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Pseudo-monotonicity and degenerate elliptic operators of second order *

Youssef Akdim & Elhoussine Azroul

Abstract

Extending the theory of pseudo-monotone mappings in weighted Sobolev spaces, we prove some existence results for degenerate or singular elliptic equations generated by the second-order differential operator

$$Au(x) = -\operatorname{div} a(x, u, \nabla u)) + a_0(x, u, \nabla u),$$

(in particular, when only large monotonicity is satisfied)

1 Introduction

Let Ω be a open subset of $\mathbb{R}^N$ $(N \geq 1)$ and $p > 1$ be a real number and $\omega = \{\omega_0, \omega_1, \dots, \omega_N\}$ be a collection of weight functions on Ω, i.e, each ω_i is a measurable and positive almost everywhere in Ω, and satisfying some integrability condition (see section 2 below).

Let us consider the second-order differential operator

$$Au(x) = A_1u(x) + A_0u(x) \tag{1.1}$$

where

$$A_1u(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) \tag{1.2}$$

is the top order part of A and where

$$A_0u(x) = a_0(x, u, \nabla u) \tag{1.3}$$

is the lower order part of A and where $\{a_i(x, \eta, \zeta),\ 0 \leq i \leq N\}$ are functions defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ and satisfy a suitable regularity and growth assumptions.

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Our objective in this paper, is to extend the theory of pseudo-monotone mappings in weighted Sobolev spaces. It's well known that, the essential condition which allows to do this, is the so-called Leray-Lions condition,

$$\sum_{i=1}^{N}(a_i(x,\eta,\zeta)-a_i(x,\eta,\bar{\zeta}))(\zeta_i-\bar{\zeta}_i)>0, \tag{1.4}$$

for a.e. $x\in\Omega$, all $\eta\in\mathbb{R}$ and all $\zeta\neq\bar{\zeta}\in\mathbb{R}^N$ (resp. the so-called weak Leray-Lions condition,

$$\sum_{i=1}^{N}(a_i(x,\eta,\zeta)-a_i(x,\eta,\bar{\zeta}))(\zeta_i-\bar{\zeta}_i)\geq 0, \tag{1.5}$$

for a.e. $x\in\Omega$, all $(\eta,\zeta,\bar{\zeta})\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$). Let us state the following assumptions:

(H1) The expression

$$|||u|||_X=\Big(\sum_{i=1}^{N}\int_\Omega|\frac{\partial u(x)}{\partial x_i}|^p\omega_i(x)\,dx\Big)^{1/p}$$

is a norm on $X=W_0^{1,p}(\Omega,\omega)$ equivalent to the usual norm (2.3)(see section 2). There exist a weight function $\bar{\omega}$ on Ω and a parameter q, $1<q<\infty$, such that the (Hardy) inequality

$$\Big(\int_\Omega|u(x)|^q\bar{\omega}(x)\Big)^{1/q}\leq c\Big(\sum_{i=1}^{N}\int_\Omega|\frac{\partial u(x)}{\partial x_i}|^p\omega_i(x)\,dx\Big)^{1/p} \tag{1.6}$$

holds for every $u\in W_0^{1,p}(\Omega,\omega)$ with a constant $c>0$ independent of u, and moreover, the imbedding expressed by (1.6) is compact, i.e.

$$W_0^{1,p}(\Omega,\omega)\hookrightarrow\hookrightarrow L^q(\Omega,\bar{\omega}). \tag{1.7}$$

(H2) Each $a_i(x,\eta,\zeta)$ $(1\leq i\leq N)$ is a Carathéodory function and

$$|a_i(x,\eta,\zeta)|\leq C_i\omega_i^{1/p}(x)[g_i(x)+\bar{\omega}^{\frac{1}{p'}}|\eta|^{q/p'}+\sum_{j=1}^{N}\omega_j^{1/p'}(x)|\zeta_j|^{p-1}], \tag{1.8}$$

for a.e. $x\in\Omega$, some constants $C_i>0$, some functions $g_i(x)\in L^{p'}(\Omega)$, all $(\eta,\zeta)\in\mathbb{R}^{N+1}$ and all $i=1,\dots,N$.

Recently, Drabek, Kufner and Mustonen [2] proved that the mapping T_1 defined from X to its dual X^* associated to the top order part A_1 is pseudo-monotone in X, under the weak conditions (1.5), (H1), (H2). Hence, the authors obtained the existence result for the Dirichlet problem associated to the $A_1u=f\in X^*$ by assuming some degeneracy.

Our first purpose in this paper, is to extend the previous result [2] in the operator A from (1.1) where the lower order part A_0 is affine with respect to the gradient, i.e., A_0 is of the form

$$A_0u(x) = c_0(x, u(x)) + \sum_{i=1}^{N} c_i(x, u(x))\frac{\partial u(x)}{\partial x_i}, \tag{1.9}$$

where $c_i(x, \eta)$, $0 \leq i \leq N$ are some Carathéodory functions defined on $\Omega \times \mathbb{R}$ and satisfy

$$\begin{aligned} |c_0(x, \eta)| &\leq C_0\bar{\omega}^{1/q}(x)[g_0(x) + \bar{\omega}^{\frac{1}{q'}}(x)|\eta|^{\frac{q}{q'}}] \\ |c_i(x, \eta)| &\leq C_i\omega_i^{1/p}(x)\bar{\omega}^{1/q}(x)[\gamma_i(x) + \bar{\omega}^{\frac{1}{r}}(x)|\eta|^{\frac{q}{r}}] \quad \text{for all } i = 1, \dots, N, \end{aligned} \tag{1.10}$$

for a.e. $x \in \Omega$, some constants $C_0 > 0,\ C_i > 0$, some functions $g_0 \in L^{q'}(\Omega)$ and $\gamma_i(x) \in L^r(\Omega)$ with

$$\frac{1}{r} + \frac{1}{p} + \frac{1}{q} < 1 \tag{1.11}$$

and where $\bar{\omega}(x)$ and q are from (1.6). More precisely, we prove the following theorem,

