2002-Fez conference on Partial Differential Equations,
Electronic Journal of Differential Equations, Conference 09, 2002, pp 65–76.
http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Stationary Solutions for a Schrödinger-Poisson System in $\mathbb{R}^3$ *

Khalid Benmlih

Abstract

Under appropriate, almost optimal, assumptions on the data we prove existence of standing wave solutions for a nonlinear Schrödinger equation in the entire space $\mathbb{R}^3$ when the real electric potential satisfies a linear Poisson equation.

1 Introduction

Consider the time-dependent system which couples the Schrödinger equation

$$i\partial_t u = -\frac{1}{2}\Delta u + (V + \widetilde{V})u \tag{1.1}$$

with initial value $u(x,0) = u(x)$, and the Poisson equation

$$-\Delta V = |u|^2 - n^*. \tag{1.2}$$

The dopant-density n^* and the effective potential $\widetilde{V}$ are given time-independent reals functions. There are many papers dealing with the physical problem modelled by this system from which we mention Markowich, Ringhofer & Schmeiser [8]; Illner, Kavian & Lange [3]; Nier [9]; Illner, Lange, Toomire & Zweifel [4], and references therein.

In this work we are mainly concerned with the proof of standing waves (actually ground states) of (1.1)–(1.2) in the entire space $\mathbb{R}^3$, i.e. solutions of the form

$$u(x,t) = e^{i\omega t}u(x)$$

with real number ω (frequency) and real wave function u. Hence we are interested in the stationary system

$$-\frac{1}{2}\Delta u + (V + \widetilde{V})u + \omega u = 0 \quad \text{in } \mathbb{R}^3 \tag{1.3}$$

$$-\Delta V = |u|^2 - n^* \quad \text{in } \mathbb{R}^3 \tag{1.4}$$

* *Mathematics Subject Classifications:* 35J50, 35Q40.
Key words: Schrödinger equation, Poisson equation, standing wave solutions, variational methods.

Published December 28, 2002.

under appropriate, almost optimal, assumptions on $\widetilde{V}$ and n^*. We suppose first that $\widetilde{V} \in L^1_{\text{loc}}(\mathbb{R}^3)$ and $n^* \in L^{6/5}(\mathbb{R}^3)$.

Let us remark that if V_0 is such that $-\Delta V_0 = -n^*$ then $(0, V_0)$ is a solution of the system (1.3)-(1.4). But here, we deal with solutions (u, V) in $H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that $u \not\equiv 0$.

F. Nier [9] has studied the system (1.3)-(1.4). He has showed the existence of a solution for small data i.e. when $\|\widetilde{V}\|_{L^2}$ and $\|n^*\|_{L^2}$ are small enough. Conversely to our approach here, he has began by solving (1.3) for a fixed V and investigate the Poisson equation then obtained.

In this paper we solve first explicitly the Poisson equation (1.4) for a fixed u in $H^1(\mathbb{R}^3)$. Next we substitute this solution $V = V(u)$ in the Schrödinger equation (1.3) and look into the solvability of

$$-\frac{1}{2}\Delta u + (V(u) + \widetilde{V})u + \omega u = 0 \quad \text{in } \mathbb{R}^3. \tag{1.5}$$

Using the explicit formula of $V(u)$, this equation appears as a *Hartree equation* studied by P.L. Lions [6] in the case where $n^* \equiv 0$ and $\widetilde{V}(x) := -2/|x|$. The fact that $\widetilde{V}$ in [6] converges to zero at infinity plays a crucial role to prove existence of solutions. However, in this paper we show that a slight modification of the arguments used in that paper allows us to prove existence of a ground state in the case $\widetilde{V}$ satisfying (1.7), (1.9) and n^* not necessarily zero (but satisfying (1.8) and (1.9) as below).

Before giving our hypotheses on $\widetilde{V}$ and n^* let us define a decomposition which will be useful in the sequel.

