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RANDOM WALK ATTRACTED BY PERCOLATION CLUSTERS

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Abstract

Starting with a percolation model in $\mathbb{Z}^d$ in the subcritical regime, we consider a random walk described as follows: the probability of transition from x to y is proportional to some function f of the size of the cluster of y. This function is supposed to be increasing, so that the random walk is attracted by bigger clusters. For $f(t) = e^{\beta t}$ we prove that there is a phase transition in β, i.e., the random walk is subdiffusive for large β and is diffusive for small β.

1 Introduction and results

First, we describe the usual site percolation model in $\mathbb{Z}^d$. This model is defined as follows. For fixed $p \in (0,1)$, consider i.i.d. random variables $\omega(x)$, $x \in \mathbb{Z}^d$, where $\omega(x) = 1$ with probability p and $\omega(x) = 0$ with probability $1-p$. A site x is said to be open if $\omega(x) = 1$ and closed otherwise. Write $x \sim z$ if x and z are neighbors. A (self-avoiding) path from x to y is: $\gamma(x,y) = \{x_0 = x, x_1, x_2, \dots, x_n = y\}$, where $x_i \neq x_j$ if $i \neq j$ and $x_i \sim x_{i+1}$, $i = 0,\dots,n-1$. A path γ is said to be open if all the sites in γ are open. The cluster of x is defined by

$$\mathfrak{C}(x) = \{y \in \mathbb{Z}^d : \omega(y) = 1 \text{ and there is an open path } \gamma(x,y) \text{ from } x \text{ to } y\}.$$

Note that, if $\omega(x) = 0$, then $\mathfrak{C}(x) = \emptyset$. It is a well-known fact (see e.g. [9]) that there exists p_{cr} (depending on d; obviously, $p_{cr} = 1$ in dimension 1) such that if $p < p_{cr}$, then a.s. there

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is no infinite open cluster, and if $p > p_{cr}$, then a.s. there exists an infinite open cluster (also, with positive probability $|\mathfrak{C}(0)| = \infty$).
Throughout this paper we assume that $p < p_{cr}$, i.e, the model is in the (strictly) subcritical regime. Fix a parameter $\beta > 0$. The percolation configuration is regarded as random environment. Fixed the environment, we start a discrete time random walk on $\mathbb{Z}^d$ with transition probabilities

$$P^{\omega}_{xy} = \frac{e^{\beta|\mathfrak{C}(y)|}}{\sum\limits_{z\sim x} e^{\beta|\mathfrak{C}(z)|}},$$

if $x \sim y$. Since β is positive, one can note that the random walk is in some sense "attracted" by bigger clusters, and the strength of this attraction grows with β. Denote by $\xi(t)$ the position of this random walk at time t. Let $\mathbb{P}$ be the probability measure with respect to ω and P^x_ω the (so-called quenched) probability for the random walk starting from x in the fixed environment ω. Denote also $\mathbf{P}^x = \mathrm{P}^x_\omega \otimes \mathbb{P}$ ($\mathbf{P}^x$ is usually called the annealed probability); throughout the paper $\|\cdot\|$ stands for the L_∞ norm. Our main result is that there is a phase transition in β, i.e., the random walk exhibits different behaviors for large and small β: it is diffusive for small values of β and subdiffusive for large values of β.

Theorem 1.1 *Suppose that the random walk $\xi(t)$ starts from the origin. There exist β_0 and β_1 (depending on d) such that $0 < \beta_0 \le \beta_1 < \infty$ and*

(i) if $0 < \beta < \beta_0$, then

$$\lim_{t\to\infty} \frac{\log \max_{0\le s\le t} \|\xi(s)\|}{\log t} = \frac{1}{2}, \quad \mathbf{P}^0\text{-}a.s. \tag{1.1}$$

(ii) if $\beta > \beta_1$, then

$$\limsup_{t\to\infty} \frac{\log \max_{0\le s\le t} \|\xi(s)\|}{\log t} < \frac{1}{2}, \quad \mathbf{P}^0\text{-}a.s. \tag{1.2}$$