Theorem 1.1 *Assume that (H1), (H2), (1.10), (1.5) hold. Then the mapping T associated to the operator A from (1.1) and (1.9) is pseudo-monotone in X.*

Remark 1.2 *Theorem 1.1 is obviously a consequence of the more general result (Theorem 3.1, it suffices to take $I = \emptyset$).*

Remark 1.3 *About the existence of such r satisfying (1.11) see Remarks 2.1 and 4.2 below.*

The second aim of this paper, is to prove the same result of the preceding without restriction on A_0 and where (1.4) is applied. This is done in Theorem 3.1, if we take $I^c = \emptyset$.

This paper is divided into four sections. In section 2, we start our basic assumptions and we prove some preliminaries lemmas concerning some convergence and generalized Hölder's inequality in weighted Sobolev space. In section 3, we give our general main result and its proof and we study an example which illustrate our abstract hypotheses. The section 4, is devoted to the study of some particular case where $\omega_0 \equiv 1$ on Ω and where some of our hypotheses (imbedding) are satisfied.

In our work, we shall adopt many ideas from [5] (where the authors have studied the non-degenerated elliptic case). But the results are generalized and improved. concerning the existence results for higher order nonlinear degenerated (or singular) elliptic equations, we refer the reader to [3, 4, 1] (where the degree theory is used in the two first papers and where the pseudo-monotonicity is used in the last but under some restrictions on the weighted). Finally, not that our approach based on the theory of pseudo-monotone mappings can be applied in the case of non reflexive Banach space, for example in weighted Orlicz-Sobolev spaces (see [1] for related topics).

2 Preliminaries and basic assumptions

1) Weighted Sobolev spaces. Let Ω be a open subset of $\mathbb{R}^N$ $(N \geq 1)$, with finite measure, let $1 < p < \infty$, and let $\omega = \{\omega_i(x)\ 0 \leq i \leq N\}$ be a vector of weight functions, i.e. every component $\omega_i(x)$ is a measurable function which is positive a.e. in Ω. Further, we suppose that

$$\omega_i \in L^1_{\text{loc}}(\Omega) \tag{2.1}$$

and

$$\omega_i^{-\frac{1}{p-1}} \in L^1_{\text{loc}}(\Omega) \tag{2.2}$$

for any $0 \leq i \leq N$ hold in all our considerations.

Now, we denote by $W^{1,p}(\Omega, \omega)$ the space of all real-valued functions $u \in L^p(\Omega, \omega_0)$ such that the derivatives in the sense of distributions fulfil

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, \omega_i) \quad \text{for all } i = 1, \ldots, N,$$

which is a Banach space under the norm,

$$\|u\|_{1,p,\omega} = \Big(\int_\Omega |u(x)|^p \omega_0(x)\, dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\, dx \Big)^{1/p}. \tag{2.3}$$

The condition (2.1) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{1,p}(\Omega, \omega)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega, \omega)$ of $W^{1,p}(\Omega, \omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Moreover, the condition (2.2) implies that $W^{1,p}(\Omega, \omega)$ as well as $W_0^{1,p}(\Omega, \omega)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega, \omega)$ is equivalent to $W^{-1,p'}(\Omega, \omega^*)$, where $\omega^* = \{\omega_i^* = \omega_i^{1-p'}\ \forall i = 0, \ldots, N\}$, with $p' = \frac{p}{p-1}$. We shall suppose that the expression

$$|||u|||_{1,p,\omega} = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\, dx \Big)^{1/p}$$

is a norm defined on $W_0^{1,p}(\Omega, \omega)$ and it's equivalent to the norm (2.3). The reader can find conditions on the weight ω which guarantee this fact in [3]. Notice that $(X, |||.|||_X)$ is a uniformly convex (and thus reflexive) Banach space.

2) Basic assumptions. Let I be a subset of $\{1, 2, \ldots, N\}$ and I^c its complement, and let introduce the following modified versions of (1.4) and (1.5),

$$\sum_{i \in I} (b_i(x, \eta, \zeta_I) - b_i(x, \eta, \bar{\zeta}_I))(\zeta_i - \bar{\zeta}_i) > 0, \tag{2.4}$$

for a.e. $x \in \Omega$, all $\eta \in \mathbb{R}$ and all $\zeta \neq \bar{\zeta} \in \mathbb{R}^N$ and

$$\sum_{i \in I^c} (b_i(x, \eta, \zeta_{I^c}) - b_i(x, \eta, \bar{\zeta}_{I^c}))(\zeta_i - \bar{\zeta}_i) \geq 0, \tag{2.5}$$

for a.e. $x \in \Omega$, all $\eta \in \mathbb{R}$ and all $\zeta, \bar{\zeta} \in \mathbb{R}^N$ where ζ_J denoted $\zeta_J = \{\zeta_i, \quad i \in J\}$ and where $a_i(x, \eta, \zeta)$ are Carathéodory functions such that,

$$\begin{aligned} a_i(x,\eta,\zeta) &= b_i(x,\eta,\zeta_I) \quad \text{for all } i \in I, \\ a_i(x,\eta,\zeta) &= b_i(x,\eta,\zeta_{I^c}) \quad \text{for all } i \in I^c, \\ a_0(x,\eta,\zeta) &= c_0(x,\eta,\zeta_I) + \sum_{i\in I^c} c_i(x,\eta,\zeta_I)\zeta_i, \end{aligned} \tag{2.6}$$

for a.e. $x \in \Omega$, all $(\eta, \zeta) \in \mathbb{R}^{N+1}$ and where b_i $(i = 1, \ldots, N)$, c_0 and c_i $(i \in I^c)$ are functions satisfying the Carathéodory conditions (i.e. measurable in x for any fixed $\xi = (\eta, \zeta) \in \mathbb{R}^{N+1}$ and continuous in ξ for almost all fixed $x \in \Omega$).

