^{\alpha/2} \tag{2.12}$$

$$= C_\alpha \int_0^\infty \mathrm{d}w \int_0^\infty \mathrm{d}w' \int_0^\infty \mathrm{d}v\, w^{\alpha/2-1} w'^{\alpha/2-1} v^{-\alpha/2-1} e^{iwx + iw'y} (e^{iv(x+y)} - 1)$$

(where we used the formula $z^{\alpha/2} = C_\alpha \int_{t=0}^{+\infty} \frac{e^{itz}-1}{t^{\alpha/2+1}} \mathrm{d}t$ for any $z \in \mathbb{C}^+$ and $\alpha \in (0,2)$, which can be proved with the residues formula) so that (2.10) holds with $\mu = 0$ and $\tau(v,v')$ with density with respect to Lebesgue measure given by

$$C_\alpha \int_0^{+\infty} u^{-\alpha/2-1}\{(v-u)^{\alpha/2-1}(v'-u)^{\alpha/2-1}\mathbb{1}_{0 \leq u \leq v \wedge v'} - v^{\alpha/2-1}v'^{\alpha/2-1}\}\mathrm{d}u \quad (2.13)$$

Unfortunately, $\tau$ does not satisfy (2.11) as its density blows up at $v = v'$: we shall treat both case separately.

Examples of random matrices satisfying Hypothesis 2.1 are defined as follows.

**Definition 2.3** (Models of symmetric heavy tailed matrices). *Let $A = (a_{i,j})_{i,j=1,\ldots,n}$ be a random symmetric matrix with i.i.d. sub-diagonal entries.*

1. *We say that $A$ is a **Lévy matrix** of parameter $\alpha$ in $]0,2[$ when $A = X/a_n$ where the entries $x_{ij}$ of $X$ have absolute values in the domain of attraction of $\alpha$-stable distribution, more precisely*

$$\mathbb{P}\left(|x_{ij}| \geq u\right) = \frac{L(u)}{u^\alpha} \quad (2.14)$$

   *with a slowly varying function $L$, and*

$$a_n = \inf\{u : P\left(|x_{ij}| \geq u\right) \leq \frac{1}{n}\}$$

   *($a_n = \tilde{L}(n)n^{1/\alpha}$, with $\tilde{L}(\cdot)$ a slowly varying function[1]).*

2. *We say that $A$ is a **Wigner matrix with exploding moments** with parameter $(C_k)_{k \geq 1}$ whenever the entries of $A$ are centered, and for any $k \geq 1$*

$$n\,\mathbb{E}\left[(a_{ij})^{2k}\right] \underset{n \to \infty}{\longrightarrow} C_k > 0. \quad (2.15)$$

   *We assume that there exists a unique measure $m$ on $\mathbb{R}^+$ such that for all $k \geq 0$,*

$$C_{k+1} = \int x^k \mathrm{d}m(x). \quad (2.16)$$

The following proposition has been proved at Lemmas 1.3, 1.8 and 1.11 of [10].

**Proposition 2.4.** *Both Lévy matrices and Wigner matrices with exploding moments satisfy Hypothesis 2.1:*
 • *For Lévy matrices, the function $\Phi$ of (2.6) is given by formula*

$$\Phi(\lambda) = -\sigma(i\lambda)^{\alpha/2} \quad (2.17)$$

*for some constant $\sigma > 0$, the function $g$ of (2.7) is $g(y) = C_\alpha y^{\frac{\alpha}{2}-1}$, with $C_\alpha = -\sigma i^{\alpha/2}$.*
 • *For Wigner matrices with exploding moments, the function $\Phi$ of (2.6) is given by*

$$\Phi(\lambda) = \int \underbrace{\frac{e^{-i\lambda x}-1}{x}}_{:=-i\lambda \text{ for } x=0} \mathrm{d}m(x), \quad (2.18)$$

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[1]A function $\tilde{L}$ is said to be *slowly varying* $\tilde{L}$ if for any fixed $\lambda > 0$, $\tilde{L}(\lambda x)/\tilde{L}(x) \longrightarrow 1$ as $x \to \infty$.

*for $m$ the measure of (2.16), the function $g$ of (2.7) is*

$$g(y) = -\int_{\mathbb{R}^+} \underbrace{\frac{J_1(2\sqrt{xy})}{\sqrt{xy}}}_{:=1 \text{ for } xy=0} \mathrm{d}m(x), \tag{2.19}$$

*for $J_1$ the Bessel function of the first kind defined by $J_1(s) = \dfrac{s}{2} \sum_{k \geq 0} \dfrac{(-s^2/4)^k}{k!(k+1)!}$, and the measures $\tau$ and $\mu$ of (2.10) are absolutely continuous with densities*

$$\frac{\mathrm{d}\tau(v,v')}{\mathrm{d}v\mathrm{d}v'} := \int \frac{J_1(2\sqrt{vx})J_1(2\sqrt{v'x})}{\sqrt{vv'}}\mathrm{d}m(x) \quad \textit{and} \quad \frac{\mathrm{d}\mu(v)}{\mathrm{d}v} := -\int \frac{J_1(2\sqrt{vx})}{\sqrt{v}}\mathrm{d}m(x). \tag{2.20}$$

One can easily see that our results also apply to complex Hermitian matrices: in this case, one only needs to require Hypothesis 2.1 to be satisfied by the absolute value of non diagonal entries and to have $a_{11}$ going to zero as $N \to \infty$.

A Lévy matrix whose entries are truncated in an appropriate way is a Wigner matrix with exploding moments [7, 29, 38]. The recentered version[2] of the adjacency matrix of an Erdös-Rényi graph, *i.e.* of a matrix $A$ such that

$$A_{ij} = 1 \text{ with probability } p/n \text{ and } 0 \text{ with probability } 1 - p/n, \tag{2.21}$$

is also an exploding moments Wigner matrix, with $\Phi(\lambda) = p(e^{-i\lambda} - 1)$ (the measure $m$ is $p\delta_1$). In this case the fluctuations were already studied in [31].

It has been proved in [10] (see also [38, 31]) that under Hypothesis 2.1, the empirical spectral law

$$\mu_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j} \tag{2.22}$$

converges weakly in probability to a deterministic probability measure $\mu_\Phi$ that depends only on $\Phi$, *i.e.* that for any continuous bounded function $f : \mathbb{R} \to \mathbb{C}$, we have the almost sure convergence

$$\frac{1}{n} \operatorname{Tr} f(A) = \frac{1}{n} \sum_{j=1}^{n} f(\lambda_j) \xrightarrow[n\to\infty]{} \int f(x)\mathrm{d}\mu_\Phi(x). \tag{2.23}$$

We introduce $F_{\mu_\Phi}(\lambda) := \mu_\Phi((-\infty, \lambda])$, cumulative distribution function of $\mu_\Phi$, and define the set $E_\Phi \subset [0,1]$ by

$$E_\Phi := \{0\} \cup F_{\mu_\Phi}(\mathbb{R}) \cup \{1\}. \tag{2.24}$$

In the case of Lévy matrices, it has been proved in [6, Theorem 1.3] that $\mu_\Phi$ has no atoms (because it is absolutely continuous), so that $E_\Phi = [0,1]$.

We introduce the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ of $A$ and an orthogonal matrix $U = [u_{ij}]$ such that $A = U \operatorname{diag}(\lambda_1, \ldots, \lambda_n)U^*$. We assume $U$ defined in such a way that the rows of the matrix $[|u_{ij}|]$ are exchangeable (this is possible[3] because $A$ is invariant, in law, by conjugation by any permutation matrix). Then define the bivariate processes

$$B^n_{s,t} := \frac{1}{\sqrt{n}} \sum_{\substack{1 \leq i \leq ns \\ 1 \leq j \leq nt}} \left(|u_{ij}|^2 - \frac{1}{n}\right) \qquad (0 \leq s \leq 1, \quad 0 \leq t \leq 1)$$

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[2]The recentering has in fact asymptotically no effect on the spectral measure $A$ as it is a rank one perturbation.

[3]Such a matrix $U$ can be defined, for example, by choosing some orthogonal bases of all eigenspaces of $A$ with uniform distributions, independently with each other and independently of $A$ (given its eigenspaces of course).

and

$$C^n_{s,\lambda} := \frac{1}{\sqrt{n}} \sum_{\substack{1 \le i \le ns \\ 1 \le j \le n \,;\, \lambda_j \le \lambda}} \left(|u_{ij}|^2 - \frac{1}{n}\right) \qquad (0 \le s \le 1, \quad \lambda \in \mathbb{R}).$$

The following theorem is the main result of this article. We endow $D([0,1]^2)$ and $D([0,1] \times \mathbb{R})$ with the Skorokhod topology and $D([0,1] \times E_\Phi)$ with the topology induced by the Skorokhod topology on $D([0,1]^2)$ by the projection map from $D([0,1]^2)$ onto $D([0,1] \times E_\Phi)$ (see Section 4.1 of [8] for the corresponding definitions).

**Theorem 2.5.** *As $n \to \infty$, the joint distribution of the processes*

$$(B^n_{s,t})_{(s,t) \in [0,1] \times E_\Phi} \qquad and \qquad (C^n_{s,\lambda})_{(s,\lambda) \in [0,1] \times \mathbb{R}}$$

*converges weakly to the joint distribution of some centered Gaussian processes*

$$(B_{s,t})_{(s,t) \in [0,1] \times E_\Phi} \qquad and \qquad (C_{s,\lambda})_{(s,\lambda) \in [0,1] \times \mathbb{R}}$$

*vanishing on the boundaries of their domains and satisfying the relation*

$$B_{s, F_{\mu_\Phi}(\lambda)} = C_{s,\lambda} \tag{2.25}$$

*for all $s \in [0,1]$, $\lambda \in \mathbb{R}$. Moreover, the process $(B_{s,t})_{(s,t) \in [0,1] \times E_\Phi}$ is continuous.*

**Remark 2.6.** *Note that the limit of $B^n_{s,t}$ is only given here when $t \in E_\Phi$, i.e. when $t$ is not in the "holes" of $F_{\mu_\Phi}(\mathbb{R})$. But as these holes result from the existence of some atoms in the limit spectral distribution of $A$, the variations of $B^n_{s,t}$ when $t$ varies in one of these holes may especially depend on the way we choose the columns of $A$ for eigenvalues with multiplicity larger than one. By the results of [18], in the case where the atoms of $\mu_\Phi$ result in atoms (with asymptotically same weight) of $\mu_n$, the choice we made here should lead to a limit process $(B_{s,t})_{(s,t) \in [0,1]^2}$ which would interpolate $(B_{s,t})_{(s,t) \in [0,1] \times E_\Phi}$ with some Brownian bridges in these "holes", namely for when $t \in [0,1] \backslash E_\Phi$.*

The following proposition insures that the $\dfrac{1}{\sqrt{n}}$ scaling in the definitions of $B^n_{s,t}$ and $C^n_{s,\lambda}$ is the right one.