Definition 1.1 We say that g satisfies the decomposition (1.6) if:

(i) $g \in L^1_{\text{loc}}(\mathbb{R}^3)$,

(ii) $g \geq 0$, and

(iii) There exists $q_0 \in [3/2, \infty] : \forall\lambda > 0 \ \exists g_{1\lambda} \in L^{q_0}(\mathbb{R}^3), q_\lambda \in]3/2, \infty[$ and $g_{2\lambda} \in L^{q_\lambda}(\mathbb{R}^3)$ such that

$$g = g_{1\lambda} + g_{2\lambda} \quad \text{and} \quad \lim_{\lambda \to 0} \|g_{1\lambda}\|_{L^{q_0}} = 0. \tag{1.6}$$

For convenience, we use throughout this paper the following notations:

- $\|.\|$ denotes the norm $\|.\|_{L^2}$ on $L^2(\mathbb{R}^3)$,
- $\mathbb{I}_A$ denotes the characteristic function of the set $A \subset \mathbb{R}^3$,
- $[F \leq \lambda]$ denotes the set $\{x; F(x) \leq \lambda\}$ for a function F and $\lambda \in \mathbb{R}$.

Let us give now two examples of functions satisfying the conditions in Definition 1.1.

Example 1.2 The following two functions satisfy the decomposition (1.6):

(i) $g(x) := 1/|x|^\alpha$ for some $0 < \alpha < 2$.

(ii) $|g|$ where g is a function in $L^r(\mathbb{R}^3)$ for some $r > 3/2$.

Proof. To prove (i) we write, for $\lambda > 0$,

$$\frac{1}{|x|^\alpha} := \underbrace{\frac{1}{|x|^\alpha}\mathbb{I}_{[|x|>1/\lambda]}}_{g_{1\lambda}} + \underbrace{\frac{1}{|x|^\alpha}\mathbb{I}_{[|x|\le 1/\lambda]}}_{g_{2\lambda}}.$$

Elementary calculations give

$$\|g_{1\lambda}\|^{q_0}_{L^{q_0}} = \frac{4\pi}{\alpha q_0 - 3}(\lambda)^{\alpha q_0 - 3} \quad \text{and} \quad \|g_{2\lambda}\|^{q}_{L^{q}} = \frac{4\pi}{3-\alpha q}(\frac{1}{\lambda})^{3-\alpha q}.$$

Hence it suffices to choose any finite numbers q_0 , q such that $3/2 < q < 3/\alpha < q_0$.
To show (ii) write, as above,

$$|g| := \underbrace{|g|\mathbb{I}_{[|g|\le\lambda]}}_{g_{1\lambda}} + \underbrace{|g|\mathbb{I}_{[|g|>\lambda]}}_{g_{2\lambda}}.$$

It is clear that $\|g_{1\lambda}\|_{L^\infty} \le \lambda$ ($q_0 = \infty$) and $\|g_{2\lambda}\|_{L^r} \le \|g\|_{L^r}$ ($q_\lambda = r$). □

Hypotheses. In what follows we assume that

$$\widetilde{V}^+ \in L^1_{\text{loc}}(\mathbb{R}^3) \quad \text{and} \quad \widetilde{V}^- \text{ satisfies the decomposition (1.6)}, \tag{1.7}$$

where $\widetilde{V}^+(x) := \max(\widetilde{V}(x), 0)$ and $\widetilde{V}^-(x) := \max(-\widetilde{V}(x), 0)$. We suppose also that

$$n^* \in L^1 \cap L^{6/5}(\mathbb{R}^3) \tag{1.8}$$

and finally if we denote by

$$\varrho(x) := 2\widetilde{V}(x) - \frac{1}{2\pi}\int_{\mathbb{R}^3} \frac{n^*(y)}{|x-y|}\,dy$$

we assume that

$$\inf\Big\{\int_{\mathbb{R}^3} \big(|\nabla\varphi|^2 + \varrho(x)\varphi^2\big)dx, \int_{\mathbb{R}^3} |\varphi|^2 = 1\Big\} < 0. \tag{1.9}$$

Remark that in the case of [6] (where $n^* \equiv 0$ and $\widetilde{V}(x) := -2/|x|$), all the three hypotheses above are satisfied. Indeed, (1.7) and (1.8) follow from (i) of Example 1.2. Moreover, if we consider $\Phi(x) := e^{-2|x|}$ then it verifies

$$-\Delta\Phi - 4\frac{\Phi}{|x|} = -4\Phi,$$

and consequently

$$\inf\Big\{\int_{\mathbb{R}^3} |\nabla\varphi|^2 - 4\int_{\mathbb{R}^3} \frac{\varphi^2}{|x|}dx, \int_{\mathbb{R}^3} |\varphi|^2 = 1\Big\} < 0$$

i.e.(1.9) is satisfied also.