We assume the following growth conditions:

(H2') Each $a_i(x, \eta, \zeta)$ is a Carathéodory function and, that there exists some positives constants C_i, and some functions $g_i(x) \in L^{p'}(\Omega)$ $i = 1, \ldots, N$, and $g_0 \in L^{q'}(\Omega)$ and some $\gamma_i(x) \in L^r(\Omega)$ for all $i \in I^c$) such that

$$|b_i(x,\eta,\zeta_I)| \le C_i \omega_i^{1/p}(x)[g_i(x) + \bar{\omega}^{\frac{1}{p'}}|\eta|^{\frac{q}{p'}} + \sum_{j\in I} \omega_j^{\frac{1}{p'}}(x)|\zeta_j|^{p-1}] \quad \text{for } i \in I$$

$$|b_i(x,\eta,\zeta_{I^c})| \le C_i \omega_i^{1/p}(x)[g_i(x) + \bar{\omega}^{\frac{1}{p'}}(x)|\eta|^{\frac{q}{p'}} + \sum_{j\in I^c} \omega_j^{\frac{1}{p'}}(x)|\zeta_j|^{p-1}],$$
$$\text{for } i \in I^c$$

$$|c_0(x,\eta,\zeta_I)| \le C_0 \bar{\omega}^{1/q}[g_0(x) + \bar{\omega}^{\frac{1}{q'}}(x)|\eta|^{\frac{q}{q'}} + \sum_{j\in I} \omega_j^{\frac{1}{q'}}(x)|\zeta_j|^{\frac{p}{q'}}]$$

$$|c_i(x,\eta,\zeta_I)| \le C_i \omega_i^{1/p}(x)\bar{\omega}^{1/q}(x)[\gamma_i(x) + \bar{\omega}^{\frac{1}{r}}(x)|\eta|^{\frac{q}{r}} + \sum_{j\in I} \omega_j^{\frac{1}{r}}(x)|\zeta_j|^{\frac{p}{r}}],$$
$$\text{for } i \in I^c,$$

for a.e. $x \in \Omega$, all $\eta \in \mathbb{R}$, $\zeta \in \mathbb{R}^N$, with

$$\frac{1}{r} + \frac{1}{p} + \frac{1}{q} < 1. \tag{2.7}$$

Remark 2.1 *1) The such r satisfying (2.7), exists when $q > p'$ (it suffices to choose $r > \frac{pq}{pq-p-q} > 1$).*

2) If $q \le p'$, we can not found any r satisfying (2.7) (since $\frac{1}{p} + \frac{1}{p'} = 1 \le \frac{1}{p} + \frac{1}{q}$).

Before to give main general result, let us give and prove the following lemmas which are needed below.

Lemma 2.2 *Let Ω be a subset of $\mathbb{R}^N$ with finite measure and let $f \in L^p(\Omega, \sigma_1)$ $(1 < p < \infty)$, $g \in L^q(\Omega, \sigma_2)$ $(1 < q < \infty)$ where σ_1 and σ_2 are weight functions in Ω and let $h \in L^r(\Omega, \sigma_1^{-\frac{r}{p}} \sigma_2^{-\frac{r}{q}})$ $(1 < r < \infty)$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le 1$, then $fgh \in L^1(\Omega)$.*

Indeed: Let $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le 1$. By Hölder inequality we have,

$$\int_\Omega |fgh|^s \le \Big(\int_\Omega f^p\sigma_1\Big)^{s/p}\Big(\int_\Omega g^q\sigma_2\Big)^{s/q}\Big(\int_\Omega h^r\sigma_1^{-r/p}\sigma_2^{-r/q}\Big)^{s/r} < \infty,$$

then $fgh \in L^s(\Omega)$ which implies that $fgh \in L^1(\Omega)$.

Lemma 2.3 *Let $(g_n)_n$ be a sequence of $L^p(\Omega,\sigma)$ and let $g \in L^p(\Omega,\sigma)$ $(1 < p < \infty)$, where σ is a weight function in Ω. If $g_n \to g$ in measure (in particular a.e. in Ω) and is bounded in $L^p(\Omega,\sigma)$, then $g_n \to g$ in $L^q(\Omega,\sigma^{q/p})$ for all $q < p$.*

Proof. Let $\varepsilon > 0$ and set $A_n = \{x \in \Omega : |g_n(x) - g(x)|\sigma^{1/p}(x) \le (\frac{\varepsilon}{2\,\mathrm{meas}(\Omega)})^{1/q}\}$, we have

$$\begin{aligned}\int_\Omega |g_n - g|^q\sigma^{q/p}\,dx &= \int_{A_n} |g_n - g|^q\sigma^{q/p}\,dx + \int_{A_n^c} |g_n - g|^q\sigma^{q/p}\,dx\\ &\le \frac{\varepsilon}{2} + \int_{A_n^c} |g_n - g|^q\sigma^{q/p}\,dx.\end{aligned}$$