One can prove also that the same result holds for the bond percolation model in the subcritical regime. The method of the proof remains the same; the reason why we have chosen the site percolation is that for bond percolation there are some technical difficulties (easily manageable, though; they relate to the fact that, in the bond percolation model, two neighboring sites can belong to different large clusters) in the proof of the part (ii) of Theorem 1.1.
Recently much work has been done on the (simple or not) random walk on the unique infinite cluster for the supercritical (bond or site) percolation in $\mathbb{Z}^d$ (see e.g. [2, 4, 10, 15]; see also [11] for some results for the random walk on the incipient infinite cluster in dimension 2). Another related subject is the class of models (see e.g. [5, 8]) that can be described as follows. Into each edge of $\mathbb{Z}^d$ we place a random variable that represents the transition rate between the sites. The new features of the model of the present paper are, first, the fact that the random environment is not independent, and secondly, the absence of the uniform ellipticity. Speaking of uniform ellipticity, we should mention that in the paper [7] there was considered a simple symmetric one-dimensional random walk with random rates, where the time spent at site i before taking a step has an exponential distribution with mean τ_i, and τ_i's are i.i.d. positive random variables with distribution function F having a polinomial tail. One may find that there are similarities of the d-dimensional analog of the model of [7] with our model, because clusters of size n will have "density" e^{-Cn}, and the mean time spent there is roughly $e^{\beta n}$, so, thinking of clusters as "sites", we indeed obtain a polinomial tail of mean time spent at a given site. However, the facts that the random environment is no longer independent and

that here the random walk is not a time-change of the simple random walk make the model of the present paper considerably more difficult to analyze.

2 Proof of Theorem 1.1

We begin by introducing some notations and recalling a well-known fact from the percolation theory. Namely, we will use the following result (see [12, 9]): if $p < p_{cr}$, then there exists $c_1 > 0$ such that for all $N > 0$ and $x \in \mathbb{Z}^d$

$$\mathbb{P}[|\mathfrak{C}(x)| > N] \le e^{-c_1 N}. \tag{2.1}$$

Now, to prove Theorem 1.1, an important idea is to consider $\xi(t)$ in finite region. Take $\Lambda_n = (-n/2, n/2]^d$ and let the process $\xi^{(n)}(t)$ be the random walk $\xi(t)$ restricted on Λ_n.

Proof of part (i). It can be easily seen that $\xi(t)$ is reversible with the reversible measure

$$\pi(x) = e^{\beta|\mathfrak{C}(x)|} \sum_{z\sim x} e^{\beta|\mathfrak{C}(z)|}, \tag{2.2}$$

and thus the finite Markov chain $\xi^{(n)}(t)$ is also reversible, with the invariant (and reversible) measure

$$\pi^{(n)}(x) = \frac{e^{\beta|\mathfrak{C}(x)|} \sum\limits_{z\sim x} e^{\beta|\mathfrak{C}(z)|}}{Z}, \tag{2.3}$$

where

$$Z = \sum_{x\in\Lambda_n} e^{\beta|\mathfrak{C}(x)|} \sum_{z\sim x} e^{\beta|\mathfrak{C}(z)|}$$

is the normalizing constant, so that $\sum_{x\in\Lambda_n} \pi^{(n)}(x) = 1$.
Consider also a random walk $\hat{\xi}^{(n)}(t)$ that is a continuization of $\xi^{(n)}(t)$. That is, $\hat{\xi}^{(n)}(t) = \xi^{(n)}(N_t)$, where N_t is a Poisson process with rate 1, independent of anything else (in other words, $\hat{\xi}^{(n)}$ is a continuous time Markov chain with the transition rates equal to the transition probabilities of $\xi^{(n)}$). Let $\mathcal{T}_i$ be the time interval between the jumps $(i-1)$ and i of N_t, $S_n = \sum_{i=1}^n \mathcal{T}_i$, and $\hat{T}_A$ (respectively, T_A) be hitting time of set A by random walk $\hat{\xi}^{(n)}(t)$ (respectively, $\xi^{(n)}(t)$). It can be easily seen (cf., for example, Chapter 2 of [1]) that $\hat{T}_A = S_{T_A}$ and $\mathbf{E}(\hat{T}_A \mid T_A) = T_A$. Moreover, since $t^{-1}N_t \to 1$ a.s., many other results concerning $\hat{\xi}^{(n)}$ can be easily translated into the corresponding results for $\xi^{(n)}$.