**Proposition 2.7.** *If the function $\Phi(z)$ of (2.6) is not linear in $z$, then for any fixed $s \in (0,1)$, the covariance of the process $(B_{s,t})_{t \in E_\Phi}$ (hence also that of $(C_{s,\lambda})_{\lambda \in \mathbb{R}}$) is not identically null.*

**Remark 2.8.** *One could wonder if the covariance might vanish uniformly on some compact in the $t$ variable, hence giving some support to the belief that the eigenvectors could behave more alike the eigenvectors of GUE for "small" eigenvalues (in the latter case the covariance should vanish). Unfortunately, it does not seem that the covariance should be so closely related with the localization/delocalization properties of the eigenvectors (see Remark 1.1 instead). Indeed, let us consider Lévy matrices with $\alpha \in (1, 2)$. Their eigenvectors are delocalized [15], so that one could expect the covariance of the process $(B_{s,t})_{t \in E_\Phi}$ to vanish. This is in contradiction with the fact that such matrices enter our model, hence have eigenvectors satisfying Theorem 2.5 and Proposition 2.7.*

To prove Theorem 2.5, a key step will be to prove the following proposition, which also allows to make the variance of the limiting processes in Theorem 2.5 more explicit.

Let us define, for $z \in \mathbb{C} \backslash \mathbb{R}$ and $s \in [0,1]$,

$$X^n(s,z) := \frac{1}{\sqrt{n}} \left( \text{Tr}(P_s \frac{1}{z-A}) - s_n \, \text{Tr} \, \frac{1}{z-A} \right), \tag{2.26}$$

where $P_s$ denotes the diagonal matrix with diagonal entries $\mathbb{1}_{i \leq ns}$ ($1 \leq i \leq n$) and

$$s_n := \frac{1}{n} \operatorname{Tr} P_s = \frac{\lfloor ns \rfloor}{n}. \tag{2.27}$$

**Proposition 2.9.** *The distribution of the random process*

$$(X^n(s,z))_{s \in [0,1], z \in \mathbb{C} \backslash \mathbb{R}}$$

*converges weakly in the sense of finite marginals towards the distribution of a centered Gaussian process*

$$(H_{s,z})_{s \in [0,1], z \in \mathbb{C} \backslash \mathbb{R}} \tag{2.28}$$

*with a covariance given by* (4.23).

As it will appear from the proofs that the process $(C_{s,\lambda})$ of Theorem 2.5 and the process $(H_{s,z})$ from the previous proposition are linked by the formula

$$\int_{\lambda \in \mathbb{R}} \frac{C_{s,\lambda}}{(z-\lambda)^2} \mathrm{d}\lambda = -H_{s,z} \qquad (s \in [0,1],\, z \in \mathbb{C} \backslash \mathbb{R}), \tag{2.29}$$

the covariance of $C_{s,\lambda}$ (hence of $B_{s,t}$ by (2.25)) can be deduced from that of the process $H_{s,z}$ as follows (the proof of this proposition is a direct application of (2.29) and of Formula (6.6) of the Appendix).

**Proposition 2.10.** *For any $s, s' \in [0,1]$ and any $\lambda, \lambda' \in \mathbb{R}$ which are not atoms of $\mu_\Phi$, we have*

$$\mathbb{E}[C_{s,\lambda} C_{s',\lambda'}] = \frac{1}{\pi^2} \lim_{\eta \downarrow 0} \int_{-\infty}^{\lambda} \int_{-\infty}^{\lambda'} \mathbb{E}[\Im(H_{s,E+i\eta}) \Im(H_{s',E'+i\eta})] \mathrm{d}E \mathrm{d}E'. \tag{2.30}$$

*When $\lambda$ or $\lambda'$ is an atom of $\mu_\Phi$, the covariance can be obtained using* (2.30) *and the right continuity of $C_{s,\lambda}$ in $\lambda$.*

## 3  Proof of Theorem 2.5

We introduce the cumulative distribution function

$$F_n(\lambda) := \frac{1}{n} |\{j\,;\, \lambda_j \leq \lambda\}| \tag{3.1}$$

of the empirical spectral law $\mu_n$ defined at (2.22). We shall use the following formula several times: for all $s \in [0,1]$ and $\lambda \in \mathbb{R}$,

$$C_{s,\lambda}^n = B_{s,F_n(\lambda)}^n. \tag{3.2}$$

We know, by Lemma 6.1 of the appendix, that the sequence $(\mathrm{distribution}(B^n))_{n \geq 1}$ is tight and has all its accumulation points supported by the set of continuous functions on $[0,1]^2$. As $F_n$ converges to $F_{\mu_\Phi}$ in the Skorokhod topology, it follows that the sequences

$$\tilde{B}^n := (B_{s,t}^n)_{(s,t) \in [0,1] \times E_\Phi} \qquad \text{and} \qquad (C_{s,\lambda}^n = B_{s,F_n(\lambda)}^n)_{(s,\lambda) \in [0,1] \times \mathbb{R}}$$

are tight in their respective spaces. To prove the theorem, it suffices to prove that the sequence $(\mathrm{distribution}(\tilde{B}^n, C^n))_{n \geq 1}$ has only one accumulation point (which is Gaussian centered, vanishing on the boundaries, supported by continuous functions as far as the first component is concerned and satisfying (2.25)). So let

$$((\tilde{B}_{s,t})_{(s,t) \in [0,1] \times E_\Phi}, (C_{s,\lambda})_{(s,\lambda) \in [0,1] \times \mathbb{R}})$$

be a pair of random processes having for distribution such an accumulation point. By (3.2), we have

$$\tilde{B}_{s, F_{\mu_\Phi}(\lambda)} = C_{s,\lambda}$$

for all $s \in [0,1]$, $\lambda \in \mathbb{R}$. Hence it suffices to prove that the distribution of $C$ is totally prescribed and Gaussian centered.

First, let us note that one can suppose that along the corresponding subsequence, the distribution of $((B^n_{s,t})_{(s,t) \in [0,1]^2}, (C^n_{s,\lambda})_{(s,\lambda) \in [0,1] \times \mathbb{R}})$ converges weakly to the distribution of a pair $(B, C)$ of processes such that $B$ is continuous and vanishing on the boundary of $[0,1]^2$. The difference with what was supposed above is that now, $t$ varies in $[0,1]$ and not only in $E_\Phi$. Again, by (3.2), we have

$$B_{s, F_{\mu_\Phi}(\lambda)} = C_{s,\lambda} \tag{3.3}$$

for all $s \in [0,1]$, $\lambda \in \mathbb{R}$. Hence the process $C$ is continuous in $s$ and continuous in $\lambda$ at any $\lambda$ which is not an atom of the (non random) probability measure $\mu_\Phi$. Hence it follows from Lemma 6.2 of the appendix that it suffices to prove that the distribution of the process

$$\left( X(s,z) := \int_{\lambda \in \mathbb{R}} \frac{C_{s,\lambda}}{(z-\lambda)^2} \mathrm{d}\lambda \right)_{s \in [0,1], z \in \mathbb{C} \backslash \mathbb{R}}$$

is totally prescribed (and Gaussian centered). This distribution is the limit distribution, along our subsequence, of the process

$$\left( \int_{\lambda \in \mathbb{R}} \frac{C^n_{s,\lambda}}{(z-\lambda)^2} \mathrm{d}\lambda \right)_{s \in [0,1], z \in \mathbb{C} \backslash \mathbb{R}}. \tag{3.4}$$

But by Lemma 3.1 below, the process of (3.4) is simply (the opposite of) the process $(X^n(s,z))_{s,z}$, defined above at (2.26). As Proposition 2.9 states that (regardless of the subsequence considered) the distribution of the process $(X^n(s,z))_{s,z}$ converges weakly to a Gaussian centered limit, this concludes the proof of Theorem 2.5.

**Lemma 3.1.** *For any $s \in [0,1]$ and any $z \in \mathbb{C} \backslash \mathbb{R}$, we have*

$$\int_{\lambda \in \mathbb{R}} \frac{C^n_{s,\lambda}}{(z-\lambda)^2} \mathrm{d}\lambda = -X^n(s,z). \tag{3.5}$$

*Proof.* Let us introduce, for $s \in [0,1]$, the random signed measure $\nu_{s,n}$ on $\mathbb{R}$ defined by

$$\nu_{s,n} := \frac{1}{\sqrt{n}} \sum_{1 \le i \le ns} \sum_{j=1}^{n} \left( |u_{ij}|^2 - \frac{1}{n} \right) \delta_{\lambda_j}.$$

Then for any $\lambda \in \mathbb{R}$, $C^n_{s,\lambda} = \nu_{s,n}((-\infty, \lambda])$. Moreover, by Fubini's theorem, we know that for any finite signed measure $m$ on $\mathbb{R}$,

$$\int_{\lambda \in \mathbb{R}} \frac{m((-\infty, \lambda])}{(z-\lambda)^2} \mathrm{d}\lambda = -\int_{\lambda \in \mathbb{R}} \frac{\mathrm{d}m(\lambda)}{z-\lambda}. \tag{3.6}$$

Hence

$$\int_{\lambda \in \mathbb{R}} \frac{C^n_{s,\lambda}}{(z-\lambda)^2} \mathrm{d}\lambda = -\int_{\lambda \in \mathbb{R}} \frac{\mathrm{d}\nu_{s,n}(\lambda)}{z-\lambda}.$$

On the other hand, we have

$$
\begin{aligned}
X^n(s,z) &= \frac{1}{\sqrt{n}} \left( \sum_{1 \le i \le ns} \left( \frac{1}{z-A} \right)_{ii} - \frac{1}{n} \sum_{1 \le i \le ns} \sum_{j=1}^n \frac{1}{z - \lambda_j} \right) \\
&= \frac{1}{\sqrt{n}} \left( \sum_{1 \le i \le ns} \sum_{j=1}^n |u_{ij}|^2 \frac{1}{z - \lambda_j} - \frac{1}{n} \sum_{1 \le i \le ns} \sum_{j=1}^n \frac{1}{z - \lambda_j} \right) \\
&= \frac{1}{\sqrt{n}} \sum_{1 \le i \le ns} \sum_{j=1}^n \left( |u_{ij}|^2 - \frac{1}{n} \right) \frac{1}{z - \lambda_j} \\
&= \int_{\lambda \in \mathbb{R}} \frac{\mathrm{d}\nu_{s,n}(\lambda)}{z - \lambda}.
\end{aligned}
$$

This concludes the proof. $\qquad \square$

## 4  Proof of Proposition 2.9

To prove Proposition 2.9, one needs to prove that the distribution of any linear combination of the $X^n(s,z)$'s ($s \in [0,1]$, $z \in \mathbb{R}$) converges weakly. For $s = 0$ or 1, $\nu_{s,n}$ is null, as $X^n(s,z)$, hence we can focus on $s \in (0,1)$. Any such linear combination can be written

$$
M^n := \sum_{i=1}^p \alpha_i X^n(s_i, z_i),
$$

for some $\alpha_i$'s in $\mathbb{C}$, some $s_i$'s in $[0,1]$ and some complex non real numbers $z_i$.

We want to prove that $M^n$ converges in law to a certain complex centered Gaussian variable. We are going to use the CLT for martingale differences stated at Theorem 6.4 of the appendix. Indeed, for $\mathcal{F}_k^n$ the $\sigma$-algebra generated by the first $k \times k$ upper-left corner of the symmetric matrix $A$, the sequence $(M_k^n := \mathbb{E}[M^n | \mathcal{F}_k^n])_{k=0,\ldots,n}$ is a centered martingale (to see that it is centered, just use the fact that as $A$ is invariant, in law, by conjugation by any permutation matrix, for all $z$, the expectation of $(\frac{1}{z-A})_{jj}$ does not depend on $j$).