By Hölder inequality,

$$\begin{aligned}\int_{A_n^c} |g_n - g|^q\sigma^{q/p}\,dx &\le \Big(\int_\Omega |g_n - g|^p\sigma\,dx\Big)^{q/p}\Big(\mathrm{meas}(A_n^c)\Big)^{1-\frac{q}{p}}\\ &\le M\,(\mathrm{meas}(A_n^c))^{1-\frac{q}{p}},\end{aligned}$$

where M is a constant does not depend on n. On the other hand since $g_n \to g$ in measure we have

$$\mathrm{meas}(A_n^c) \to 0 \quad \text{as } n \to \infty,$$

then there exists some $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\int_{A_n^c} |g_n - g|^q\sigma^{q/p}\,dx \le \frac{\varepsilon}{2}.$$

Remark 2.4 *We can also give an other proof of the last lemma, by using the non-weighted case, i.e., $g_n\sigma^{1/p}$ is bounded in $L^p(\Omega)$ and $g_n(x)\sigma^{1/p}(x) \to g(x)\sigma^{1/p}(x)$, in measure, hence $g_n\sigma^{1/p} \to g\sigma^{1/p}$ in $L^q(\Omega)$ for all $q < p$.*

The following lemma is a generalization of [7, Lemma 3.2] in weighted spaces.

Lemma 2.5 *Let $g \in L^q(\Omega,\sigma)$ and let $g_n \in L^q(\Omega,\sigma)$, with $\|g_n\|_{q,\sigma} \le c$ $(1 < q < \infty)$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^q(\Omega,\sigma)$, where $\rightharpoonup$ denotes weak convergence.*

Proof. Since $g_n\sigma^{1/q}$ is bounded in $L^q(\Omega)$ and $g_n(x)\sigma^{1/q}(x) \to g(x)\sigma^{1/q}(x)$, a.e. in Ω, by the [7, Lemma 3.2], we have

$$g_n\sigma^{1/q} \rightharpoonup g\sigma^{1/q} \quad \text{in } L^q(\Omega).$$

Moreover for all $\varphi \in L^{q'}(\Omega, \sigma^{1-q'})$, we have $\varphi\sigma^{-1/q} \in L^{q'}(\Omega)$, then

$$\int_\Omega g_n\varphi\,dx \to \int_\Omega g\varphi\,dx, \quad \text{i.e. } g_n \rightharpoonup g \text{ in } L^q(\Omega, \sigma).$$

Lemma 2.6 *Let $g_n \in L^p(\Omega, \sigma_1)$ and let $g \in L^p(\Omega, \sigma_1)$ $(1 < p < \infty)$. If $g_n \rightharpoonup g$ in $L^p(\Omega, \sigma_1)$, then*

$$g_n v \rightharpoonup gv \quad \text{in } L^s(\Omega, \sigma_1^{s/p}\sigma_2^{s/q}) \text{ for any } v \in L^q(\Omega, \sigma_2),$$

with $q > 1$ and $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$.

Proof. Let$\varphi \in L^{s'}(\Omega, \sigma_1^{\frac{s}{p}(1-s')}\sigma_2^{\frac{s}{q}(1-s')})$. For any $v \in L^q(\Omega, \sigma_2)$ we have, $v\varphi \in L^{p'}(\Omega, \sigma_1^{1-p'})$. Indeed, since $\frac{1}{p'} = \frac{1}{s'} + \frac{1}{q}$, we have by Hölder's inequality,

$$\begin{aligned}
&\int_\Omega |v\varphi|^{p'}\sigma_1^{1-p'}(x)\,dx \\
&= \int_\Omega |v\sigma_2^{1/q}(x)|^{p'}|\varphi|^{p'}\sigma_1^{1-p'}(x)\sigma_2^{-p'/q}(x)\,dx \\
&\le \Big(\int_\Omega |v|^q\sigma_2(x)\,dx\Big)^{p'/q}\Big(\int_\Omega |\varphi|^{s'}\sigma_1^{\frac{s'}{p'}(1-p')}(x)\sigma_2^{-s'/q}(x)\,dx\Big)^{p'/s'} \\
&= \Big(\int_\Omega |v|^q\sigma_2\,dx\Big)^{p'/q}\Big(\int_\Omega |\varphi|^{s'}\sigma_1^{\frac{s}{p}(1-s')}(x)\sigma_2^{\frac{s}{q}(1-s')}(x)\,dx\Big)^{p'/s'} < \infty.
\end{aligned}$$

Finally, since $g_n \rightharpoonup g$ in $L^p(\Omega, \sigma_1)$, then

$$\int_\Omega g_n v\varphi\,dx \to \int_\Omega gv\varphi\,dx \text{ i.e. } g_n v \rightharpoonup gv \text{ in } L^s(\Omega, \sigma_1^{s/p}\sigma_2^{s/q})\ \forall v \in L^q(\Omega, \sigma_2).$$

Lemma 2.7 *Let Ω be a subset of $\mathbb{R}^N$ with finite measure and let $1 \le p \le q$ then, we have the continuous imbedding $L^q(\Omega, \sigma) \hookrightarrow L^p(\Omega, \sigma^{p/q})$ where σ is a weight function in Ω.*

The proof of this lemma can be deduced easily from Hölder's inequality.