Remark 2.1 *Using this technique, it is elementary to obtain that Theorem 1.1 holds for $\xi(t)$ iff it holds for $\hat{\xi}(t)$, where $\hat{\xi}(t)$ is the continuization of $\xi(t)$ defined in the same way.*

So, now we consider the finite continuous time Markov chain $\hat{\xi}^{(n)}(t)$. Denote by λ the spectral gap of $\hat{\xi}^{(n)}(t)$.

Lemma 2.1 *There exist $c_{14} > 0$ and $n^* = n^*(\omega)$ such that for all $n > n^*$ we have $\lambda \ge c_{14} n^{-2}$, $\mathbb{P}$-a.s.*

Proof of Lemma 2.1. The idea is to use Theorem 3.2.1 from [14] to prove the lemma. For each pair $x, y \in \Lambda_n$, we will choose exactly one path $\gamma(x,y)$ (connecting x and y) in a way described below. Let $|\gamma(x,y)|$ be the length of $\gamma(x,y)$ (i.e. the number of edges in $\gamma(x,y)$). Denote by $\mathcal{E}(\Lambda_n)$ the set of edges of $\mathbb{Z}^d \cap \Lambda_n$. For an edge $u = \langle z_1, z_2\rangle$ let $Q(u) = (P^\omega_{z_1 z_2}\pi^{(n)}(z_1) + P^\omega_{z_2 z_1}\pi^{(n)}(z_2))/2$. According to Theorem 3.2.1 of [14], it holds that $\lambda \geq 1/A$, where

$$A = \max_{u\in\mathcal{E}(\Lambda_n)} \Big\{ \frac{1}{Q(u)} \sum_{x,y\in\Lambda_n:\ \gamma(x,y)\ni u} |\gamma(x,y)|\pi^{(n)}(x)\pi^{(n)}(y) \Big\}. \tag{2.4}$$

Here, we have, for $u = \langle z_1, z_2\rangle$,

$$Q(u) = \frac{e^{\beta(|\mathfrak{C}(z_1)|+|\mathfrak{C}(z_2)|)}}{Z}$$

and for each pair $x = (x^{(1)},\dots,x^{(d)}), y = (y^{(1)},\dots,y^{(d)}) \in \Lambda_n$ we choose the path $\gamma(x,y)$ in the following way. Let $e_1,\dots,e_d$ be the coordinate vectors. Denote $\Delta_i = y^{(i)} - x^{(i)}$ and let $\mathrm{sgn}(\Delta_i)$ be the sign of Δ_i. Suppose for definiteness that $x^{(d)} \leq y^{(d)}$ (so that $\mathrm{sgn}(\Delta_d) \geq 0$). We take then

$$\begin{aligned} \gamma(x,y) \quad = \quad & (x, x+e_d, \dots, x+\Delta_d e_d, x + \mathrm{sgn}(\Delta_{d-1})e_{d-1} + \Delta_d e_d, \dots, \\ & x + \Delta_{d-1}e_{d-1} + \Delta_d e_d, \dots, x + \Delta_1 e_1 + \dots + \Delta_d e_d = y), \end{aligned}$$

so, first we successively change the d-th coordinate of x to obtain the d-th coordinate of y, then we do the same with $(d-1)$-th coordinate, and so on. With this construction it is clear that the length of $\gamma(x,y)$ is at most dn. For an edge

$$u = \langle (x^{(1)},\dots,x^{(d)}), (x^{(1)},\dots,x^{(d-1)}, x^{(d)}+1)\rangle$$

define

$$I_u = \{(z^{(1)},\dots,z^{(d)}) \in \Lambda_n :\ \ z^{(1)} = x^{(1)},\dots,z^{(d-1)} = x^{(d-1)},\ z^{(d)} \leq x^{(d)}\}$$

and

$$R_u = \{(z^{(1)},\dots,z^{(d)}) \in \Lambda_n :\ \ z^{(d)} > x^{(d)}\}$$

(for the edges of other directions the computations are quite analogous). We have then

$$\begin{aligned} & \sum_{x,y\in\Lambda_n:\ \gamma(x,y)\ni u} |\gamma(x,y)|\pi^{(n)}(x)\pi^{(n)}(y) \\ & \qquad \leq \quad dn \sum_{x\in I_u} \pi^{(n)}(x) \sum_{y\in R_u} \pi^{(n)}(y) \\ & \qquad \leq \quad dn \sum_{x\in I_u} \pi^{(n)}(x), \end{aligned} \tag{2.5}$$

as $\sum_{y\in R_u} \pi^{(n)}(y) \leq 1$.