Our main result is the following. We prove that the Schrödinger–Poisson system (1.3)-(1.4) has a ground state, minimizing the energy functional corresponding to (1.5), given by (see Lemma 2.2):

$$E(\varphi) := \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} |\nabla V(\varphi)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \widetilde{V}\varphi^2 dx + \frac{\omega}{2}\int_{\mathbb{R}^3} \varphi^2 dx \quad (1.10)$$

Theorem 1.3 *Under the assumptions* (1.7)*,* (1.8)*, and* (1.9) *there exists* $\omega_* > 0$ *such that for all* $0 < \omega < \omega_*$ *the equation* (1.5) *has a nonnegative solution* $u \not\equiv 0$ *which minimizes the functional* E*:*

$$E(u) = \min_{\varphi \in H^1(\mathbb{R}^3)} E(\varphi).$$

The remainder of this paper is organized as follows: In section 2 we present some preliminary lemmas which will be useful in the sequel. In section 3, we conclude by proving our main result.

2 Preliminary results

In this section we present a few preliminary lemmas which shall be required in several proofs. Recall (cf. [7, Theorem I.1] or [10, p.151]) that $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ for the norm

$$\|\varphi\|_{\mathcal{D}^{1,2}} = \Big(\int_{\mathbb{R}^3} |\nabla\varphi|^2 \, dx\Big)^{1/2}.$$

By a Sobolev inequality, $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is continuously embedded in $L^6(\mathbb{R}^3)$, an equivalent characterization is

$$\mathcal{D}^{1,2}(\mathbb{R}^3) := \left\{\varphi \in L^6(\mathbb{R}^3); |\nabla\varphi| \in L^2(\mathbb{R}^3)\right\}.$$

For the solvability of the Poisson equation (1.3) we state the following lemma.

Lemma 2.1 *For all* $f \in L^{6/5}(\mathbb{R}^3)$*, the equation*

$$-\Delta W = f \quad \text{in } \mathbb{R}^3 \quad (2.1)$$

has a unique solution $W \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ *given by*

$$W(f)(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\, dy\,. \quad (2.2)$$

Proof. The existence and the uniqueness of the solution of (2.1) follow from corollary 3.1.4 of reference [5], by minimizing on $\mathcal{D}^{1,2}(\mathbb{R}^3)$ the functional

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx - \int_{\mathbb{R}^3} fv dx.$$

For this, using Hölder's and Sobolev's inequalities we check easily that J is coercive (that is $J(v_n) \to +\infty$ as $\|v_n\|_{\mathcal{D}^{1,2}} \to \infty$), strictly convex, lower semi-continuous and C^1 on $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Hence J attains its minimum at $W \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ which is the unique solution of (2.1).

By uniqueness, W is the Newtonian potential of f and has (cf. [1, p.235]) an explicit formula given by (2.2). Furthermore, multiplying (2.1) by W and integrating we obtain

$$\|\nabla W\|^2 = \int_{\mathbb{R}^3} f(x)W(x)dx.$$

After using Hölder and Sobolev inequalities we get

$$\|\nabla W\| \ \leq \ S_*^{1/2}\|f\|_{L^{6/5}} \tag{2.3}$$

where S_* is the best Sobolev constant in

$$\|v\|^2_{L^6(\mathbb{R}^3)} \leq S_*\|\nabla v\|^2_{L^2(\mathbb{R}^3)}. \tag{2.4}$$

Hence the linear mapping $f \mapsto W$ is continuous from $L^{6/5}(\mathbb{R}^3)$ into $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

□

Now in order to find a solution of equation (1.5), we are going to show that the operator

$$v \mapsto -\frac{1}{2}\Delta v + (W(|v|^2 - n^*) + \widetilde{V})v + \omega v$$

is the derivative of a functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ and hence equation (1.5) has a variational structure. To this end, we have the following lemma (see also [3])

Lemma 2.2 *Let $n^* \in L^{6/5}(\mathbb{R}^3)$. For $\varphi \in H^1(\mathbb{R}^3)$ we denote by $V(\varphi) := W(|\varphi|^2 - n^*)$ the unique solution of* (2.1) *when $f := |\varphi|^2 - n^*$. Define*

$$I(\varphi) := \frac{1}{4}\int_{\mathbb{R}^3} |\nabla V(\varphi)|^2 dx.$$

Then I is C^1 on $H^1(\mathbb{R}^3)$ and its derivative is given by

$$\langle I'(\varphi), \psi\rangle = \int_{\mathbb{R}^3} V(\varphi)\varphi\psi dx \quad \forall \psi \in H^1(\mathbb{R}^3). \tag{2.5}$$