3 Main general result

Under the previous assumptions, the differential operator (1.1) (with coefficients satisfying (2.6), generates a mapping T from $X = W_0^{1,p}(\Omega, \omega)$ to its dual X^*

through the formula,

$$\begin{aligned}\langle Tu, v\rangle = &\int_\Omega \sum_{i\in I} b_i(x,u,\zeta_I(\nabla u))\frac{\partial v}{\partial x_i}\,dx + \int_\Omega \sum_{i\in I^c} b_i(x,u,\zeta_{I^c}(\nabla u))\frac{\partial v}{\partial x_i}\,dx \\ &+ \int_\Omega c_0(x,u,\zeta_I(\nabla u))v\,dx + \int_\Omega \sum_{i\in I^c} c_i(x,u,\zeta_I(\nabla u))\frac{\partial u}{\partial x_i} v\,dx,\end{aligned} \tag{3.1}$$

for all $u, v \in X$ and where $\langle , \rangle$ denotes the duality pairing between X^* and X. When we have adopted the notation $\zeta_J(\nabla u) = \{\frac{\partial u}{\partial x_i},\ i \in J\}$.

We recall that the mapping T is well defined and bounded, this can be easily seen by Lemma 2.2 and Hölder's inequality.

Definition A bounded mapping T from X to X^* is called pseudo-monotone if for any sequence $u_n \in X$ with $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle T\,u_n, u_n - u\rangle \le 0$, one has

$$\liminf_{n\to\infty}\langle Tu_n, u_n - v\rangle \ge \langle Tu, u - v\rangle \quad \text{for all } v \in X.$$

Theorem 3.1 *Assume that (H1), (H2'), (2.4) and (2.5) hold. Then the corresponding mapping T defined by (3.1) is pseudo-monotone in $X = W_0^{1,p}(\Omega,\omega)$.*

Remark 3.2 1) When $I = \emptyset$, the previous theorem applies in particular to operators like (1.1) with A_0 affine with respect to the gradient variable, this gives from (1.5) a sufficient condition (theorem 1.1 in the introduction).
2) When $I = \emptyset$ and $A_0 \equiv 0$, we immediately obtain [2, Proposition 1].
3) When $I^c = \emptyset$, we obtain [1, Theorem 7.4] and when $A_0 \equiv 0$, $I = \emptyset$, we give in [1, Theorem 7.2].
4) Theorem 3.1 generalizes [5, Theorem 3.1] in the weighted case.

Applying the previous theorem, we obtain the following existence results, which generalizes the corresponding (cf. [1, 2]).

Corollary 3.3 *Let Ω be a bounded open subset of $\mathbb{R}^N$ and assume the hypotheses in Theorem 3.1. Also assume the degenerate ellipticity condition*

$$\sum_{i=0}^N a_i(x,\xi)\xi_i \ge C_0 \sum_{i=1}^N \omega_i(x)|\xi_i|^p$$

for a.e. $x \in \Omega$, some $C_0 > 0$ and all $\xi \in \mathbb{R}^{N+1}$. Then for any $f \in X^*$ the Dirichlet associated problem

$$\begin{aligned}&\int_\Omega \sum_{i\in I} b_i(x,u,\zeta_I(\nabla u))\frac{\partial v}{\partial x_i}\,dx + \int_\Omega \sum_{i\in I^c} b_i(x,u,\zeta_{I^c}(\nabla u))\frac{\partial v}{\partial x_i}\,dx \\ &\quad + \int_\Omega c_0(x,u,\zeta_I(\nabla u))v\,dx + \int_\Omega \sum_{i\in I^c} c_i(x,u,\zeta_I(\nabla u))\frac{\partial u}{\partial x_i} v\,dx = \int_\Omega fv\,dx\end{aligned}$$

for all $v \in X$ has at least one solution $u \in X$.

Proof of Theorem 3.1. Let $(u_n)_n$ be a sequence in X such that:

$$u_n \rightharpoonup u \text{ in } X \tag{3.2}$$

and

$$\limsup_{n\to\infty} \langle Tu_n, u_n - u\rangle \le 0, \tag{3.3}$$

i.e.

$$\begin{aligned} \limsup_{n\to\infty} \quad \{ \quad & \int_\Omega \sum_{i\in I} b_i(x, u_n, \zeta_I(\nabla u_n))(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})\, dx \\ & + \int_\Omega \sum_{i\in I^c} b_i(x, u_n, \zeta_{I^c}(\nabla u_n))(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})\, dx \\ & + \int_\Omega c_0(x, u_n, \zeta_I(\nabla u_n))(u_n - u)\, dx \\ & + \int_\Omega \sum_{i\in I^c} c_i(x, u_n, \zeta_I(\nabla u_n))\frac{\partial u_n}{\partial x_i}(u_n - u)\, dx\} \le 0. \end{aligned}$$

a) We shall prove that

$$\langle Tu_n, v\rangle \to \langle Tu, v\rangle \quad \text{as } n \to \infty \text{ for all } v \in X. \tag{3.4}$$

First step. We show that

$$\lim_{n\to\infty} \int_\Omega \sum_{i\in I} (b_i(x, u_n, \zeta_I(\nabla u_n)) - b_i(x, u_n, \zeta_I(\nabla u)))(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})\, dx = 0 \tag{3.5}$$

and

$$\lim_{n\to\infty} \int_\Omega \sum_{i\in I^c} (b_i(x, u_n, \zeta_{I^c}(\nabla u_n)) - b_i(x, u_n, \zeta_{I^c}(\nabla u)))(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})\, dx = 0. \tag{3.6}$$

Indeed: First, we can choose q_1 such that $1 < q_1 < r$, and $\frac{1}{q_1} + \frac{1}{p} + \frac{1}{q} < 1$ (due to $\frac{1}{r} + \frac{1}{p} + \frac{1}{q} < 1$). It follows from the compact imbedding (1.7) that, for a subsequence,

$$\begin{aligned} u_n \to u \text{ in } L^q(\Omega, \bar\omega) \\ u_n(x) \to u(x) \text{ } a.e. \text{ in } \Omega. \end{aligned} \tag{3.7}$$