Now our goal is to prove that with large probability, for all such u, $\sum_{x\in I_u} \pi^{(n)}(x)$ is of order n/Z. Denote

$$\tilde{I}_u = \{(z^{(1)},\dots,z^{(d)}) \in \Lambda_n :\ \ z^{(1)} = x^{(1)},\dots,z^{(d-1)} = x^{(d-1)}\}.$$

Note that $I_u \subset \tilde{I}_u$, so we will concentrate on obtaining an upper bound for $\sum_{x\in\tilde{I}_u} \pi^{(n)}(x)$. It is important to observe that the variables $\pi^{(n)}(x)$ are not independent. For the sake of simplicity, suppose that $\sqrt{n}$ is an integer, the general case can be treated analogously. Divide $\tilde{I}_u$ into $\sqrt{n}$ equal (connected) parts of size $\sqrt{n}$, denote $x_{ij} := (x^{(1)}, \dots, x^{(d-1)}, (i-1)\sqrt{n}+j)$ and write

$$\sum_{x\in\tilde{I}_u} \pi^{(n)}(x) = \sum_{j=1}^{\sqrt{n}} \sum_{i=-\frac{\sqrt{n}}{2}+1}^{\frac{\sqrt{n}}{2}} \pi^{(n)}(x_{ij}).$$

For fixed j, let $B_i = \mathbf{1}_{\{|\mathfrak{C}(x_{ij})|<\sqrt{n}/2\}}$, and $B = \bigcap_{i=1}^{\sqrt{n}} B_i$. By (2.1), we have $\mathbb{P}[B_i] \geq 1 - e^{-c_1\sqrt{n}}$ which implies that $\mathbb{P}[B] \geq 1 - e^{-c_2\sqrt{n}}$ for some $c_2 > 0$, so

$$\begin{aligned}\mathbb{P}\Big[\sum_{i=-\frac{\sqrt{n}}{2}+1}^{\frac{\sqrt{n}}{2}} \pi^{(n)}(x_{ij}) \geq \frac{k\sqrt{n}}{Z}\Big] \\ \leq \mathbb{P}\Big[\Big(\sum_{i=-\frac{\sqrt{n}}{2}+1}^{\frac{\sqrt{n}}{2}} \pi^{(n)}(x_{ij}) \geq \frac{k\sqrt{n}}{Z}\Big)\mathbf{1}_B\Big] + e^{-c_2\sqrt{n}}.\end{aligned}$$

Now, it is important to note that the variables $e^{\beta|\mathfrak{C}(x_{ij})|}\sum_{z\sim x_{ij}} e^{\beta|\mathfrak{C}(z)|}\mathbf{1}_B = \pi(x_{ij})\mathbf{1}_B$ (recall (2.2) and (2.3)), $i = 1, \dots, \sqrt{n}$, are independent. We have also

$$e^{\beta|\mathfrak{C}(x_{ij})|}\sum_{z\sim x_{ij}} e^{\beta|\mathfrak{C}(z)|} \leq 2de^{2\beta|\mathfrak{C}(\tilde{x}_{ij})|},$$

where $\tilde{x}_{ij}$ satisfies $|\mathfrak{C}(\tilde{x}_{ij})| = \max_{z\sim x_{ij}}\{|\mathfrak{C}(x_{ij})|, |\mathfrak{C}(z)|\}$.
For $y = n^\alpha$ ($\alpha > 0$ will be chosen later), using (2.1), we have

$$\sum_{i=-\frac{\sqrt{n}}{2}+1}^{\frac{\sqrt{n}}{2}} \mathbb{P}[2de^{2\beta|\mathfrak{C}(\tilde{x}_{ij})|} > y] \leq \sqrt{n}e^{-\frac{c_3\log y}{2\beta}} = \sqrt{n}y^{-\frac{c_3}{2\beta}} = n^{\frac{1}{2}-\frac{c_3\alpha}{2\beta}}. \tag{2.6}$$