Then, denoting $\mathbb{E}[\,\cdot\,|\mathcal{F}_k^n]$ by $\mathbb{E}_k$, and defining

$$
Y_k := (\mathbb{E}_k - \mathbb{E}_{k-1})(M^n)
$$

(which depends implicitly on $n$), we need to prove that for any $\varepsilon > 0$,

$$
L^n(\varepsilon) := \sum_{k=1}^n \mathbb{E}(|Y_k|^2 \mathbb{1}_{|Y_k| \ge \varepsilon}) \xrightarrow[n \to \infty]{} 0, \tag{4.1}
$$

and that the sequences

$$
\sum_{k=1}^n \mathbb{E}_{k-1}(|Y_k|^2) \qquad \text{and} \qquad \sum_{k=1}^n \mathbb{E}_{k-1}(Y_k^2)
$$

converge in probability towards some deterministic limits. As $\overline{X^n(s,z)} = X^n(s,\overline{z})$, it is in fact enough to fix $s, s' \in (0,1)$ and $z, z' \in \mathbb{C} \backslash \mathbb{R}$ and to prove that for

$$
Y_k := (\mathbb{E}_k - \mathbb{E}_{k-1})(X^n(s,z)) \qquad \text{and} \qquad Y_k' := (\mathbb{E}_k - \mathbb{E}_{k-1})(X^n(s',z')), \tag{4.2}
$$

we have (4.1) for any $\varepsilon > 0$ and that

$$
\sum_{k=1}^n \mathbb{E}_{k-1}(Y_k Y_k')
$$

converges in probability towards a deterministic constant. We introduce the notation

$$G := \frac{1}{z - A} \qquad \text{and} \qquad G' := \frac{1}{z' - A}.$$

Recall that $P_s$ denotes the diagonal matrix with diagonal entries $\mathbb{1}_{i \leq ns}$ $(1 \leq i \leq n)$. Let $A^{(k)}$ be the symmetric matrix with size $n - 1$ obtained by removing the $k$-th row and the $k$-th column of $A$. The matrix $P_s^{(k)}$ is defined in the same way out of $P_s$. Set $G^{(k)} := \frac{1}{z - A^{(k)}}$. Note that $\mathbb{E}_k G^{(k)} = \mathbb{E}_{k-1} G^{(k)}$, so that $Y_k$, which is equal to $\frac{1}{\sqrt{n}} (\mathbb{E}_k - \mathbb{E}_{k-1}) (\mathrm{Tr}(P_s G) - s_n \mathrm{Tr}\, G)$, can be rewritten

$$Y_k = \frac{1}{\sqrt{n}} (\mathbb{E}_k - \mathbb{E}_{k-1}) \left( (\mathrm{Tr}(P_s G) - \mathrm{Tr}(P_s^{(k)} G^{(k)})) - s_n (\mathrm{Tr}\, G - \mathrm{Tr}\, G^{(k)}) \right)$$

Then, (4.1) is obvious by Formula (6.8) of the appendix (indeed, $L^n(\varepsilon)$ is null for $n$ large enough). Let us now apply Formula (6.7) of the appendix. We get

$$Y_k = \frac{1}{\sqrt{n}} (\mathbb{E}_k - \mathbb{E}_{k-1}) \left( \frac{\mathbb{1}_{k \leq ns} - s_n + \mathbf{a}_k^* G^{(k)} (P_s^{(k)} - s_n) G^{(k)} \mathbf{a}_k}{z - a_{kk} - \mathbf{a}_k^* G^{(k)} \mathbf{a}_k} \right). \tag{4.3}$$

Following step by step Paragraph 3.2 of [10], one can neglect the non diagonal terms in the expansions of the quadratic forms in (4.3), *i.e.* replace $Y_k$ by $\frac{1}{\sqrt{n}} (\mathbb{E}_k - \mathbb{E}_{k-1}) (f_k)$, with

$$f_k := f_k(z, s) = \frac{\mathbb{1}_{k \leq ns} - s_n + \sum_j \mathbf{a}_k(j)^2 \{ G^{(k)} (P_s^{(k)} - s_n) G^{(k)} \}_{jj}}{z - \sum_j \mathbf{a}_k(j)^2 G_{jj}^{(k)}}. \tag{4.4}$$

In other words,

$$\sum_{k=1}^n \mathbb{E}_{k-1} (Y_k Y_k') = \frac{1}{n} \sum_{k=1}^n \mathbb{E}_{k-1} \left[ (\mathbb{E}_k - \mathbb{E}_{k-1}) (f_k) (\mathbb{E}_k - \mathbb{E}_{k-1}) (f_k') \right] + o(1), \tag{4.5}$$

where $f_k'$ is defined as $f_k$ in (4.4), replacing the function $s$ by $s'$ and $z$ by $z'$.

Let us denote by $\mathbb{E}_{\mathbf{a}_k}$ the expectation with respect to the randomness of the $k$-th column of $A$ (*i.e.* the conditional expectation with respect to the $\sigma$-algebra generated by the $a_{ij}$'s such that $k \notin \{i, j\}$). Note that $\mathbb{E}_{k-1} = \mathbb{E}_{\mathbf{a}_k} \circ \mathbb{E}_k = \mathbb{E}_k \circ \mathbb{E}_{\mathbf{a}_k}$, hence

$$\mathbb{E}_{k-1} \left[ (\mathbb{E}_k - \mathbb{E}_{k-1}) (f_k) (\mathbb{E}_k - \mathbb{E}_{k-1}) (f_k') \right] = \mathbb{E}_k [\mathbb{E}_{\mathbf{a}_k} (f_k \times f_k'')] - \mathbb{E}_k \, \mathbb{E}_{\mathbf{a}_k} f_k \times \mathbb{E}_k \, \mathbb{E}_{\mathbf{a}_k} f_k', \tag{4.6}$$

where $f_k''$ is defined as $f_k'$ replacing the matrix $A$ by the matrix

$$A' = [a_{ij}']_{1 \leq i, j \leq N} \tag{4.7}$$

defined by the the fact that the $a_{ij}'$'s such that $i > k$ or $j > k$ are i.i.d. copies of $a_{11}$ (modulo the fact that $A'$ is symmetric), independent of $A$ and for all other pairs $(i, j)$, $a_{ij}' = a_{ij}$.

For each $s \in (0, 1)$ let us define $\mathbb{C}_s^2$ to be the set of pairs $(z, \tilde{z})$ of complex numbers such that

$$\left( \Im z > 0 \text{ and } -\frac{\Im z}{1 - s} < \Im \tilde{z} < \frac{\Im z}{s} \right) \qquad \text{or} \qquad \left( \Im z < 0 \text{ and } \frac{\Im z}{s} < \Im \tilde{z} < -\frac{\Im z}{1 - s} \right).$$

Note that $\mathbb{C}_s^2$ is the set of pairs $(z, \tilde{z})$ of complex numbers such that $\Im z \neq 0$ and both $\Im(z + (1 - s)\tilde{z})$ and $\Im(z - s\tilde{z})$ have the same sign as $\Im z$. At last, in the next lemma,

$$\partial_{\tilde{z}} = \frac{\partial}{\partial \tilde{z}}$$

is not to be taken for the usual notation $\partial_{\bar{z}}$.

**Lemma 4.1.** *For any fixed $z \in \mathbb{C}\backslash\mathbb{R}$ and any fixed $s \in (0,1)$, as $n, k \longrightarrow \infty$ in such a way that $k/n \longrightarrow u \in [0,1]$, we have the convergence in probability*

$$\lim_{N \rangle \infty} \mathbb{E}_{\mathbf{a}_k}[f_k(z,s)] = L_u(z,s) := -\int_0^{+\infty} \frac{1}{t} \partial_{\tilde{z}, \tilde{z}=0} e^{i \operatorname{sgn}_z t(z+\tilde{z}(\mathbb{1}_{u \leq s}-s))} e^{\rho_{z,\tilde{z},s}(t)} dt,$$

*where for $s \in (0,1)$ fixed, $(z, \tilde{z}, t) \longmapsto \rho_{z,\tilde{z},s}(t)$ is the unique function defined on $\mathbb{C}_s^2 \times \mathbb{R}_+$, analytic in its two first variables and continuous in its third one, taking values into $\{z \in \mathbb{C}\,;\, \Re z \leq 0\}$, solution of*

$$\rho_{z,\tilde{z},s}(t) = t \int_0^\infty g(ty)(s e^{iy \operatorname{sgn}_z \tilde{z}} + (1-s)) e^{iy \operatorname{sgn}_z (z-s\tilde{z})} e^{\rho_{z,\tilde{z},s}(y)} dy$$

*where $g$ is the function introduced at (2.7).*

*Proof.* We use the fact that for $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\frac{1}{z} = -i \operatorname{sgn}_z \times \int_0^{+\infty} e^{\operatorname{sgn}_z itz} dt, \tag{4.8}$$

where $\operatorname{sgn}_z$ has been defined above by $\operatorname{sgn}_z = \operatorname{sgn}(\Im z)$. Hence by (4.4),

$$f_k = -i \operatorname{sgn}_z \int_0^{+\infty} \{\mathbb{1}_{k \leq ns} - s_n + \sum_j (G^{(k)}(P_s^{(k)} - s_n)G^{(k)})_{jj} \mathbf{a}_k(j)^2\} e^{i \operatorname{sgn}_z t(z - \sum_j (G^{(k)})_{jj} \mathbf{a}_k(j)^2)} dt.$$

Let us define $G^{(k)}(\cdot, \cdot)$ on $\mathbb{C}_{s_n}^2$ by

$$G^{(k)}(z, \tilde{z}) := \frac{1}{z + \tilde{z}(P_s^{(k)} - s_n) - A^{(k)}} \tag{4.9}$$

(note that $G^{(k)}(\cdot, \cdot)$ is well defined by the remark following the definition of $\mathbb{C}_s^2$). Then for any fixed $z \in \mathbb{C}\backslash\mathbb{R}$,

$$G_k(z)(P_s^{(k)} - s_n)G_k(z) = -\partial_{\tilde{z}, \tilde{z}=0} G^{(k)}(z, \tilde{z}).$$

Hence

$$\{\mathbb{1}_{k \leq ns} - s_n + \sum_j (G^{(k)}(P_s^{(k)} - s_n)G^{(k)})_{jj} \mathbf{a}_k(j)^2\} e^{i \operatorname{sgn}_z t(z - \sum_j (G^{(k)})_{jj} \mathbf{a}_k(j)^2)}$$

$$= \frac{1}{it \operatorname{sgn}_z} \partial_{\tilde{z}, \tilde{z}=0} e^{i \operatorname{sgn}_z t\{z + \tilde{z}(\mathbb{1}_{k \leq ns} - s_n) - \sum_j (G^{(k)}(z,\tilde{z}))_{jj} \mathbf{a}_k(j)^2\}}$$

and

$$f_k = -\int_0^{+\infty} \frac{1}{t} \partial_{\tilde{z}, \tilde{z}=0} e^{i \operatorname{sgn}_z t\{z + \tilde{z}(\mathbb{1}_{k \leq ns} - s_n) - \sum_j G^{(k)}(z,\tilde{z})_{jj} \mathbf{a}_k(j)^2\}} dt. \tag{4.10}$$

Let us now compute $\mathbb{E}_{\mathbf{a}_k}(f_k)$. One can permute $\mathbb{E}_{\mathbf{a}_k}$ and $\int_0^{+\infty}$ because $z \in \mathbb{C}\backslash\mathbb{R}$ is fixed and for each $j$, $-G^{(k)}(z, \tilde{z})_{jj}$ has imaginary part with the same sign as $z$ for $\tilde{z}$ small enough. Hence for $\phi_n$ defined as in (2.5) by $\phi_n(\lambda) = \mathbb{E}\, e^{-i\lambda a_{11}^2}$, we have

$$\mathbb{E}_{\mathbf{a}_k}(f_k) = -\int_0^{+\infty} \frac{1}{t} \partial_{\tilde{z}, \tilde{z}=0} e^{i \operatorname{sgn}_z t(z + \tilde{z}(\mathbb{1}_{k \leq ns} - s_n))} \prod_j \phi_n(\operatorname{sgn}_z t G^{(k)}(z, \tilde{z})_{jj}) dt$$