Proof. Note that if $\varphi \in H^1(\mathbb{R}^3)$ then, by interpolation, $|\varphi|^2 \in L^{6/5}(\mathbb{R}^3)$. So taking $f = |\varphi|^2 - n^*$ and multiplying the equation (2.1) by $V(\varphi) := W(|\varphi|^2 - n^*)$ we deduce that $\|\nabla V(\varphi)\|^2 = \int f(x)V(\varphi)(x)dx$, and hence in view of (2.2) we get

$$I(\varphi) = \frac{1}{16\pi}\int\int \frac{(|\varphi|^2 - n^*)(x)(|\varphi|^2 - n^*)(y)}{|x-y|}\,dx\,dy. \tag{2.6}$$

Using this expression, we show easily that (2.5) holds for the Gâteaux differential of I i.e. for all $\varphi,\ \psi \in H^1(\mathbb{R}^3)$

$$\lim_{t\to 0^+}\frac{I(\varphi + t\psi) - I(\varphi)}{t} = \int_{\mathbb{R}^3} V(\varphi)\varphi\psi\,dx,$$

and that the mapping $\varphi \mapsto \varphi V(\varphi)$ is continuous on $H^1(\mathbb{R}^3)$. Thus I is Frechet differentiable and C^1 on $H^1(\mathbb{R}^3)$ and its derivative satisfies (2.5). □

At certain steps of our proof of Theorem 1.3, we need some estimates for which we will use the next inequalities.

Lemma 2.3 *(i) If $\theta \in L^r(\mathbb{R}^3)$ for some $r \geq 3/2$ then $\forall\delta > 0, \exists C_\delta > 0$ such that*

$$\int_{\mathbb{R}^3}\theta(x)|\varphi(x)|^2dx \leq \delta\|\nabla\varphi\|^2 + C_\delta\|\varphi\|^2 \quad \forall\varphi \in H^1(\mathbb{R}^3) \tag{2.7}$$

(ii) For all $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ and $y \in \mathbb{R}^3$ one has

$$\int_{\mathbb{R}^3}\frac{|\varphi(x)|^2}{|x-y|^2}dx \leq 4\|\nabla\varphi\|^2 \tag{2.8}$$

(iii) For any $\delta > 0$ and all $y \in \mathbb{R}^3$

$$\int_{\mathbb{R}^3}\frac{|\varphi(x)|^2}{|x-y|}dx \leq \delta\|\nabla\varphi\|^2 + \frac{4}{\delta}\|\varphi\|^2 \quad \forall\varphi \in H^1(\mathbb{R}^3) \tag{2.9}$$

Proof. In order to prove (i) we show first that (2.7) holds for any $\theta \in L^\infty + L^{3/2}$ and conclude since $L^r(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3)$ for all $r \geq 3/2$. Let $\theta = \theta_1 + \theta_2$ with $\theta_1 \in L^\infty$ and $\theta_2 \in L^{3/2}$. Then for each $\lambda > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^3}\theta(x)|\varphi(x)|^2dx &\leq \|\theta_1\|_{L^\infty}\|\varphi\|^2 + \lambda\int_{[|\theta_2|\leq\lambda]}|\varphi|^2dx + \int_{[|\theta_2|>\lambda]}|\theta_2||\varphi|^2dx \\ &\leq (\|\theta_1\|_{L^\infty} + \lambda)\,\|\varphi\|^2 + \|\theta_2\|_{L^{3/2}([|\theta_2|>\lambda])}\|\varphi\|^2_{L^6} \\ &\leq (\|\theta_1\|_{L^\infty} + \lambda)\,\|\varphi\|^2 + S_*\|\theta_2^\lambda\|_{L^{3/2}}\|\nabla\varphi\|^2\end{aligned}$$

where S_* is the best Sobolev constant in (2.4) and θ_2^λ denotes $\theta_2^\lambda := \theta_2\mathbb{I}_{[|\theta_2|>\lambda]}$. It is clear that $|\theta_2^\lambda| \leq |\theta_2|$ for all $\lambda > 0$ and that $\theta_2^\lambda \to 0$ pointwise a.e. when $\lambda \to +\infty$. Since $\theta_2 \in L^{3/2}$ then by Lebesgue convergence theorem we infer that $\|\theta_2^\lambda\|_{L^{3/2}}$ converges to zero. Hence for any $\delta > 0$ there exists $K_\delta > 0$ such that if $\lambda \geq K_\delta$ one has $S_*\|\theta_2^\lambda\|_{L^{3/2}} \leq \delta$. Choosing $C_\delta := \|\theta_1\|_{L^\infty} + K_\delta$ we deduce that (2.7) holds for all $\theta \in L^\infty(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3)$.