According to Corollary 1.5 from [13], if $X_1, \dots, X_k$ are independent random variables, $S_k = \sum_{i=1}^k X_i$, $F_i(x) = \mathbb{P}[X_i < x]$, then for any set $y_1, \dots, y_k$ of positive numbers and any t, $t \in (0,1]$,

$$\mathbb{P}[S_k \geq x] \leq \sum_{i=1}^k \mathbb{P}[X_i > y_i] + \Big(\frac{eA_t^+}{xy^{t-1}}\Big)^{\frac{x}{y}}, \tag{2.7}$$

where $y \geq \max\{y_1, \dots, y_k\}$ and

$$A_t^+ = \sum_{i=1}^k \int_0^\infty u^t dF_i(u).$$

Denote $\bar{F}_i(u) = 1 - F_i(u)$. We apply Corollary 1.5 from [13] to random variables $2de^{2\beta|\mathfrak{C}(\tilde{x}_{ij})|}\mathbf{1}_B$, $i = -\frac{\sqrt{n}}{2}+1, \dots, \frac{\sqrt{n}}{2}$, with $x = k\sqrt{n}$, $y_i \equiv y = n^\alpha$, and $t = 1$. First term of the right-hand side of (2.7) was estimated in (2.6). For the second term, we have

$$\begin{aligned} A_1^+ &= \sum_{i=-\frac{\sqrt{n}}{2}+1}^{\frac{\sqrt{n}}{2}} \int_0^\infty u dF_i(u) \\ &= -\sqrt{n}\int_0^\infty u d\bar{F}_i(u) \\ &= -\sqrt{n}\int_0^\infty \bar{F}_i(u) du \\ &\leq c_4\sqrt{n}\int_1^\infty u^{-\frac{c_3}{2\beta}} du \\ &= c_5\sqrt{n}, \end{aligned} \tag{2.8}$$

as β is small, thus

$$\Big(\frac{eA_t^+}{xy^{t-1}}\Big)^{\frac{x}{y}} \leq \Big(\frac{ec_5\sqrt{n}}{k\sqrt{n}}\Big)^{\frac{k\sqrt{n}}{n^\alpha}} = \Big(\frac{ec_5}{k}\Big)^{kn^{\frac{1}{2}-\alpha}} \tag{2.9}$$

so, to guarantee that $\Big(\frac{eA_t^+}{xy^{t-1}}\Big)^{\frac{x}{y}} \to 0$ as $n \to \infty$, it is sufficient to take k large enough and $\beta/c_3 < \alpha < 1/2$.
We proved that for β sufficiently small

$$\mathbb{P}\Big[\sum_{i=-\frac{\sqrt{n}}{2}+1}^{\frac{\sqrt{n}}{2}} \pi^{(n)}(x_{ij}) \geq \frac{k\sqrt{n}}{Z}\Big] \leq c_7 n^{-\frac{1}{2}(\frac{c_3\alpha}{\beta}-1)} \leq c_7 n^{-\frac{c_{10}}{\beta}+\frac{1}{2}}. \tag{2.10}$$

Thus,

$$\begin{aligned} \mathbb{P}\Big[\sum_{x\in\tilde{I}_u} \pi^{(n)}(x) \geq \frac{c_8 n}{Z}\Big] &= \mathbb{P}\Big[\sum_{j=1}^{\sqrt{n}} \sum_{i=-\frac{\sqrt{n}}{2}+1}^{\frac{\sqrt{n}}{2}} \pi^{(n)}(x_{ij}) \geq \frac{c_8 n}{Z}\Big] \\ &\leq \sum_{j=1}^{\sqrt{n}} \mathbb{P}\Big[\sum_{i=-\frac{\sqrt{n}}{2}+1}^{\frac{\sqrt{n}}{2}} \pi^{(n)}(x_{ij}) \geq \frac{k\sqrt{n}}{Z}\Big] \\ &\leq c_7 n^{-\frac{c_{10}}{\beta}+1} \end{aligned} \tag{2.11}$$