Now, by (2.6), we have the uniform convergence on compact sets $n(\phi_n - 1) \longrightarrow \Phi$ as $n \to \infty$. As $\Re(i \operatorname{sgn}_z z) < 0$, the integrals are well dominated at infinity. Moreover, the integral

$$\int_0^{+\infty} \frac{1}{t} \partial_{\tilde{z}, \tilde{z}=0} e^{i \operatorname{sgn}_z t(z + \tilde{z}(\mathbb{1}_{k \leq ns} - s_n))} e^{\frac{1}{n} \sum_j \Phi(\operatorname{sgn}_z t G^{(k)}(z,\tilde{z})_{jj})} dt$$

is well converging at the origin as the derivative in $\tilde{z}$ is of order $t$. Indeed, $G^{(k)}(z,\tilde{z})_{jj}$ takes its values in $\mathbb{C}^-$ and is uniformly bounded, and $\Phi$ is analytic on $\mathbb{C}^-$. By Lemma 6.3, it follows that

$$\mathbb{E}_{\mathbf{a}_k}(f_k) = -\int_0^{+\infty} \frac{1}{t}\partial_{\tilde{z},\tilde{z}=0}e^{i\,\mathrm{sgn}_z\,t(z+\tilde{z}(\mathbb{1}_{k\leq ns}-s_n))}e^{\frac{1}{n}\sum_j \Phi(\mathrm{sgn}_z\,tG^{(k)}(z,\tilde{z})_{jj})}\mathrm{d}t + o(1)\,.$$

We therefore basically need to compute the asymptotics of

$$\rho^n_{z,\tilde{z},s}(t) := \frac{1}{n}\sum_j \Phi(\mathrm{sgn}_z\,tG^{(k)}(z,\tilde{z})_{jj}).$$

Note that by definition of $\Phi$, for any $\lambda \in \overline{\mathbb{C}^-}$, $\Re\Phi(\lambda) \leq 0$. Thus $\rho^n_{z,\tilde{z},s}(t)$ is analytic in $z \in \mathbb{C}\backslash\mathbb{R}$, and uniformly bounded on compact subsets of $\mathbb{C}\backslash\mathbb{R}$ and takes values in $\{z \in \mathbb{C}\,;\, \Re z \leq 0\}$. By Montel's theorem, all limit points of this function for uniform convergence on compact subsets will satisfy the same property. Now, notice by Schur complement formula and the removal of the non diagonal terms (Lemma 7.7 of [10] again), that for $n \gg 1$,

$$G^{(k)}(z,\tilde{z})_{jj} = \frac{1}{z + \tilde{z}(\mathbb{1}_{k\leq ns} - s_n) - \sum_\ell a_{j\ell}^2 G^{(k,j)}(z,\tilde{z})_{\ell\ell}} + o(1) \qquad (4.11)$$

where $G^{(k,j)}$ is the resolvent where two rows and columns have been suppressed. We can now proceed to write that by invariance of the law of $A$ by conjugation by permutation matrices, for all $j$,

$$\mathbb{E}[\Phi(\mathrm{sgn}_z\,tG^{(k)}(z,\tilde{z})_{jj})] = \begin{cases} \mathbb{E}[\Phi(\mathrm{sgn}_z\,tG^{(k)}(z,\tilde{z})_{11})] & \text{if } j \leq ns, \\[2ex] \mathbb{E}[\Phi(\mathrm{sgn}_z\,tG^{(k)}(z,\tilde{z})_{nn})] & \text{if } j > ns, \end{cases}$$

so that by concentration arguments, see [10, Appendix], $\rho^n_{z,\tilde{z},s}(t)$ self-averages and for $n \gg 1$, with very large probability,

$$\begin{aligned} \rho^n_{z,\tilde{z},s}(t) &= \mathbb{E}[\frac{1}{n}\sum_j \Phi(\mathrm{sgn}_z\,tG^{(k)}(z,\tilde{z})_{jj})] + o(1) \\ &= s_n\,\mathbb{E}[\Phi(\mathrm{sgn}_z\,tG^{(k)}(z,\tilde{z})_{11})] + (1-s_n)\,\mathbb{E}[\Phi(\mathrm{sgn}_z\,tG^{(k)}(z,\tilde{z})_{nn})] + o(1). \end{aligned}$$

On the other side, using (4.11), the function $g$ introduced in the hypothesis at (2.7) and a change of variable $y \to y/t$, we have (using Lemma 6.3 twice)

$$\begin{aligned} \mathbb{E}[\Phi(\mathrm{sgn}_z\,tG^{(k)}(z,\tilde{z})_{11})] &= t\int_0^\infty g(ty)e^{iy\,\mathrm{sgn}_z(z+\tilde{z}(1-s_n))}\prod_j \phi_n(y\,\mathrm{sgn}_z\,G^{(k,1)}(z,\tilde{z})_{jj})\mathrm{d}y \\ &= t\int_0^\infty g(ty)\exp(iy\,\mathrm{sgn}_z(z+\tilde{z}(1-s_n)))e^{\rho^n_{z,\tilde{z},s}(y)}\mathrm{d}y \\ &\quad + o(1) \qquad\qquad (4.12) \\ \mathbb{E}[\Phi(\mathrm{sgn}_z\,tG^{(k)}(z,\tilde{z})_{nn})] &= t\int_0^\infty g(ty)e^{iy\,\mathrm{sgn}_z(z-s_n\tilde{z})}\prod_j \phi_n(\mathrm{sgn}_z\,yG^{(k,1)}(z,\tilde{z})_{jj})\mathrm{d}y \\ &= t\int_0^\infty g(ty)\exp(iy\,\mathrm{sgn}_z(z-s_n\tilde{z}))e^{\rho^n_{z,\tilde{z},s}(y)}\mathrm{d}y + o(1) \quad (4.13) \end{aligned}$$

so that we deduce that the limit points $\rho_{z,\tilde{z},s}(t)$ of $\rho^n_{z,\tilde{z},s}(t)$ satisfy

$$\rho_{z,\tilde{z},s}(t) = t\int_0^\infty g(ty)(se^{iy\,\mathrm{sgn}_z\,\tilde{z}} + (1-s))e^{iy\,\mathrm{sgn}_z(z-s\tilde{z})}e^{\rho_{z,\tilde{z},s}(y)}\mathrm{d}y.$$

Let us now prove that for each fixed $s \in (0,1)$, there exists a unique function satisfying this equation and the conditions stated in the lemma. So let us suppose that we have two solutions $\rho_{z,\tilde{z},s}(t)$ and $\tilde{\rho}_{z,\tilde{z},s}(t)$ with non positive real parts. Then

$$\Delta_{z,\tilde{z}}(t) := \rho_{z,\tilde{z},s}(t) - \tilde{\rho}_{z,\tilde{z},s}(t)$$

satisfies

$$\Delta_{z,\tilde{z}}(t) = t \int_0^\infty g(ty)(se^{iy \operatorname{sgn}_z \tilde{z}} + (1-s))e^{iy \operatorname{sgn}_z (z - s\tilde{z})}(e^{\rho_{z,\tilde{z},s}(y)} - e^{\tilde{\rho}_{z,\tilde{z},s}(y)})\mathrm{d}y.$$

Let $\delta(z, \tilde{z}) := \min\{\operatorname{sgn}_z \Im(z + (1-s)\tilde{z}),\ \operatorname{sgn}_z \Im(z - s\tilde{z})\} > 0$. We have

$$|(se^{iy \operatorname{sgn}_z \tilde{z}} + (1-s))e^{iy \operatorname{sgn}_z (z-s\tilde{z})}| \le e^{-\delta(z,\tilde{z})y},$$

hence

$$|\Delta_{z,\tilde{z}}(t)| \quad \le \quad t \int_0^\infty |g(ty)|e^{-\delta(z,\tilde{z})y}|\Delta_{z,\tilde{z}}(y)|\mathrm{d}y$$

Thus by the hypothesis made on $g$ at (2.8),

$$|\Delta_{z,\tilde{z}}(t)| \quad \le \quad Kt^{\gamma+1} \underbrace{\int_0^\infty y^\gamma e^{-\delta(z,\tilde{z})y}|\Delta_{z,\tilde{z}}(y)|\mathrm{d}y}_{:=I_1(z,\tilde{z})} + Kt^{\kappa+1} \underbrace{\int_0^\infty y^\kappa e^{-\delta(z,\tilde{z})y}|\Delta_{z,\tilde{z}}(y)|\mathrm{d}y}_{:=I_2(z,\tilde{z})}$$

It follows that the numbers $I_1(z,\tilde{z})$ and $I_2(z,\tilde{z})$ defined above satisfy

$$I_1(z,\tilde{z}) \quad \le \quad K\left(I_1(z,\tilde{z})\int_0^\infty y^{2\gamma+1}e^{-\delta(z,\tilde{z})y}\mathrm{d}y + I_2(z,\tilde{z})\int_0^\infty y^{\gamma+\kappa+1}e^{-\delta(z,\tilde{z})y}\mathrm{d}y\right),$$

$$I_2(z,\tilde{z}) \quad \le \quad K\left(I_1(z,\tilde{z})\int_0^\infty y^{\gamma+\kappa+1}e^{-\delta(z,\tilde{z})y}\mathrm{d}y + I_2(z,\tilde{z})\int_0^\infty y^{2\kappa+1}e^{-\delta(z,\tilde{z})y}\mathrm{d}y\right).$$

For $\delta(z,\tilde{z})$ large enough, the integrals above are all strictly less that $\frac{1}{4K}$, so $I_1(z,\tilde{z}) = I_2(z,\tilde{z}) = 0$. It follows that for $\Im z$ large enough and $\Im \tilde{z}$ small enough, both solutions coincide. By analytic continuation, unicity follows. $\qquad\square$

Getting back to (4.5) and (4.6), we shall now, as in [10], analyze

$$L_k^n(s, z; s', z') := \mathbb{E}_{\mathbf{a}_k}(f_k \times f_k''). \tag{4.14}$$

Let us first define the measure

$$\tilde{\tau} := \tau + \delta_0 \otimes \mu + \mu \otimes \delta_0 \tag{4.15}$$

on $(\mathbb{R}^+)^2$ for $\tau$ and $\mu$ the measures introduced at (2.10) or at Remark 2.2. We always have, for $x, y \in \mathbb{C}^+$,

$$\Phi(x^{-1} + y^{-1}) = \iint_{(\mathbb{R}_+)^2} e^{i(xv + yv')}\mathrm{d}\tilde{\tau}(v, v') \tag{4.16}$$