By (H2'), the sequences $\{c_0(x, u_n, \zeta_I(\nabla u_n))\}$ (resp. $\{c_i(x, u_n, \zeta_I(\nabla u_n))\frac{\partial u_n}{\partial x_i}$ $(i \in I^c)\}$) remains bounded in $L^{q'}(\Omega, \bar\omega^{1-q'})$ (resp. $L^{\tilde s}(\Omega, \bar\omega^{\frac{-\tilde s}{q}})$ with $\frac{1}{\tilde s} = \frac{1}{p} + \frac{1}{r}$). Indeed,

$$\begin{aligned} & \int_\Omega |\bar\omega^{-1/q} c_i(x, u_n, \zeta_I(\nabla u_n))\frac{\partial u_n}{\partial x_i}|^{\tilde s} \\ \le \quad & \Big(\int_\Omega \omega_i^{-r/p}\bar\omega^{-r/q}|c_i(x, u_n, \zeta_I(\nabla u_n))|^r\Big)^{\tilde s/r}\Big(\int_\Omega |\frac{\partial u_n}{\partial x_i}|^p\omega_i\Big)^{\tilde s/p} < c. \end{aligned}$$

Thanks to Lemma 2.7 and since $q' \le \tilde{s}$, we have

$$L^{\tilde{s}}(\Omega, \bar{\omega}^{-\tilde{s}/q}) \hookrightarrow L^{q'}(\Omega, \bar{\omega}^{-q'/q}),$$

then $\{c_i(x, u_n, \zeta_I(\nabla u_n))\frac{\partial u_n}{\partial x_i}\ (i \in I^c)\})$ is bounded in $L^{q'}(\Omega, \bar{\omega}^{1-q'}))$. Hence, using (3.7) we conclude that

$$\lim_{n\to\infty} \int_\Omega c_0(x, u_n, \zeta_I(\nabla u_n))(u_n - u)\, dx = 0 \tag{3.8}$$

and

$$\lim_{n\to\infty} \int_\Omega \sum_{i\in I^c} c_i(x, u_n, \zeta_I(\nabla u_n))\frac{\partial u_n}{\partial x_i}(u_n - u)\, dx = 0. \tag{3.9}$$

On the other hand, in virtue of (3.7) and continuity of the Nemytskii operators (see [3]), we have

$$\begin{aligned} b_i(x, u_n, \zeta_I(\nabla u)) &\to b_i(x, u, \zeta_I(\nabla u)) \quad \text{in } L^{p'}(\Omega, \omega_i^*),\ i \in I \\ b_i(x, u_n, \zeta_{I^c}(\nabla u)) &\to b_i(x, u, \zeta_{I^c}(\nabla u)) \quad \text{in } L^{p'}(\Omega, \omega_i^*),\ i \in I^c, \end{aligned}$$

which implies

$$\lim_{n\to\infty} \int_\Omega \sum_{i\in I} b_i(x, u_n, \zeta_I(\nabla u))(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})\, dx = 0 \tag{3.10}$$

and

$$\lim_{n\to\infty} \int_\Omega \sum_{i\in I^c} b_i(x, u_n, \zeta_{I^c}(\nabla u))(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})\, dx = 0. \tag{3.11}$$

Combining (2.4), (2.5), (3.3), (3.8), (3.9), (3.10) and (3.11) we conclude the assertions (3.5) and (3.6).

Second step. For to prove of the relation (3.4) it suffices to show the following assertions:

(i) For every $v \in X$,

$$\lim_{n\to\infty} \int_\Omega c_0(x, u_n, \zeta_I(\nabla u_n))v\, dx = \int_\Omega c_0(x, u, \zeta_I(\nabla u))v\, dx. \tag{3.12}$$

(ii) For every $v \in X$,

$$\lim_{n\to\infty} \int_\Omega \sum_{i\in I} b_i(x, u_n, \zeta_I(\nabla u_n))\frac{\partial v}{\partial x_i}\, dx = \int_\Omega \sum_{i\in I} b_i(x, u, \zeta_I(\nabla u))\frac{\partial v}{\partial x_i}\, dx. \tag{3.13}$$

(iii) For every $v \in X$,

$$\lim_{n\to\infty} \int_\Omega \sum_{i\in I^c} c_i(x, u_n, \zeta_I(\nabla u_n))\frac{\partial u_n}{\partial x_i}v\, dx = \int_\Omega \sum_{i\in I^c} c_i(x, u, \zeta_I(\nabla u))\frac{\partial u}{\partial x_i}v\, dx. \tag{3.14}$$

(iv) For every $v \in X$,

$$\lim_{n\to\infty}\int_\Omega \sum_{i\in I^c} b_i(x,u_n,\zeta_{I^c}(\nabla u_n))\frac{\partial v}{\partial x_i}\,dx = \int_\Omega \sum_{i\in I^c} b_i(x,u,\zeta_{I^c}(\nabla u))\frac{\partial v}{\partial x_i}\,dx. \tag{3.15}$$

Proof of (i)and (ii). Invoking Landes [6, Lemma 6], we obtain from (3.5) and the strict monotonicity (2.4) that,

$$\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{a.e. in } \Omega \text{ for each } i \in I, \tag{3.16}$$

which gives

$$c_0(x,u_n,\zeta_I(\nabla u_n)) \to c_0(x,u,\zeta_I(\nabla u)) \quad \text{a.e. in } \Omega,$$
$$b_i(x,u_n,\zeta_I(\nabla u_n)) \to b_i(x,u,\zeta_I(\nabla u)) \quad \text{a.e. in } \Omega\ \forall i \in I.$$

The growth conditions (H2') imply that, the sequences $\{c_0(x,u_n,\zeta_I(\nabla u_n))\}$ (resp. $\{b_i(x,u_n,\zeta_I(\nabla u_n)) \quad i\in I\}$) remains bounded in $L^{q'}(\Omega,\bar\omega^{1-q'})$ (resp. $L^{p'}(\Omega,\omega_i^*)$). Hence by Lemma 2.5 we conclude (i) and (ii).