So, using (2.11) in (2.5) and (2.4), we have

$$A \leq \max_{u\in\mathcal{E}(\Lambda_n)} \Big\{\frac{c_9 Z}{e^{\beta(|\mathfrak{C}(z_1)|+|\mathfrak{C}(z_2)|)}}\frac{n^2}{Z}\Big\} \leq c_9 n^2, \tag{2.12}$$

as $e^{\beta(|\mathfrak{C}(z_1)|+|\mathfrak{C}(z_2)|)} \geq 1$, with probability at least $1-c_7 n^{-\frac{c_{10}}{\beta}+1}$. Since β can be made arbitrarily small, Borel-Cantelli lemma implies that for almost all environments for n large enough it holds that $A \leq c_{13}n^2$ and thus $\lambda \geq c_{14}n^{-2}$. Lemma 2.1 is proved. □

Now, using Lemma 2.1.4 from [14] with $f(x) = \mathbf{1}_{\{\|x\| \geq n/4\}}$, where, as before, $\|\cdot\|$ is the L_∞ norm, we prove (1.1). By Lemma 2.1.4 from [14] we have that

$$\|H_t f - \pi^{(n)}(f)\|_2^2 \leq e^{-2\lambda t}\,\mathrm{Var}_\pi^{(n)}(f).$$

In what follows we show that $\pi^{(n)}(f)$ is of constant order. We have

$$\pi^{(n)}(f) = \sum_{x\in\Lambda_n,\, \|x\|\geq n/4} \frac{e^{\beta|\mathfrak{C}(x)|}\sum\limits_{z\sim x} e^{\beta|\mathfrak{C}(z)|}}{Z}.$$

Since $|\mathfrak{C}(x)| \geq 0$, it is easy to obtain that for all ω it holds

$$\sum_{x\in\Lambda_n,\, \|x\|\geq n/4} e^{\beta|\mathfrak{C}(x)|}\sum_{z\sim x} e^{\beta|\mathfrak{C}(z)|} \geq \frac{n^d}{2}. \tag{2.13}$$

Using the same kind of argument as in the proof of Lemma 2.1, one can easily see that for all n

$$\mathbb{P}\Big[\sum_{x\in\Lambda_n,\, \|x\|< n/4} e^{\beta|\mathfrak{C}(x)|}\sum_{z\sim x} e^{\beta|\mathfrak{C}(z)|} \geq c_{15}n^d\Big] \leq c''_{15} n^{-\frac{c'_{15}}{\beta}}, \tag{2.14}$$

where c'_{15}, c''_{15} depend only on c_{15}. Thus, with probability at least $1 - c''_{15}n^{-\frac{c'_{15}}{\beta}}$ we have $\pi^{(n)}(f) \geq const$. Then, using that $\mathrm{Var}_\pi^{(n)}(f) \leq 1$, taking $t = c_{16}n^2$ for c_{16} large enough yields that the random walk $\hat{\xi}^{(n)}(t)$, and thus $\xi^{(n)}(t)$, will be at distance of order n from the origin (as both random walks start from 0) after a time of order n^2 with probability bounded away from 0.

Now, for any fixed $\varepsilon > 0$, divide the time interval $(0, t]$ into t^ε intervals of length $t^{1-\varepsilon}$. Borel-Cantelli lemma implies then that for t large enough there will be at least one time interval such that at the end of this interval $\xi^{(n)}$ will be at distance at least $t^{\frac{1}{2}-\frac{\varepsilon}{2}}$ from the origin. Since $\varepsilon > 0$ is arbitrary, we proved that

$$\liminf_{t\to\infty} \frac{\log \max_{0\leq s\leq t}\|\xi(s)\|}{\log t} \geq \frac{1}{2}, \quad \mathbf{P}^0\text{-a.s.} \tag{2.15}$$

It remains to prove that

$$\limsup_{t\to\infty} \frac{\log \max_{0\leq s\leq t}\|\xi(s)\|}{\log t} \leq \frac{1}{2}, \quad \mathbf{P}^0\text{-a.s.} \tag{2.16}$$