**Lemma 4.2.** *Let us fix $s_1, s_2 \in (0,1)$. As $k, n \longrightarrow \infty$ in such a way that $k/n$ tends to $u \in [0,1]$, the quantity $L_k^n(s_1, z; s_2, z')$ defined at (4.14) converges in probability to the deterministic limit*

$$L_u(s_1, z; s_2, z') :=$$

$$\iint_{\mathbb{R}_+^2} \partial_{\tilde{z}, \tilde{z}=0} \partial_{\tilde{z}', \tilde{z}'=0} e^{i \operatorname{sgn}_z t(z + \tilde{z}(\mathbb{1}_{u \le s_1} - s_1)) + i \operatorname{sgn}_{z'} t'(z' + \tilde{z}'(\mathbb{1}_{u \le s_2} - s_2)) + \rho_u(s_1, t, z, \tilde{z}; s_2, t', z', \tilde{z}')} \frac{\mathrm{d}t\mathrm{d}t'}{tt'}$$

*where the function*

$$(t, z, \tilde{z}, t', z', \tilde{z}') \in \mathbb{R}_+ \times \mathbb{C}^2_{s_1} \times \mathbb{R}_+ \times \mathbb{C}^2_{s_2} \longmapsto \rho_u(s_1, t, z, \tilde{z}; s_2, t', z', \tilde{z}')$$

*is characterized as follows :*

$$\rho_u(s_1, t, z, \tilde{z}; s_2, t', z', \tilde{z}') = \rho_u(s_2, t', z', \tilde{z}'; s_1, t, z, \tilde{z}) \tag{4.17}$$

*and if, for example, $s_1 \le s_2$, then for $\gamma_1 = s_1, \gamma_2 = s_2 - s_1$, $\gamma_3 = 1 - s_2$ and $\tilde{\tau} = \tau + \delta_0 \otimes \mu + \mu \otimes \delta_0$,*

$$
\begin{aligned}
\rho_u(s_1, t_1, z_1, \tilde{z}_1; s_2, t_2, z_2, \tilde{z}_2) &= u \sum_{\beta=1}^3 \gamma_\beta \iint_{\mathbb{R}_+^2} e^{\sum_{r=1,2} \operatorname{sgn}_{z_r} \frac{iv_r}{t_r} \{z_r + \tilde{z}_r (\mathbb{1}_{\beta \le r} - s_r)\}} \times \\
&\qquad e^{\rho_u(s_1, \frac{v_1}{t_1}, z_1, \tilde{z}; s_2, \frac{v_2}{t_2}, z_2, \tilde{z}_2)} \mathrm{d}\tilde{\tau}(v, v') + \\
&\qquad \sum_{r=1,2} t_r \int_0^\infty g(t_r y) \{ (s_r - u)^+ e^{iy \operatorname{sgn}_{z_r} \tilde{z}_r} + 1 - \max(s_r, u) \} \\
&\qquad e^{iy \operatorname{sgn}_{z_r} (z_r - s_r \tilde{z}_r)} e^{\rho_{z_r, \tilde{z}_r, s_r}(y)} \mathrm{d}y
\end{aligned}
$$

*(the characterization of $\rho_u(s_1, t, z, \tilde{z}; s_2, t', z', \tilde{z}')$ when $s_2 \le s_1$ can be deduced from the previous equation and (4.17)).*

**Proof.** Of course, $L_k^n(s_1, z; s_2, z') = L_k^n(s_2, z'; s_1, z)$. Let us suppose for example that $s_1 \le s_2$. We use the definition of $G^{(k)}(z, \tilde{z})$ given at (4.9) for $s$ replaced by $s_1$ and define in the same way, for $(z', \tilde{z}) \in \mathbb{C}^2_{s_{2,n}}$,

$$G'^{(k)}(z', \tilde{z}) := \frac{1}{z' + \tilde{z}(P_{s_2}^{(k)} - s_{2,n}) - A'^{(k)}}$$

with $s_{i,n} := \frac{\lfloor ns_i \rfloor}{n}$ $(i = 1, 2)$.

First, recall the following formula for $f_k$ established at Equation (4.10):

$$f_k = -\int_0^{+\infty} \frac{1}{t} \partial_{\tilde{z}, \tilde{z}=0} e^{i \operatorname{sgn}_z t \{z + \tilde{z}(\mathbb{1}_{k \le ns} - s_n) - \sum_{j \ne k} G^{(k)}(z, \tilde{z})_{jj} \mathbf{a}_k(j)^2\}} \mathrm{d}t.$$

In the same way, we find

$$f_k'' = -\int_0^{+\infty} \frac{1}{t'} \partial_{\tilde{z}, \tilde{z}=0} e^{i \operatorname{sgn}_{z'} t' \{z' + \tilde{z}(\mathbb{1}_{k \le ns'} - s_n') - \sum_{j \ne k} G'^{(k)}(z', \tilde{z})_{jj} \mathbf{a}_k'(j)^2\}} \mathrm{d}t.$$

As the $\mathbf{a}_k(j)$ and the $\mathbf{a}_k'(j)$ are identical when $j \le k$ and independent when $j > k$, we have

$$
\begin{aligned}
L_k^n(s_1, z_1; s_2, z_2) &= \iint_{\mathbb{R}_+^2} \partial_{\tilde{z}_1, \tilde{z}_1=0} \partial_{\tilde{z}_2, \tilde{z}_2=0} e^{i \operatorname{sgn}_{z_1} t_1 \{z_1 + \tilde{z}_1 (\mathbb{1}_{k \le ns_1} - s_{1,n})\} + i \operatorname{sgn}_{z_2} t_2 \{z_2 + \tilde{z}_2 (\mathbb{1}_{k \le ns_2} - s_{2,n})\}} \times \\
&\qquad \prod_{j \le k} \phi_n(\operatorname{sgn}_{z_1} t_1 G^{(k)}(z_1, \tilde{z}_1)_{jj} + \operatorname{sgn}_{z_2} t_2 G'^{(k)}(z_2, \tilde{z}_2)_{jj}) \times \\
&\qquad \prod_{j > k} \phi_n(\operatorname{sgn}_{z_1} t_1 G^{(k)}(z_1, \tilde{z}_1)_{jj}) \phi_n(\operatorname{sgn}_{z_2} t_2 G'^{(k)}(z_2, \tilde{z}_2)_{jj}) \frac{\mathrm{d}t_1 \mathrm{d}t_2}{t_1 t_2}
\end{aligned}
$$

Then using the usual off-diagonal terms removal and Lemma 6.3, we get

$$
\begin{aligned}
L_k^n(s_1, z_1; s_2, z_2) &= \iint_{\mathbb{R}_+^2} \partial_{\tilde{z}_1, \tilde{z}_1=0} \partial_{\tilde{z}_2, \tilde{z}_2=0} e^{\sum_{r=1,2} i \operatorname{sgn}_{z_r} t_r \{z_r + \tilde{z}_r (\mathbb{1}_{k \le ns_r} - s_{r,n})\}} \times \\
&\qquad \exp(\rho_k^n(s_1, t_1, z_1, \tilde{z}_1; s_2, t_2, z_2, \tilde{z}_2)) \frac{\mathrm{d}t_1 \mathrm{d}t_2}{t_1 t_2} + o(1)
\end{aligned}
$$

with

$$\rho_k^n(s_1, t_1, z_1, \tilde{z}_1; s_2, t_2, z_2, \tilde{z}_2) \quad := \quad \frac{1}{n} \sum_{j \leq k} \Phi(\text{sgn}_{z_1} t_1 G^{(k)}(z_1, \tilde{z}_1)_{jj} + \text{sgn}_{z_2} t_2 G'^{(k)}(z_2, \tilde{z}_2)_{jj})$$

$$+ \quad \frac{1}{n} \sum_{j > k} \{\Phi(\text{sgn}_{z_1} t_1 G^{(k)}(z_1, \tilde{z}_1)_{jj}) + \Phi(\text{sgn}_{z_2} t_2 G'^{(k)}(z_2, \tilde{z}_2)_{jj})\}$$

By the Schur formula, (4.16) and the off-diagonal terms removal, we have

$$\Phi(\text{sgn}_{z_1} t_1 G^{(k)}(z_1, \tilde{z}_1)_{jj} + \text{sgn}_{z_2} t_2 G'^{(k)}(z_2, \tilde{z}_2)_{jj}) = \qquad (4.18)$$

$$\iint_{\mathbb{R}_+^2} e^{\text{sgn}_z \frac{iv_1}{t_1} \{z_1 + \tilde{z}_1(\mathbb{1}_{j \leq ns_1} - s_1) - \sum_{\ell \notin \{k,j\}} a_{j\ell}^2 G^{(k,j)}(z_1, \tilde{z}_1)_{\ell\ell}\}}$$

$$\times e^{\text{sgn}_{z_2} \frac{iv_2}{t_2} \{z_2 + \tilde{z}_2(\mathbb{1}_{j \leq ns_2} - s_2) - \sum_{\ell \notin \{k,j\}} (a'_{j\ell})^2 G'^{(k,j)}(z_2, \tilde{z}_2)_{\ell\ell}\}} d\tilde{\tau}(v_1, v_2) + o(1).$$

By (4.18) and using the concentration around the expectation, we get, for $j < k$,

$$\Phi(\text{sgn}_{z_1} t_1 G^{(k)}(z_1, \tilde{z}_1)_{jj} + \text{sgn}_{z_2} t_2 G'^{(k)}(z_2, \tilde{z}_2)_{jj}) = o(1) + \qquad (4.19)$$

$$\iint_{\mathbb{R}_+^2} e^{\text{sgn}_{z_1} \frac{iv_1}{t_1} \{z_1 + \tilde{z}_1(\mathbb{1}_{j \leq ns_1} - s_1)\} + \text{sgn}_{z_2} \frac{iv_2}{t_2} \{z_2 + \tilde{z}_2(\mathbb{1}_{j \leq ns_2} - s_2)\}} e^{\rho_k^n(s_1, \frac{v_1}{t_1}, z_1, \tilde{z}; s_2, \frac{v_2}{t_2}, z_2, \tilde{z}_2)} d\tilde{\tau}(v, v').$$

Now, using the proof of Lemma 4.1 (especially (4.12) and (4.13)) and the fact that $k/n \longrightarrow u$, we get

$$\frac{1}{n} \sum_{j > k} \Phi(\text{sgn}_{z_1} t_1 G^{(k)}(z_1, \tilde{z}_1)_{jj}) \quad = \quad o(1) + (s_1 - u)^+ \mathbb{E}(\Phi(\text{sgn}_{z_1} t_1 G^{(k)}(z_1, \tilde{z}_1)_{11}) +$$

$$(1 - \max(s_1, u)) \mathbb{E}(\Phi(\text{sgn}_{z_1} t_1 G^{(k)}(z_1, \tilde{z}_1)_{nn})$$

$$= \quad o(1) + t_1 \int_0^\infty g(t_1 y) \{(s_1 - u)^+ e^{iy \, \text{sgn}_{z_1} \tilde{z}_1} + 1 - \max(s_1, u)\}$$

$$e^{iy \, \text{sgn}_{z_1}(z_1 - s_1 \tilde{z}_1)} e^{\rho_{z_1, \tilde{z}_1, s_1}(y)} dy \qquad (4.20)$$

In the same way,

$$\frac{1}{n} \sum_{j > k} \Phi(\text{sgn}_{z_2} t_2 G^{(k)}(z_2, \tilde{z}_2)_{jj}) \quad = \quad o(1) + t_2 \int_0^\infty g(t_2 y) \{(s_2 - u)^+ e^{iy \, \text{sgn}_{z_2} \tilde{z}_2} + 1 - \max(s_2, u)\}$$

$$e^{iy \, \text{sgn}_{z_2}(z_2 - s_2 \tilde{z}_2)} e^{\rho_{z_2, \tilde{z}_2, s_2}(y)} dy$$