Proof of (iii). Similarly, by (3.7) and (3.16) we can write,

$$c_i(x,u_n,\zeta_I(\nabla u_n)) \to c_i(x,u,\zeta_I(\nabla u)) \quad \text{a.e. in } \Omega \text{ for all } i \in I^c.$$

And by the growth conditions (H2') also $\{c_i(x,u_n,\zeta_I(\nabla u_n)),\ i\in I^c\}$ is bounded in $L^r(\Omega,\omega_i^{-\frac{r}{p}}\bar\omega^{-\frac{r}{q}})$, then in virtue of Lemma 2.3, we have

$$c_i(x,u_n,\zeta_I(\nabla u_n)) \to c_i(x,u,\zeta_I(\nabla u)) \quad \text{in } L^{q_1}(\Omega,\omega_i^{\frac{-q_1}{p}}\bar\omega^{\frac{-q_1}{q}}) \quad \forall i\in I^c.$$

Let $s>1$ such that $\frac{1}{s}=\frac{1}{p}+\frac{1}{q}$. Since $\frac{1}{s'}+\frac{1}{s}=1>\frac{1}{s}+\frac{1}{q_1}$ i.e. $s'<q_1$, we have (as in the proof of Lemma 2.7),

$$\int_\Omega |v|^{s'}\omega_i^{-s'/p}\bar\omega^{-s'/q}\,dx \le \Big(\int_\Omega |v|^{q_1}\omega_i^{-q_1/p}\bar\omega^{\frac{-q_1}{q}}\,dx\Big)^{s'/q_1}(\text{meas}(\Omega))^{1-\frac{s'}{q_1}}$$

for all $v\in L^{q_1}(\Omega,\omega_i^{-q_1/p}\bar\omega^{-q_1/q})$. Then

$$L^{q_1}(\Omega,\omega_i^{\frac{-q_1}{p}}\bar\omega^{\frac{-q_1}{q}}) \hookrightarrow L^{s'}(\Omega,\omega_i^{\frac{-s'}{p}}\bar\omega^{\frac{-s'}{q}}),$$

which implies

$$c_i(x,u_n,\zeta_I(\nabla u_n)) \to c_i(x,u,\zeta_I(\nabla u)) \quad \text{in } L^{s'}(\Omega,\omega_i^{-s'/p}\bar\omega^{-s'/q})\ \ \forall i\in I^c.$$

On the other hand, from Lemma 2.6 we obtain,

$$\frac{\partial u_n}{\partial x_i}v \rightharpoonup \frac{\partial u}{\partial x_i}v \quad \text{in } L^s(\Omega,\omega_i^{s/p}\bar\omega^{s/q}),$$

for any $v \in L^q(\Omega, \bar{\omega})$ and so for any $v \in X$,

$$\lim_{n\to\infty} \int_\Omega \sum_{i\in I^c} c_i(x, u_n, \zeta_I(\nabla u_n)) \frac{\partial u_n}{\partial x_i} v\, dx = \int_\Omega \sum_{i\in I^c} c_i(x, u, \zeta_I(\nabla u)) \frac{\partial u}{\partial x_i} v\, dx$$

for any $v \in X$.

Proof of (iv). As before, the growth conditions (H2') implies that, the sequence $\{b_i(x, u_n, \zeta_{I^c}(\nabla u_n)) \ \ i \in I^c\}$ is bounded in $L^{p'}(\Omega, \omega_i^*)$. Next, we show that,

$$\int_\Omega \sum_{i\in I^c} \{b_i(x, u, \zeta_{I^c}(v)) - h_i\}(v_i - \frac{\partial u}{\partial x_i})\, dx \geq 0 \quad \text{for all } v = (v_i) \in \prod_{i=1}^N L^p(\Omega, \omega_i), \tag{3.17}$$

here h_i stands for the weak limit of $\{b_i(x, u_n, \zeta_{I^c}(\nabla u_n)),\ i \in I^c\}$ in $L^{p'}(\Omega, \omega_i^{1-p'})$. Indeed by (3.6) we have

$$\limsup_{n\to\infty} \int_\Omega \sum_{i\in I^c} b_i(x, u_n, \zeta_{I^c}(\nabla u_n)) \frac{\partial u_n}{\partial x_i}\, dx \leq \int_\Omega \sum_{i\in I^c} h_i \frac{\partial u}{\partial x_i}\, dx \tag{3.18}$$

and from (2.5), we obtain for any $v = (v_i) \in \prod_{i=1}^N L^p(\Omega, \omega_i)$,

$$\begin{aligned} &\int_\Omega \sum_{i\in I^c} b_i(x, u_n, \zeta_{I^c}(\nabla u_n)) \frac{\partial u_n}{\partial x_i}\, dx \\ &\geq \int_\Omega \sum_{i\in I^c} b_i(x, u_n, \zeta_{I^c}(\nabla u_n)) v_i\, dx + \int_\Omega \sum_{i\in I^c} b_i(x, u_n, \zeta_{I^c}(v))(\frac{\partial u_n}{\partial x_i} - v_i)\, dx. \end{aligned}$$