It is a well-known fact that a reversible Markov chain with a "well-behaved" reversible measure cannot go much farther than $t^{1/2}$ by time t, see [3, 6, 11, 16]. By Theorem 1 from [6] we have, for any $\varepsilon > 0$

$$\begin{aligned} \mathrm{P}_\omega[\|\xi_n\| \geq n^{1/2+\varepsilon}] \quad &\leq \quad 2e^{-\frac{n^{2(1/2+\varepsilon)}}{2n}} \sum_{y:\|y\|\geq n^{1/2+\varepsilon}} \frac{e^{\beta|\mathfrak{C}(y)|}\sum_{z'\sim y} e^{\beta|\mathfrak{C}(z')|}}{e^{\beta|\mathfrak{C}(0)|}\sum_{z\sim 0} e^{\beta|\mathfrak{C}(z)|}} \\ &\leq \quad c_{20}e^{-n^\varepsilon} \quad \mathbb{P}\text{ -a.s.} \end{aligned} \tag{2.17}$$

for all n large enough. To obtain the bound (2.17) we have used the fact that, due to (2.1),

$$\mathbb{P}[\max_{x\in\Lambda_n} e^{\beta|\mathfrak{C}(x)|} \geq n^{\frac{\varepsilon}{2}}] \leq n^{-\frac{c_{21}\varepsilon}{\beta}}$$

for some $c_{21} > 0$. Borel-Cantelli lemma and (2.17) imply (2.16) and thus the part (i) of Theorem 1.1 is proved.

Proof of part (ii). For $x \in \mathbb{Z}^d$ let $T_0(x) = 0$, $T_0'(x) = 0$, and define

$$\begin{aligned} T_i'(x) &= \min\{t \geq T_{i-1}(x) + T_{i-1}'(x) :\ \xi(t) \in \mathfrak{C}(x)\} \\ T_i(x) &= \min\{t > T_i'(x) :\ \xi(t) \notin \mathfrak{C}(x)\} - T_i'(x), \end{aligned}$$

$i = 1, 2, 3, \ldots$, where $T_i(x)$ is defined if $\min\{t \geq T_k(x) :\ \xi(t) \in \mathfrak{C}(x)\}$, $k \leq i$, are finite. In words, $T_i'(x)$ is the moment of ith entry to the cluster of x, and $T_i(x)$ is the time spent there (i.e., after $T_i'(x)$ and before going out of $\mathfrak{C}(x)$). It is important to note that the cluster $\mathfrak{C}(x)$ is surrounded by sites with $\omega(\cdot) = 0$. Comparing $T_i(x)$ with geometric random variable with parameter $(2d-1+e^{\beta|\mathfrak{C}(x)|})^{-1}$, one can easily see that if $|\mathfrak{C}(x)| \geq \delta \log n$, then $\mathbf{E}T_i(x) \geq c_{17} n^{\beta\delta}$. Moreover, it is elementary to obtain that for $\varepsilon > 0$ and for any $\delta > 0$ we can choose β large enough so that with probability bounded away from 0 we have

$$T_i(x) \geq c_{18} n^{2+\varepsilon} \tag{2.18}$$

for x such that $|\mathfrak{C}(x)| \geq \delta \log n$.

Now, we use a dynamic construction of the percolation environment usually called the *generation method* (see [12]). That is, we proceed in the following way: we assign generation index 0 to the origin, and put $\omega(0) = 1$ (the origin is open) with probability p or $\omega(0) = 0$ (closed) with probability $1 - p$. If $\omega(0) = 0$, the process stops. If $\omega(0) = 1$, then to all $x \sim 0$ we assign generation index 1, and put $\omega(x) = 1$ with probability p or $\omega(x) = 0$ with probability $1 - p$, independently. Suppose that the we constructed m generations of the process. Let Y_i be the set of sites with generation index i and $Y^m = \{0\} \cup Y_1 \cup \ldots \cup Y_m$. Denote by Y_{m+1} the set of neighbors of the open sites in Y_m which do not belong to Y^m. Assign to the sites from Y_{m+1} the generation index $m + 1$ and a value 1 or 0 in a way described above. If $Y_m \neq \emptyset$ and $\omega(y) = 0$ for all $y \in Y_{m+1}$, then the process stops. Note that for subcritical percolation this process stops a.s., and what we obtain at the moment when the process stops is the cluster of the origin surrounded by 0-s.