Summing up, we get that any limit point $\rho_u(s_1, t_1, z_1, \tilde{z}_1; s_2, t_2, z_2, \tilde{z}_2)$ of

$$\rho_k^n(s_1, t_1, z_1, \tilde{z}_1; s_2, t_2, z_2, \tilde{z}_2)$$

satisfies

$$\rho_u(s_1, t_1, z_1, \tilde{z}_1; s_2, t_2, z_2, \tilde{z}_2) \quad = \quad u \sum_{\beta=1}^3 \gamma_\beta \iint_{\mathbb{R}_+^2} e^{\sum_{r=1,2} \text{sgn}_{z_r} \frac{iv_r}{t_r} \{z_r + \tilde{z}_r(\mathbb{1}_{\beta \leq r} - s_r)\}} \times$$

$$e^{\rho_u(s_1, \frac{v_1}{t_1}, z_1, \tilde{z}; s_2, \frac{v_2}{t_2}, z_2, \tilde{z}_2)} d\tilde{\tau}(v, v') +$$

$$\sum_{r=1,2} t_r \int_0^\infty g(t_r y) \{(s_r - u)^+ e^{iy \, \text{sgn}_{z_r} \tilde{z}_r} + 1 - \max(s_r, u)\}$$

$$e^{iy \, \text{sgn}_{z_r}(z_r - s_r \tilde{z}_r)} e^{\rho_{z_r, \tilde{z}_r, s_r}(y)} dy$$

The proof of the fact that under analyticity hypotheses, the limit points are uniquely prescribed by the above equations goes along the same lines as the proofs in Section 5.2 of [10], sketched as follows. First, we have to consider separately the case where $\Phi$ satisfies (2.9) and the case where $\Phi$ satisfies (2.10). In the case where $\Phi$ satisfies (2.9), the proof is very similar to the proof of the corresponding case in Section 5.2 of [10] and to the detailed proof of the uniqueness for Lemma 4.1 of the present paper, using (2.11) instead of (2.8). The case where $\Phi$ satisfies (2.10) is a little more delicate. As in Lemma 5.1 of [10], one first needs to notice that considered as functions of $t, t', t''$, the limit points satisfy an Hölder bound, using essentially the facts that for any $2\kappa \in (0, \alpha/2)$

$$\limsup_{n \geq 1} \mathbb{E}\Big[ \Big( \sum_{i=1}^{n} |a_{1i}|^2 \Big)^{2\kappa} \Big] < \infty \,, \tag{4.21}$$

and that for any $\beta \in (\alpha/1, 1]$, there exists a constant $c = c(\alpha, \beta)$ such that for any $x, y$ in $\mathbb{C}^-$,

$$|x^{\frac{\alpha}{2}} - y^{\frac{\alpha}{2}}| \leq c|x - y|^\beta \, (|x| \wedge |y|)^{\frac{\alpha}{2} - \beta} \,. \tag{4.22}$$

Then one has to interpret the equation satisfied by the limit points as a fixed point equation for a strictly contracting function in a space of Hölder functions: the key argument, to prove that the function is contracting, is to use the estimates given in Lemma 5.7 of [15]. □

This concludes the proof of Proposition 2.9 and it follows from this that the covariance of the process $H_{s,z}$ is given by

$$C(s, z; s', z') := \mathbb{E}[H_{s,z} H_{s',z'}] = \int_0^1 \mathrm{d}u (L_u(s, z; s', z') - L_u(z, s) L_u(z', s')) \tag{4.23}$$

with the functions $L$ defined in Lemmas 4.1 and 4.2.

## 5 Proof of Proposition 2.7

Let us now prove that the limit covariance of $(C_{s,\lambda})$ is not identically zero (hence this is also the case for $(B_{s,t})$ by (2.25)). Using Lemma 6.1 and (3.2), one easily sees that $(C_{s,\lambda}^n)$ is uniformly bounded in $L^4$. It follows that

$$\mathrm{Var}(C_{s,\lambda}) = \mathbb{E}[(C_{s,\lambda})^2] = \lim_{n \to \infty} \mathbb{E}[(C_{s,\lambda}^n)^2]. \tag{5.1}$$

Thus we shall prove that the limit of $\mathbb{E}[(C_{s,\lambda}^n)^2]$ is not identically zero.

*Preliminary computation:* For $x_1, \ldots, x_n \in \mathbb{C}$ such that $x_1 + \cdots + x_n = 0$, for any $0 \leq \ell \leq n$, we have

$$\sum_{i=1}^{\ell} x_i = \sum_{i=1}^{n} \alpha_i x_i \qquad \text{for } \alpha_i := \begin{cases} 1 - \frac{\ell}{n} & \text{if } i \leq \ell, \\ -\frac{\ell}{n} & \text{if } i > \ell, \end{cases} \tag{5.2}$$

and that $\alpha_1 + \cdots + \alpha_n = 0$. Note also that for $(X_1, \ldots, X_n)$ an exchangeable random vector and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that $\alpha_1 + \cdots + \alpha_n = 0$, we have

$$\mathbb{E} \sum_{i,i'} \alpha_i \alpha_{i'} X_i X_{i'} = \sum_i \alpha_i^2 \mathbb{E}[X_1(X_1 - X_2)]. \tag{5.3}$$

It follows from (5.2) and (5.3) that if the coordinates of an exchangeable random vector $(X_1, \ldots, X_n)$ sum up to zero, then for any $0 \leq \ell \leq n$,

$$\mathbb{E} \sum_{\substack{1 \leq i \leq \ell \\ 1 \leq i' \leq \ell}} X_i X_{i'} = n \left( \frac{\ell}{n} - \frac{\ell^2}{n^2} \right) \mathbb{E}[X_1(X_1 - X_2)]. \tag{5.4}$$

Let us now fix $s \in (0,1)$ and $\lambda \in \mathbb{R}$ and apply our preliminary computation (5.4) with $X_i = \sum_{j \, ; \, \lambda_j \leq \lambda} (|u_{ij}|^2 - n^{-1})$ and $\ell = \lfloor ns \rfloor$. For $s_n := \lfloor ns \rfloor / n$, we get

$$\mathrm{Var}(C_{s,\lambda}^n) = \mathbb{E}[(C_{s,\lambda}^n)^2] = \left( s_n - s_n^2 \right) \mathbb{E}[X_1(X_1 - X_2)] \tag{5.5}$$

Note also that as each $|u_{ij}|^2$ has expectation $n^{-1}$,

$$\mathbb{E}[X_1(X_1 - X_2)] = \mathbb{E}\Big[ \sum_{\substack{j \, ; \, \lambda_j \leq \lambda \\ j' \, ; \, \lambda_{j'} \leq \lambda}} |u_{1j}|^2 |u_{1j'}|^2 - |u_{1j}|^2 |u_{2j'}|^2 \Big]$$

Moreover, by exchangeability of the rows of $U$ (which is true even conditionally to the $\lambda_j$'s) and the fact that its columns have norm one, for any $j, j'$,

$$n(n-1) \, \mathbb{E}[\mathbb{1}_{\lambda_j, \lambda_{j'} \leq \lambda} |u_{1j}|^2 |u_{2j'}|^2] + n \, \mathbb{E}[\mathbb{1}_{\lambda_j, \lambda_{j'} \leq \lambda} |u_{1j}|^2 |u_{1j'}|^2] = 1,$$

so that

$$\mathbb{E}[X_1(X_1 - X_2)] = O\left(\frac{1}{n}\right) + \left(1 - \frac{1}{n}\right) \mathbb{E}\Big[ \sum_{j,j' \, ; \, \lambda_j, \lambda_{j'} \leq \lambda} (|u_{1j}|^2 |u_{1j'}|^2 - n^{-2}) \Big].$$

By (5.5), we deduce that

$$\mathbb{E}[(C_{s,\lambda}^n)^2] = O\left(\frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)\left(s_n - s_n^2\right) \mathbb{E}\Big[ \sum_{j,j' \, ; \, \lambda_j, \lambda_{j'} \leq \lambda} (|u_{1j}|^2 |u_{1j'}|^2 - n^{-2}) \Big],$$

so that

$$\mathbb{E}[(C_{s,\lambda}^n)^2] = O\left(\frac{1}{n}\right) + \left(s_n - s_n^2\right) \mathbb{E}\Big[\Big( \sum_{j \, ; \, \lambda_j \leq \lambda} (|u_{1j}|^2 - n^{-1}) \Big)^2\Big] \tag{5.6}$$

Moreover, for $\mu_n$, $\mu_{n,\mathbf{e}_1}$ the random probability measures introduced in (1.4), we have

$$\sum_{j \, ; \, \lambda_j \leq \lambda} (|u_{1j}|^2 - n^{-1}) = (\mu_{n,\mathbf{e}_1} - \mu_n)((-\infty, \lambda])$$

Hence by (5.6),

$$\mathbb{E}[(C_{s,\lambda}^n)^2] = O\left(\frac{1}{n}\right) + \left(s_n - s_n^2\right) \mathbb{E}[\{(\mu_{n,\mathbf{e}_1} - \mu_n)((-\infty, \lambda])\}^2]. \tag{5.7}$$

Let us now suppose that for a certain $s \in (0,1)$, we have $\mathrm{Var}(C_{s,\lambda}) = 0$ for all $\lambda \in \mathbb{R}$. To conclude the proof, we shall now exhibit a contradiction. By (5.1) and (5.7), we know that for all $\lambda$, $\mathbb{E}[\{(\mu_{n,\mathbf{e}_1} - \mu_n)((-\infty, \lambda])\}^2] \longrightarrow 0$ as $n \to \infty$, hence $\mathbb{E}[|(\mu_{n,\mathbf{e}_1} - \mu_n)((-\infty, \lambda])|] \longrightarrow 0$. As $\mu_{n,\mathbf{e}_1}$, $\mu_n$ are probability measures, for any $\lambda \in \mathbb{R}$,

$$|(\mu_{n,\mathbf{e}_1} - \mu_n)((-\infty, \lambda])| \leq 2.$$

Thus for any $z \in \mathbb{C} \backslash \mathbb{R}$, by dominated convergence, as $n \to \infty$,

$$\int_{\lambda \in \mathbb{R}} \frac{\mathbb{E}[|(\mu_{n,\mathbf{e}_1} - \mu_n)((-\infty, \lambda])|]}{|z - \lambda|^2} \mathrm{d}\lambda \longrightarrow 0.$$

We deduce the convergence in probability

$$\int_{\lambda \in \mathbb{R}} \frac{(\mu_{n,\mathbf{e}_1} - \mu_n)((-\infty, \lambda])}{(z - \lambda)^2} \mathrm{d}\lambda \longrightarrow 0.$$

But by (3.6), for any $z \in \mathbb{C} \backslash \mathbb{R}$, with the notation $G(z) := (z - A)^{-1}$,

$$
\begin{aligned}
\int_{\lambda \in \mathbb{R}} \frac{(\mu_{n,\mathbf{e}_1} - \mu_n)((-\infty, \lambda])}{(z - \lambda)^2} \mathrm{d}\lambda &= -\int_{\lambda \in \mathbb{R}} \frac{\mathrm{d}(\mu_{n,\mathbf{e}_1} - \mu_n)(\lambda)}{z - \lambda} \\
&= \frac{1}{n} \operatorname{Tr} G(z) - G(z)_{11}.
\end{aligned}
$$

We deduce the convergence in probability, for any fixed $z \in \mathbb{C}^+$,

$$
\frac{1}{n} \operatorname{Tr}(G(z)) - G(z)_{11} \longrightarrow 0. \tag{5.8}
$$

By (2.23), we deduce that $G(z)_{11}$ converges in probability to the Stieltjes transform $G_{\mu_\Phi}(z)$ of the limit empirical spectral law $\mu_\Phi$ of $A$. By the Schur complement formula (see [1, Lem. 2.4.6]) and the asymptotic vanishing of non diagonal terms in the quadratic form (Lemma 7.7 of [10]), we deduce the convergence in probability

$$
z - \sum_{j=2}^{n} |a_{1j}|^2 G^{(1)}(z)_{jj} \longrightarrow 1/G_{\mu_\Phi}(z), \tag{5.9}
$$

where $A^{(1)}$ is the matrix obtained after having removed the first row and the first column to $A$ and $G^{(1)}(z) := (z - A^{(1)})^{-1}$.