Regarding (ii), (2.8) is the classical Hardy inequality (see [2]).
Finally, to show (iii) for all $\delta > 0$ and any $y \in \mathbb{R}$, we write

$$\int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x-y|} dx = \int_{|x-y|<\frac{\delta}{4}} \frac{|\varphi(x)|^2}{|x-y|^2} |x-y| dx + \int_{|x-y|\geq\frac{\delta}{4}} \frac{|\varphi(x)|^2}{|x-y|} dx$$
$$\leq \frac{\delta}{4} \int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x-y|^2} dx + \frac{4}{\delta} \int_{\mathbb{R}^3} |\varphi(x)|^2 dx$$

and (2.9) holds by using Hardy inequality (2.8). □

Remark 2.4 Note that $\widetilde{V}^-$ satisfies the inequality (2.7) i.e. $\forall \delta > 0 \exists C_\delta > 0$ such that

$$\int_{\mathbb{R}^3} \widetilde{V}^-(x)|\varphi(x)|^2 dx \leq \delta \|\nabla \varphi\|^2 + C_\delta \|\varphi\|^2 \quad \forall \varphi \in H^1(\mathbb{R}^3). \tag{2.10}$$

Indeed, by (1.7) $\widetilde{V}^-$ satisfies the decomposition (1.6). Then for a fixed $\lambda > 0$ we have

$$\widetilde{V}^- = \widetilde{V}^-_{1\lambda} + \widetilde{V}^-_{2\lambda}$$

where for $i = 1, 2$, $\widetilde{V}^-_{i\lambda} \in L^s(\mathbb{R}^3)$ for some $s \in [3/2, \infty]$ ($s = q_0$ or $s = q_\lambda$). Hence by Lemma 2.3 each $\widetilde{V}^-_{i\lambda}$ satisfies the inequality (2.7) and consequently $\widetilde{V}^-$ also.

To finish this section we state the following convergence Lemma.

Lemma 2.5 *Let $\psi \in L^r(\mathbb{R}^3)$ for some $r > 3/2$. If $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$ then*

$$\int_{\mathbb{R}^3} \psi(x) v_n^2(x) dx \to 0 \quad as \quad n \to +\infty$$

Proof. Consider the subset of $\mathbb{R}^3$, $A_\lambda := [|\psi| > \lambda]$ and a compact subset K of A_λ suitably chosen later. We write

$$\int_{\mathbb{R}^3} |\psi|(x) v_n^2(x) dx = \int_{\mathbb{R}^3 - A_\lambda} |\psi| v_n^2 dx + \int_{A_\lambda - K} |\psi| v_n^2 dx + \int_K |\psi| v_n^2\, dx$$
$$\leq \lambda \|v_n\|^2 + \|\psi\|_{L^r(A_\lambda - K)} \|v_n\|^2_{L^{2r'}(\mathbb{R}^3)} + \|\psi\|_{L^r(\mathbb{R}^3)} \|v_n\|^2_{L^{2r'}(K)}$$
$$\leq \lambda C_0 + C_1 \|\psi\|_{L^r(A_\lambda - K)} + \|\psi\|_{L^r(K)} \|v_n\|^2_{L^{2r'}(K)}$$

where $\frac{1}{r'} + \frac{1}{r} = 1$. In the last inequality we used that $(v_n)_n$ is bounded in $H^1(\mathbb{R}^3)$ (note that $2 < 2r' < 6$). For a given arbitrary $\delta > 0$, we fix first λ such that $\lambda C_0 \leq \dfrac{\delta}{3}$. Next we choose a compact subset $K \subset A_\lambda$ such that

$$C_1 \|\psi\|_{L^r(A_\lambda - K)} \leq \frac{\delta}{3}$$

and finally since $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ and $2 < 2r' < 6$ then up a subsequence $\|v_n\|^2_{L^{2r'}(K)}$ converges to 0 and therefore there exists $N_\delta \in \mathbb{N}$ such that for all $n \geq N_\delta$ we get

$$\|\psi\|_{L^r(K)}\|v_n\|^2_{L^{2r'}(K)} \leq \frac{\delta}{3}$$

which completes the proof. □

3 Proof of Theorem 1.3

Now we are in position to prove our main result. To this end, we shall minimize the energy functional