So, first we construct the environment within the set $H_1 = \mathfrak{C}(0) \cup \partial\mathfrak{C}(0)$, where

$$\partial\mathfrak{C}(0) = \{y :\ y \notin \mathfrak{C}(0), y \sim x \text{ for some } x \in \mathfrak{C}(0)\}$$

(note that $\omega(y) = 0$ for any $y \in \partial\mathfrak{C}(0)$) and we know nothing yet about the environment out of the set H_1. For an arbitrary set $H \subset \mathbb{Z}^d$ denote

$$\begin{aligned} H^\circ = \{x \notin H :\ &\text{for any infinite path } \gamma(x) \text{ starting from } x \text{ it holds} \\ &\text{that } \gamma(x) \cap H \neq \emptyset\} \end{aligned}$$

(i.e., H° is the set of the holes within the set H) and let

$$G_1 = H_1 \cup H_1^\circ.$$

Then, choose $\omega(x)$ for $x \in H_1^\circ$ and start the random walk $\xi(t)$ from the origin. Let

$$\tau_1 = \min\{t :\ \xi(t) \notin G_1\}.$$

Note that $\mathfrak{C}(\xi(\tau_1)) \cap \mathfrak{C}(0) = \emptyset$, and construct, using the above method

$$H_2 = \mathfrak{C}(\xi(\tau_1)) \cup \partial\mathfrak{C}(\xi(\tau_1))$$

and

$$G_2 = G_1 \cup H_2 \cup (G_1 \cup H_2)^\circ.$$

Then, define

$$\tau_2 = \min\{t : \ \xi(t) \notin G_2\},$$

and so on. For all i, we have

$$\mathbb{P}[\mathfrak{C}(\xi(\tau_i)) \geq \delta \log n] \geq p^{\delta \log n}$$

where p is the percolation parameter. This is so due to the fact that, to have $\mathfrak{C}(\xi(\tau_i)) \geq \delta \log n$, it is sufficient to choose a path of length $\delta \log n$ emanating from $\xi(\tau_i)$ which does not intersect G_i (it is possible by the construction of τ_i, since $\xi(\tau_i)$ cannot be completely surrounded by points of G_i), and such path will be open with probability $p^{\delta \log n}$. Thus, for any $\varepsilon > 0$ (one can take the same ε from (2.18)), we can choose δ small enough (take δ such that $\delta \log p^{-1} < \varepsilon$) so that

$$\mathbf{P}^0[\mathfrak{C}(\xi(\tau_i)) \geq \delta \log n \mid \mathcal{F}_i] \geq n^{-\varepsilon}, \tag{2.19}$$

where $\mathcal{F}_i$ is the σ-algebra generated by $\{\omega(x),\ x \in G_i\}$ and $\{\xi(m),\ m \leq \tau_i\}$. Fix $\theta > 0$ in such a way that $1 - \theta > \varepsilon$. Note that, as $p < p_{cr}$, using (2.1) and Borel-Cantelli lemma, for n large enough $\min\{k : \ \xi(\tau_k) \notin \Lambda_n\}$ (the number of times that we repeat the basic step in the above construction) will be of order at least $n^{1-\theta}$ for all n large enough, $\mathbf{P}^0$-a.s. (recall that $\Lambda_n = (-n/2, n/2]^d$; with overwhelming probability all the clusters inside Λ_n will be of sizes at most n^θ). On each step, by (2.19), with probability at least $n^{-\varepsilon}$ the random walk enters the cluster of size at least $\delta \log n$. By (2.18), it stays in that cluster (if β is large enough) for at least $n^{2+\varepsilon}$ time units with large probability. If $1 - \theta > \varepsilon$, with overwhelming probability on some step (of the above construction) the random walk will delay (in the corresponding cluster) for more than $n^{2+\varepsilon}$ time units before going out of Λ_n. In other words, we will have

$$\max_{s \leq n^{2+\varepsilon}} \|\xi(s)\| \leq \frac{dn}{2},$$

which implies (1.2). This concludes the proof of Theorem 1.1. □

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