It follows that the (implicitly depending on $n$) random variable $X = \sum_{j=2}^{n} |a_{1j}|^2 G^{(1)}(z)_{jj}$ converges in probability to a deterministic limit as $n \to \infty$. Let us show that this is not possible if $\Phi$ is not linear.

Let $\mathbb{E}_G$ denote the integration with respect to the randomness of the first row of $A$. The random variable $X$ takes values in $\mathbb{C}^-$. For any $t \geq 0$, by Lemma 6.3, we have

$$
\mathbb{E}[e^{-itX}] = \prod_{j=2}^{n} \phi_n(t G^{(1)}(z)_{jj}) = (1 + o(1)) \exp\{\frac{1}{n-1} \sum_{j=2}^{n} \Phi(t G^{(1)}(z)_{jj})\} \tag{5.10}
$$

By Equation (20) of [10], we know that

$$
\frac{1}{n-1} \sum_{j=2}^{n} \Phi(t G^{(1)}(z)_{jj}) \longrightarrow \rho_z(t), \tag{5.11}
$$

where $\rho_z$ is a continuous function on $\mathbb{R}^+$ satisfying (by Theorem 1.9 of [10]):

$$
\rho_z(\lambda) = \lambda \int_0^{+\infty} g(\lambda y) e^{iyz + \rho_z(y)} \mathrm{d}y. \tag{5.12}
$$

By (5.10) and (5.11), as $n \to \infty$,

$$
\mathbb{E}[e^{-itX}] \longrightarrow e^{\rho_z(t)}.
$$

but we already saw that $X$ converges in probability to a constant, hence there is $c_z \in \mathbb{C}$ such that for all $t$, $\rho_z(t) = c_z t$. From (5.12) and (2.7), we deduce that for all $\lambda \geq 0$,

$$
c_z \lambda = \lambda \int_0^{+\infty} g(\lambda y) e^{iyz + c_z y} \mathrm{d}y = \int_0^{+\infty} g(t) e^{i \frac{z - ic_z}{\lambda} t} \mathrm{d}t = \Phi(\frac{\lambda}{z - ic_z}).
$$

As we supposed that $\Phi$ is not linear, by analytic continuation, this is a contradiction.

Note that the fact that $E[(C_{s,\lambda}^n)^2]$ does not go to zero could also be deduced from [14] in the Lévy case according to (5.7).

# 6 Appendix

## 6.1 A tightness lemma for bivariate processes

Let us endow the space $D([0,1]^2)$ with the Skorokhod topology (see [8] for the definitions).

Let $M = [m_{ij}]_{1 \le i,j \le n}$ be a random bistochastic matrix depending implicitly on $n$. We define the random process of $D([0,1]^2)$

$$S^n(s,t) := \frac{1}{\sqrt{n}} \sum_{\substack{1 \le i \le ns \\ 1 \le j \le nt}} \left( m_{ij} - \frac{1}{n} \right).$$

**Lemma 6.1.** *Let us suppose that $M$ is, in law, invariant under left multiplication by any permutation matrix. Then the process $S^n$ is $C$-tight in $D([0,1]^2)$, i.e. the sequence $(\mathrm{distribution}(S^n))_{n \ge 1}$ is tight and has all its accumulation points supported by the set of continuous functions on $[0,1]^2$. Moreover, the process $S^n$ is uniformly bounded in $L^4$.*

*Proof.* Let us prove that for all $0 \le s < s' \le 1$, $0 \le t < t' \le 1$,

$$\mathbb{E}[(\Delta_{s,s',t,t'} S^n)^4] \le \frac{7}{n} + 6(s'-s)^2(t'-t)^2(1-(s'-s))^2, \tag{6.1}$$

where $\Delta_{s,s',t,t'} S^n$ denotes the increment of $S^n$ on $[s,s'] \times [t,t']$, *i.e.*

$$\Delta_{s,s',t,t'} S^n := \frac{1}{\sqrt{n}} \sum_{\substack{ns < i \le ns' \\ nt \le j \le nt'}} (m_{ij} - \frac{1}{n}). \tag{6.2}$$

As $S^n$ vanishes on the boundary on $[0,1]^2$, according to [11, Th. 3], (6.1) will imply the lemma.

To prove (6.1), we fix $0 \le s < s' \le 1$, $0 \le t < t' \le 1$. Let us now introduce some notation (where the dependence on $n$ will by dropped for readability). We define the sets

$$I := \{i \in \{1,\ldots,n\} \,;\, ns < i \le ns'\} \text{ and } J := \{j \in \{1,\ldots,n\} \,;\, nt < j \le nt'\},$$

the numbers $(\alpha_i)_{1 \le i \le n}$ defined by

$$\alpha_i := \begin{cases} -\dfrac{1}{\sqrt{n}} \dfrac{|I|}{n} & \text{if } i \notin I \\[2mm] \dfrac{1}{\sqrt{n}} \left( 1 - \dfrac{|I|}{n} \right) & \text{if } i \in I \end{cases}$$

and the exchangeable random vector (implicitly depending on $n$) $(X_1, \ldots, X_n)$ defined by

$$X_i = \sum_{j \in J} m_{ij}.$$

Note that

$$\Delta_{s,s',t,t'} S^n \;=\; \frac{1}{\sqrt{n}} \left( \left( \sum_{i \in I} X_i \right) - \frac{|I||J|}{n} \right)$$

and that as columns of $M$ sum up to one, $|J| = \sum_{j \in J} \sum_{i=1}^{n} m_{ij} = \sum_{i=1}^{n} X_i$, hence

$$\Delta_{s,s',t,t'} S^n = \frac{1}{\sqrt{n}} \left( \sum_{i \in I} X_i - \frac{|I|}{n} \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \alpha_i X_i.$$

Thus by exchangeability of the $X_i$'s, we have

$$
\begin{aligned}
\mathbb{E}[(\Delta_{s,s',t,t'} S^n)^4] \;=\;\; & \mathbb{E}(X_1^4)\operatorname{sum}_4(\alpha) + 4\,\mathbb{E}(X_1^3 X_2)\operatorname{sum}_{3,1}(\alpha) + 3\,\mathbb{E}(X_1^2 X_2^2)\operatorname{sum}_{2,2}(\alpha) \\
& + 6\,\mathbb{E}(X_1^2 X_2 X_3)\operatorname{sum}_{2,1,1}(\alpha) + \mathbb{E}(X_1 X_2 X_3 X_4)\operatorname{sum}_{1,1,1,1}(\alpha),
\end{aligned}
$$

with

$$
\operatorname{sum}_4(\alpha) := \sum_{i=1}^n \alpha_i^4, \quad \operatorname{sum}_{3,1}(\alpha) := \sum_{i\neq j} \alpha_i^3 \alpha_j, \quad \operatorname{sum}_{2,2}(\alpha) := \sum_{i\neq j} \alpha_i^2 \alpha_j^2,
$$

$$
\operatorname{sum}_{2,1,1}(\alpha) := \sum_{\substack{i,j,k \\ \text{pairwise} \neq}} \alpha_i^2 \alpha_j \alpha_k, \quad \operatorname{sum}_{1,1,1,1}(\alpha) := \sum_{\substack{i,j,k,\ell \\ \text{pairwise} \neq}} \alpha_i \alpha_j \alpha_k \alpha_\ell.
$$

As the $\alpha_i$'s sum up to zero, we have

$$
\begin{aligned}
\operatorname{sum}_{3,1}(\alpha) &= \sum_i \Big(\alpha_i^3 \sum_{j\neq i} \alpha_j\Big) = -\operatorname{sum}_4(\alpha), \\
\operatorname{sum}_{2,1,1}(\alpha) &= \sum_i \Big(\alpha_i^2 \sum_{j\neq i}\big(\alpha_j \sum_{k\notin\{i,j\}} \alpha_k\big)\Big) = \sum_i \Big(\alpha_i^2 \sum_{j\neq i}\big(\alpha_j(-\alpha_i - \alpha_j)\big)\Big) \\
&= -\operatorname{sum}_{3,1}(\alpha) - \operatorname{sum}_{2,2}(\alpha) = \operatorname{sum}_4(\alpha) - \operatorname{sum}_{2,2}(\alpha) \\
\operatorname{sum}_{1,1,1,1}(\alpha) &= -3\operatorname{sum}_{2,1,1}(\alpha) = 3\operatorname{sum}_{2,2}(\alpha) - 3\operatorname{sum}_4(\alpha)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbb{E}[(\Delta_{s,s',t,t'} S^n)^4] \;=\;\; & \operatorname{sum}_4(\alpha)\big(\mathbb{E}(X_1^4) - 4\,\mathbb{E}(X_1^3 X_2) + 6\,\mathbb{E}(X_1^2 X_2 X_3) - 3\,\mathbb{E}(X_1 X_2 X_3 X_4)\big) \\
& + \operatorname{sum}_{2,2}(\alpha)\big(3\,\mathbb{E}(X_1^2 X_2^2) - 6\,\mathbb{E}(X_1^2 X_2 X_3) + 3\,\mathbb{E}(X_1 X_2 X_3 X_4)\big),
\end{aligned}
$$

Now, as for all $i$, $|\alpha_i| \leq \frac{1}{\sqrt{n}}$, we have $\operatorname{sum}_4(\alpha) \leq \frac{1}{n}$, and as for all $i$, $0 \leq X_i \leq 1$ (because the rows of $s$ sum up to one), we have

$$
\mathbb{E}[(\Delta_{s,s',t,t'} S^n)^4] \;\leq\; \frac{7}{n} + 3\operatorname{sum}_{2,2}(\alpha)\big(\mathbb{E}(X_1^2 X_2^2) + \mathbb{E}(X_1 X_2 X_3 X_4)\big).
$$

To conclude the proof of (6.1), we shall prove that

$$
\operatorname{sum}_{2,2}(\alpha) \leq (s' - s)^2 \tag{6.3}
$$

and

$$
E(X_1^2 X_2^2) + \mathbb{E}(X_1 X_2 X_3 X_4) \leq 2(t' - t)^2. \tag{6.4}
$$

Let us first check (6.3). We have

$$
\operatorname{sum}_{2,2}(\alpha) \leq \Big(\sum_i \alpha_i^2\Big)^2 = \left\{(n - |I|)\frac{|I|^2}{n^3} + |I|\frac{1}{n}(1 - |I|/n)^2\right\}^2 = \left\{\frac{|I|}{n}\Big(1 - \frac{|I|}{n}\Big)\right\}^2,
$$

which gives (6.3). Let us now check (6.4). As $0 \leq X_i \leq 1$, it suffices to prove that

$$
\mathbb{E}(X_1 X_2) \leq (t' - t)^2. \tag{6.5}
$$

We have

$$
\mathbb{E}(X_1 X_2) = \sum_{j,j'\in J} \mathbb{E}(m_{1j} m_{2j'}),
$$

so it suffices to prove that uniformly on $j, j' \in \{1,\ldots,n\}$, $\mathbb{E}(m_{1j} m_{2j'}) \leq \dfrac{1}{n(n-1)}$. We get this as follows: using the exchangeability of the rows of $M$ and the fact that its rows sum up to one, we have, for any $j, j' \in \{1,\ldots,n\}$,

$$
1 = \mathbb{E}\Big(\big(\sum_i m_{ij}\big)\big(\sum_{i'} m_{i'j'}\big)\Big) = n(n-1)\,\mathbb{E}(m_{1j} m_{2j'}) + n\,\mathbb{E}(m_{1j} m_{1j'}).
$$