$$E(\varphi) := \frac{1}{4}\int |\nabla\varphi|^2 dx + I(\varphi) + \frac{1}{2}\int \widetilde{V}\varphi^2 dx + \frac{\omega}{2}\int \varphi^2 dx$$

whose critical points correspond, on account of Lemma 2.2, to solutions of (1.5). Using (2.6), we may decompose $E(\varphi)$ as

$$E(\varphi) = E_1(\varphi) - E_2(\varphi) + E_3(\varphi) + E(0) \tag{3.1}$$

where

$$\begin{aligned}
E_1(\varphi) :=& \frac{1}{4}\int |\nabla\varphi|^2\, dx + \frac{1}{2}\int \widetilde{V}^+\varphi^2 dx + \frac{\omega}{2}\int \varphi^2\, dx \\
E_2(\varphi) :=& \frac{1}{2}\int \widetilde{V}^-\varphi^2\, dx + \frac{1}{8\pi}\int\int \frac{n^*(y)}{|x-y|}\varphi^2(x)\, dx\, dy \\
E_3(\varphi) :=& \frac{1}{16\pi}\int\int \frac{\varphi^2(x)\varphi^2(y)}{|x-y|}\, dx\, dy \\
E(0) :=& \frac{1}{16\pi}\int\int \frac{n^*(x)n^*(y)}{|x-y|}\, dx\, dy.
\end{aligned}$$

The proof of Theorem 1.3 is divided into the four following Lemmas:

Lemma 3.1 *Let $\omega > 0$ and $c \in \mathbb{R}$. If the set $[E \leq c]$ is bounded in $L^2(\mathbb{R}^3)$ then it is also bounded in $H^1(\mathbb{R}^3)$.*

Proof. By the expression (3.1), $E(\varphi) \leq c$ implies in particular

$$\frac{1}{4}\|\nabla\varphi\|^2 - E_2(\varphi) \leq c_0 \tag{3.2}$$

where $c_0 := c - E(0)$ and since the other terms are nonnegative. To estimate $E_2(\varphi)$ we use (2.9) which gives for any $\delta > 0$,

$$\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{n^*(y)}{|x-y|}\varphi^2(x)dxdy \leq \left(\delta\|\nabla\varphi\|^2 + \frac{4}{\delta}\|\varphi\|^2\right)\|n^*\|_{L^1}.$$

Using this inequality, Remark 2.4 and choosing δ such that $\delta\big(\frac{1}{2}+\frac{\|n^*\|_{L^1}}{8\pi}\big)<\frac{1}{8}$ we obtain

$$E_2(\varphi)\le\frac{1}{8}\|\nabla\varphi\|^2+K_0\|\varphi\|^2 \tag{3.3}$$

where K_0 is a positive constant. In Consequence (3.2) gives

$$\frac{1}{8}\|\nabla\varphi\|^2\le K_0\|\varphi\|^2+c_0.$$

Lemma 3.2 *For all $\omega>0$ and $c\in\mathbb{R}$ the set $[E\le c]$ is bounded in $L^2(\mathbb{R}^3)$.*

Proof. Assume by contradiction that there exists a sequence $(u_j)_j\subset H^1(\mathbb{R}^3)$ such that $E(u_j)\le c$ and $\|u_j\|\to+\infty$. Let $v_j:=u_j/\|u_j\|$ then $\|v_j\|=1$ and from $E(u_j)\le c$ we get

$$\frac{1}{4}\int|\nabla v_j|^2dx-E_2(v_j)+E_3(v_j)\|u_j\|^2+\frac{\omega}{2}\le\frac{c_0}{\|u_j\|^2}. \tag{3.4}$$

By using the estimate (3.3) for $\varphi:=v_j$ we obtain

$$\frac{1}{8}\|\nabla v_j\|^2+E_3(v_j)\|u_j\|^2+\frac{\omega}{2}\le\frac{c_0}{\|u_j\|^2}+K_0. \tag{3.5}$$

Since ω and $E_3(v_j)$ are nonnegative, this inequality implies that $(v_j)_j$ is bounded in $H^1(\mathbb{R}^3)$ and that $E_3(v_j)\|u_j\|^2$ is also bounded; i.e.

$$\Big(\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{v_j^2(x)v_j^2(y)}{|x-y|}dxdy\Big)\|u_j\|^2\le c_1.$$

Let then $v\in H^1(\mathbb{R}^3)$ be such that for a subsequence of v_j, noted again v_j, we have $v_j\rightharpoonup v$ weakly in $H^1(\mathbb{R}^3)$, $v_j\to v$ pointwise almost everywhere and v_j^2 converging to v^2 strongly in $L^p_{\text{loc}}(\mathbb{R}^3)$ for any $1\le p<3$. By Fatou's Lemma we deduce that

$$\begin{aligned}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{v^2(x)v^2(y)}{|x-y|}dxdy &\le\liminf_{j\to+\infty}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{v_j^2(x)v_j^2(y)}{|x-y|}\,dx\,dy\\ &\le\liminf_{j\to+\infty}\frac{c_1}{\|u_j\|^2}=0\end{aligned}$$

and therefore $v\equiv0$. On the other hand, it follows from (3.4) that

$$\frac{\omega}{2}-E_2(v_j)\le\frac{c_0}{\|u_j\|^2}. \tag{3.6}$$

Set

$$h(x):=\widetilde{V}^-(x)+V^*(x) \tag{3.7}$$

where $V^*(x):=\frac{1}{4\pi}\int\frac{n^*(y)}{|x-y|}dy$ is the Newtonian potential of n^* given by Lemma 2.1 . Then (3.6) is equivalent to

$$\omega-\int_{\mathbb{R}^3}h(x)v_j^2(x)dx\le\frac{2c_0}{\|u_j\|^2}\,. \tag{3.8}$$

Using successively the hypothesis (1.7) and Lemma 2.5 we may show that

$$\int_{\mathbb{R}^3} h(x)v_j^2(x)dx \to 0 \quad \text{as } j \to +\infty. \tag{3.9}$$

Passing to the limit in (3.8) we infer that $\omega \leq 0$ which is a contradiction. In conclusion, any $(u_j)_j \subset H^1(\mathbb{R}^3)$ such that $E(u_j) \leq c$ is bounded in $L^2(\mathbb{R}^3)$. □

Lemma 3.3 *For any $\omega > 0$ the functional E is weakly lower semi-continuous on $H^1(\mathbb{R}^3)$ and attains its minimum on $H^1(\mathbb{R}^3)$ at $u \geq 0$.*

Proof. First, to show that the functional E is weakly lower semi-continuous, remark that in the expression (3.1) the term E_1 and E_3 are continuous and convex (therefore weakly lower semi-continuous). Then we just have to prove that $u \mapsto \int_{\mathbb{R}^3} h(x)u^2(x)dx$ is weakly sequentially continuous on $H^1(\mathbb{R}^3)$ where h is defined by (3.7). Consider $u_j \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$ and write

$$\int h(x)u_j^2(x)dx = \int h(x)(u_j - u)^2 dx + 2\int h(x)u(u_j - u)dx + \int h(x)u^2 dx.$$

Taking $(u_j - u)$ instead of v_j in (3.9) we infer that

$$\int_{\mathbb{R}^3} h(x)(u_j - u)^2 dx \to 0 \quad \text{as } j \to \infty.$$

Moreover, similarly to the proof of (3.9) we show that

$$\int_{\mathbb{R}^3} h(x)u(u_j - u)dx \to 0 \quad \text{as } j \to \infty,$$

and consequently

$$\int_{\mathbb{R}^3} h(x)u_j^2(x)dx \to \int_{\mathbb{R}^3} h(x)u^2(x)dx \quad \text{as} \quad j \to \infty.$$

This means that $u \mapsto \int_{\mathbb{R}^3} h(x)u^2(x)dx$ is weakly sequentially continuous on $H^1(\mathbb{R}^3)$ and therefore E is weakly lower semi-continuous on $H^1(\mathbb{R}^3)$.

Next, if we denote $\mu := \inf\left\{E(\varphi); \varphi \in H^1(\mathbb{R}^3)\right\}$ and $(u_n)_n \subset H^1(\mathbb{R}^3)$ a minimizing sequence then by Lemmas 3.1 and 3.2, $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$ and therefore there exists $u \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$. The functional E being weakly lower semi-continuous on $H^1(\mathbb{R}^3)$ we have

$$E(u) \leq \liminf_{n \to +\infty} E(u_n) = \mu$$

and consequently $E(u) = \mu$. Since E is C^1 on $H^1(\mathbb{R}^3)$ then $E^{'}(u) = 0$ and in view of Lemma 2.2, u is a solution of the equation (1.5).