This concludes the proof. $\qquad\square$

## 6.2 Injectivity of the Cauchy transform for certain classes of functions

**Lemma 6.2.** *Let $f$ be a real valued bounded càdlàg function on $\mathbb{R}$ vanishing at infinity with at most countably many discontinuity points. Then $f$ is entirely determined by the function*

$$K_f(z) := \int \frac{f(\lambda)}{(z-\lambda)^2} \mathrm{d}\lambda \qquad (z \in \mathbb{C}\backslash\mathbb{R}).$$

*More precisely, for any $\lambda \in \mathbb{R}$, we have*

$$f(\lambda) = \lim_{\substack{\tilde{\lambda}\downarrow\lambda \\ f \text{ is cont. at } \tilde{\lambda}}} \lim_{\eta\downarrow 0} \frac{1}{\pi} \int_{-\infty}^{\tilde{\lambda}} \Im K_f(E + i\eta) \mathrm{d}E. \tag{6.6}$$

*Proof.* Let us introduce the Cauchy transform of $f$, defined, on $\mathbb{C}\backslash\mathbb{R}$, by $H_f(z) := \int \frac{f(\lambda)}{z-\lambda} \mathrm{d}\lambda$. It is well known that at any $\tilde{\lambda}$ where $f$ is continuous, we have

$$f(\tilde{\lambda}) = \lim_{\eta\downarrow 0} -\frac{1}{\pi}\Im H_f(\tilde{\lambda}+i\eta).$$

Then, the result follows because for all $\tilde{\lambda} \in \mathbb{R}$, $\eta > 0$,

$$-H_f(\tilde{\lambda}+i\eta) = \int_{-\infty}^{\tilde{\lambda}} K_f(E+i\eta)\mathrm{d}E.$$

$\square$

## 6.3 A lemma about large products and the exponential function

The following lemma helps controlling the error terms in the proof of Proposition 2.9 (in these cases, $M$ always has order one).

**Lemma 6.3.** *Let $u_i$, $i = 1,\ldots,n$, be some complex numbers and set*

$$P := \prod_{i=1}^{n}(1 + \frac{u_i}{n}) \qquad S := \frac{1}{n}\sum_{i=1}^{n}u_i.$$

*There is a universal constant $R > 0$ such that for $M := \max_i |u_i|$,*

$$\frac{M}{n} \leq R \implies |P - e^S| \leq \frac{M^2}{n}e^{|S| + \frac{M^2}{n}}.$$

*Proof.* Let $L(z)$ be defined on $B(0,1)$ by $\log(1+z) = z + z^2 L(z)$ and $R > 0$ be such that on $B(0,R)$, $|L(z)| \leq 1$. If $\frac{M}{n} \leq R$, we have

$$P = \prod_i \exp\left\{\frac{u_i}{n} + \frac{u_i^2}{n^2}L(\frac{u_i}{n})\right\} = e^S \exp\left\{\sum_i \frac{u_i^2}{n^2}L(\frac{u_i}{n})\right\},$$

which allows to conclude easily, as for any $z$, $|e^z - 1| \leq |z|e^{|z|}$. $\square$

## 6.4 CLT for martingales

Let $(\mathcal{F}_k)_{k\geq 0}$ be a filtration such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and let $(M_k)_{k\geq 0}$ be a square-integrable complex-valued martingale starting at zero with respect to this filtration. For $k \geq 1$, we define the random variables

$$Y_k := M_k - M_{k-1} \qquad v_k := \mathbb{E}[|Y_k|^2 \,|\, \mathcal{F}_{k-1}] \qquad \tau_k := \mathbb{E}[Y_k^2 \,|\, \mathcal{F}_{k-1}]$$

and we also define

$$v := \sum_{k \geq 1} v_k \qquad \tau := \sum_{k \geq 1} \tau_k \qquad L(\varepsilon) := \sum_{k \geq 1} \mathbb{E}[|Y_k|^2 \mathbb{1}_{|Y_k| \geq \varepsilon}].$$

Let now everything depend on a parameter $n$, so that $\mathcal{F}_k = \mathcal{F}_k^n, M_k = M_k^n, Y_k = Y_k^n, v = v^n, \tau = \tau^n, L(\varepsilon) = L^n(\varepsilon), \ldots$

Then we have the following theorem. It is proved in the real case at [12, Th. 35.12]. The complex case can be deduced noticing that for $z \in \mathbb{C}$, $\Re(z)^2, \Im(z)^2$ and $\Re(z)\Im(z)$ are linear combinations of $z^2, \overline{z}^2, |z|^2$.

**Theorem 6.4.** *Suppose that for some constants $v \geq 0, \tau \in \mathbb{C}$, we have the convergence in probability*

$$v^n \xrightarrow[n \to \infty]{} v \qquad \tau^n \xrightarrow[n \to \infty]{} \tau$$

*and that for each $\varepsilon > 0$,*

$$L^n(\varepsilon) \xrightarrow[n \to \infty]{} 0.$$

*Then we have the convergence in distribution*

$$M_n^n \xrightarrow[n \to \infty]{} Z,$$

*where $Z$ is a centered complex Gaussian variable such that $\mathbb{E}(|Z|^2) = v$ and $\mathbb{E}(Z^2) = \tau$.*

### 6.5  Some linear algebra lemmas

Let $\| \cdot \|_\infty$ denote the operator norm of matrices associated with the canonical Hermitian norm.

**Lemma 6.5.** *Let $A = [a_{ij}]$ be an $n \times n$ Hermitian matrix, $z \in \mathbb{C} \backslash \mathbb{R}$, $G := (z - A)^{-1}$, $P$ be a diagonal matrix. For $1 \leq k \leq n$ we denote by $A^{(k)}$, $P^{(k)}$ be the matrices with size $n - 1$ obtained by removing the $k$-th row and the $k$-th column of $A$ and $P$ and set $G^{(k)} := (z - A^{(k)})^{-1}$. Then*

$$\operatorname{Tr}(PG) - \operatorname{Tr}(P^{(k)}G^{(k)}) = \frac{P_{kk} + \mathbf{a}_k^* G^{(k)} P^{(k)} G^{(k)} \mathbf{a}_k}{z - a_{kk} - \mathbf{a}_k^* G^{(k)} \mathbf{a}_k}, \tag{6.7}$$

*with $\mathbf{a}_k$ the $k$-th column of $A$ where the diagonal entry has been removed. Moreover,*

$$|\operatorname{Tr}(PG) - \operatorname{Tr}(P^{(k)}G^{(k)})| \leq \frac{5\|P\|_\infty}{|\Im z|}. \tag{6.8}$$

*Proof.* • Let us first prove (6.7). By linearity, one can suppose that $P$ has only one nonzero diagonal entry, say the $i$th one, equal to one. Using the well known formula

$$((z - A)^{-1})_{ii} - \mathbb{1}_{i \neq k}((z - A^{(k)})^{-1})_{ii} = \frac{G_{ki}G_{ik}}{G_{kk}},$$

we have

$$
\begin{aligned}
\operatorname{Tr}(PG) - \operatorname{Tr}(P^{(k)}G^{(k)}) &= ((z - A)^{-1})_{ii} - \mathbb{1}_{i \neq k}((z - A^{(k)})^{-1})_{ii} \\
&= \frac{G_{ki}G_{ik}}{G_{kk}} \\
&= \frac{((z - A)^{-1}P(z - A)^{-1})_{kk}}{((z - A)^{-1})_{kk}} \\
&= \frac{\partial_{t|t=0}((z - A - tP)^{-1})_{kk}}{((z - A)^{-1})_{kk}}
\end{aligned}
$$

Let $\log$ denote the determination of the $\log$ on $\mathbb{C}\backslash\mathbb{R}^-$ vanishing at one. Then

$$
\begin{aligned}
\mathrm{Tr}(PG) - \mathrm{Tr}(P^{(k)}G^{(k)}) &= \partial_{t|_{t=0}} \log\{((z-A-tP)^{-1})_{kk}\} \\
&= \partial_{t|_{t=0}} \log \frac{1}{z - a_{kk} - t\mathbb{1}_{k=i} - \mathbf{a}_k^*(z - X^{(k)} - tP^{(k)})^{-1}\mathbf{a}_k} \\
&= -\partial_{t|_{t=0}} \log(z - a_{kk} - t\mathbb{1}_{k=i} - \mathbf{a}_k^*(z - X^{(k)} - tP^{(k)})^{-1}\mathbf{a}_k) \\
&= \frac{-\partial_{t|_{t=0}}(z - a_{kk} - t\mathbb{1}_{k=i} - \mathbf{a}_k^*(z - X^{(k)} - tP^{(k)})^{-1}\mathbf{a}_k)}{(z - a_{kk} - t\mathbb{1}_{k=i} - \mathbf{a}_k^*(z - X^{(k)} - tP^{(k)})^{-1}\mathbf{a}_k)_{|_{t=0}}} \\
&= \frac{P_{kk} + \mathbf{a}_k^* G^{(k)} P G^{(k)} \mathbf{a}_k}{z - a_{kk} - \mathbf{a}_k^* G^{(k)} \mathbf{a}_k}
\end{aligned}
$$

• Let us now prove (6.8) (the proof does not use (6.7)). One can suppose that $k=1$. Let us introduce

$$
\tilde{A} := \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A^{(1)} & \\ 0 & & & \end{bmatrix}
$$

and define $\tilde{G}$ and $\tilde{G}^{(1)}$ as $G$ and $G^{(1)}$ with $\tilde{A}$ instead of $A$. We have

$$
|\mathrm{Tr}(PG) - \mathrm{Tr}(P^{(1)}G^{(1)})| \leq |\mathrm{Tr}(P(G-\tilde{G}))| + |\mathrm{Tr}(P\tilde{G}) - \mathrm{Tr}(P^{(1)}\tilde{G}^{(1)})| + |\mathrm{Tr}(P^{(1)}(G^{(1)} - \tilde{G}^{(1)}))|.
$$

Let us treat the terms of the RHT separately.

The third term is null because $\tilde{A}^{(1)} = A^{(1)}$. We have

$$
|\mathrm{Tr}(P(G-\tilde{G}))| \leq \|P(G-\tilde{G})\|_\infty \mathrm{rank}(G-\tilde{G})
$$

which is $\leq \frac{4\|P\|_\infty}{|\Im z|}$ by the resolvent formula. At last, as $P$ is diagonal and the matrix $z - \tilde{A}$ can be inverted by blocs, we have

$$
|\mathrm{Tr}(P\tilde{G}) - \mathrm{Tr}(P^{(1)}\tilde{G}^{(1)})| = |P_{11}\tilde{G}_{11}| \leq \frac{\|P\|_\infty}{|\Im z|}.
$$

$\square$

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