Let us remark finally that by a simple inspection we have $E(|u|) \leq E(u)$ and therefore we may assume that $u \geq 0$. □

Lemma 3.4 *There exists $\omega_* > 0$ such that if $0 < \omega < \omega_*$ then $E(u) < E(0)$ and thus $u \not\equiv 0$.*

Proof. Assuming (1.9), there exist $\mu_1 < 0$ and $\varphi_1 \in H^1(\mathbb{R}^3)$ such that $\int |\varphi_1|^2 = 1$ and

$$\int_{\mathbb{R}^3} |\nabla\varphi_1|^2 dx + \int_{\mathbb{R}^3} \varrho(x)\varphi_1^2(x)dx < \mu_1.$$

From (3.1) we observe that

$$\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx + \int_{\mathbb{R}^3} \varrho(x)\varphi^2(x)dx = 4E_1(\varphi) - 4E_2(\varphi) - 2\omega \int_{\mathbb{R}^3} \varphi^2(x)dx.$$

Then the last inequality gives

$$E_1(\varphi) - E_2(\varphi) - \frac{\omega}{2} < \frac{\mu_1}{4}.$$

Now, for $t > 0$ and using again (3.1) we compute easily

$$\begin{aligned} E(t\varphi_1) - E(0) &= t^2 E_1(\varphi_1) - t^2 E_2(\varphi_1) + t^4 E_3(\varphi_1) \\ &< \frac{t^2}{4}\left[(\mu_1 + 2\omega) + 4t^2 E_3(\varphi_1)\right]. \end{aligned}$$

Hence, if $(\mu_1 + 2\omega) < 0$ there exists $t_* > 0$ small enough such that for all $0 < t \le t_*$,

$$(\mu_1 + 2\omega) + 4t^2 E_3(\varphi_1) < 0\,.$$

In other words, setting $\omega_* := -\mu_1/2$ then if $0 < \omega < \omega_*$ we have $E(t\varphi_1) < E(0)$ for $0 < t \le t_*$. Since $E(u) := \inf\{E(\varphi); \varphi \in H^1(\mathbb{R}^3)\}$, this implies that $E(u) < E(0)$ and consequently $u \not\equiv 0$. The proof of Theorem 1.3 is thus complete. □

Remark 3.5 If n^* is nonnegative then we may replace the assumption (1.9) by the next one

$$\inf\left\{\int |\nabla\varphi|^2 dx + 2\int \widetilde{V}(x)\varphi^2 dx; \int |\varphi|^2 = 1\right\} < 0$$

which does not depend on n^* and implies obviously (1.9).

Acknowledgments. The author acknowledges the hospitality of Laboratoire de Mathématiques Appliquées de Versailles (France) where a part of this work was done. He is grateful to Otared Kavian for very valuable discussions and suggestions.

References

[1] D. Gilbard & N.S. Trudinger: *Elliptic Partial Differential Equations of Second Order*; 2nd edition, Springer, Berlin 1983.

[2] G.H. Hardy, J.E. Littlewood & G. Pólya : *Inequalities*; Cambridge University Press, London 1952.

[3] R. Illner, O. Kavian & H. Lange: Stationary solutions of quasi-linear Schrödinger–Poisson systems, J. Differential equations 145(1998), 1-16.

[4] R. Illner, H. Lange, B. Toomire & P. Zweifel: On quasi-linear Schrödinger–Poisson systems; Math. Meth. Appl. Sci., 20 (1997), 1223-1238.

[5] O. Kavian: *Introduction à la Théorie des Points Critiques.* Springer-Verlag, Berlin, 1993.

[6] P. L. Lions: Some remarks on Hartree equation; Nonlinear Anal., Theory Methods Appl. 5 (1981), 1245-1256.

[7] P. L. Lions: The concentration-compactness principle in the calculus of variations, the limit case, Part 1; Revista Mathematica Iberoamericana, 1 (1985), 145-201.

[8] P. A. Markowich, C. Ringhofer & C. Schmeiser: *Semiconductor Equations* ; Springer, Wien 1990.

[9] F. Nier: Schrödinger–Poisson systems in dimension $d \leq 3$, the whole space case; Proceedings of the Royal Society of Edinburgh, 123A (1993), 1179-1201.

[10] M. Struwe: *Variational methods, Application to nonlinear PDE & Hamiltonian systems*; 2nd edition, Springer, Berlin 1996.

Khalid Benmlih
Department of Economic Sciences, University of Fez
P.O. Box 42A, Fez, Morocco.
E-mail: kbenmlih@hotmail.com