Letting $n \to \infty$ we conclude by (3.18) that,

$$\int_\Omega \sum_{i\in I^c} h_i \frac{\partial u}{\partial x_i}\, dx \geq \int_\Omega \sum_{i\in I^c} h_i v_i\, dx + \int_\Omega \sum_{i\in I^c} b_i(x, u, \zeta_{I^c}(v))(\frac{\partial u}{\partial x_i} - v_i)\, dx$$

and hence (3.17) follows. Choosing $v = \nabla u + t\tilde{w}$ with $t > 0, \tilde{w} = (\tilde{w}_i) \in \prod_{i=1}^N L^p(\Omega, \omega_i)$ and letting $t \to 0$ we obtain,

$$h_i = b_i(x, u, \zeta_{I^c}(\nabla u)) \quad \text{a.e. in } \Omega,$$

which gives,

$$\lim_{n\to\infty} \int_\Omega \sum_{i\in I^c} b_i(x, u_n, \zeta_{I^c}(\nabla u_n)) \frac{\partial v}{\partial x_i}\, dx = \int_\Omega \sum_{i\in I^c} b_i(x, u, \zeta_{I^c}(\nabla u)) \frac{\partial v}{\partial x_i}\, dx$$

for all $v \in X$. □

b) We shall prove that

$$\liminf_{n\to\infty} \langle Tu_n, u_n \rangle \geq \langle Tu, u \rangle \tag{3.19}$$

by (2.4) and (2.5) we have

$$\begin{aligned}
&\int_\Omega \sum_{i\in I} b_i(x,u_n,\zeta_I(\nabla u_n))\frac{\partial u_n}{\partial x_i}\,dx + \int_\Omega \sum_{i\in I^c} b_i(x,u_n,\zeta_{I^c}(\nabla u_n))\frac{\partial u_n}{\partial x_i}\,dx \\
&\geq \int_\Omega \sum_{i\in I} b_i(x,u_n,\zeta_I(\nabla u_n))\frac{\partial u}{\partial x_i}\,dx + \int_\Omega \sum_{i\in I} b_i(x,u_n,\zeta_I(\nabla u))(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})\,dx \\
&+ \int_\Omega \sum_{i\in I^c} b_i(x,u_n,\zeta_{I^c}(\nabla u_n))\frac{\partial u}{\partial x_i}\,dx + \int_\Omega \sum_{i\in I^c} b_i(x,u_n,\zeta_{I^c}(\nabla u))(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})\,dx,
\end{aligned}$$

then letting $n \to \infty$, and using (3.8) and (3.9), we conclude (3.19).

Example Many ideas in this example have adapted from the corresponding examples 1-2 in [2]. We shall suppose that the weight functions satisfy: $\omega_{i_0}(x) \equiv 0$ for some $i_0 \in I^c$, and $\omega_i(x) = \omega(x)$, $x \in \Omega$, for all $i \in I \cup I^c$ and $i \neq i_0$ with $\omega(x) > 0$ a.e. in Ω. Then, we can consider the Hardy inequality in the form

$$\Big(\int_\Omega |u(x)|^q \bar{\omega}(x)\,dx\Big)^{1/q} \leq c\Big(\sum_{i\neq i_0}\int_\Omega |\frac{\partial u}{\partial x_i}|^p \omega\Big)^{1/p} \tag{3.20}$$

for every $u \in X$ with a constant $c > 0$ independent of u and for some $q \geq p'$.

Let us consider the Carathéodory functions:

$$\begin{aligned}
b_i(x,\eta,\zeta_I) &= \omega|\zeta_i|^{p-1}\operatorname{sgn}\zeta_i + \omega_0 A_0(\eta) \quad \text{for } i \in I \\
b_i(x,\eta,\zeta_{I^c}) &= \omega|\zeta_i|^{p-1}\operatorname{sgn}\zeta_i + \omega_0 A_0(\eta) \quad \text{for } i \in I^c \text{ and } i \neq i_0 \\
b_{i_0}(x,\eta,\zeta_{I^c}) &= \omega_0 A_0(\eta) \\
c_0(x,\eta,\zeta_I) &= \sum_{j\in I} \omega^{1/q'}\bar{\omega}^{1/q}|\zeta_j|^{\frac{p}{q'}} + \omega_0 B_0(\eta) \\
c_i(x,\eta,\zeta_I) &= \sum_{j\in I} \omega^{1/p+1/r}\bar{\omega}^{1/q}|\zeta_j|^{p/r} + \omega_0 B_1(\eta) \quad \text{for } i \in I^c,
\end{aligned} \tag{3.21}$$

with $1/p + 1/r + 1/q < 1$. The above functions define by (3.21) satisfies the growth conditions (H2') if we suppose that

$$\begin{aligned}
|\omega_0 A_0(\eta)| &\leq \beta_1 \omega^{1/p}\bar{\omega}^{1/p'}|\eta|^{q/p'} \\
|\omega_0 B_0(\eta)| &\leq \beta_2 \bar{\omega}|\eta|^{q/q'} \\
|\omega_0 B_1(\eta)| &\leq \beta_3 \omega^{1/p}\bar{\omega}^{1/q+1/r}|\eta|^{q/r},
\end{aligned} \tag{3.22}$$

with β_j $j = 1,2,3$ are some positive constants. In particular, let us use the special weight functions $\omega_0, \omega, \bar{\omega}$ expressed in terms of the distance to the boundary $\partial\Omega$: denote $d(x) = \operatorname{dist}(x, \partial\Omega)$ and set

$$\omega(x) = d^{\lambda}(x), \quad \omega_0(x) = d^{\lambda_0}(x), \quad \bar{\omega}(x) = d^{\mu}(x).$$

In this case the condition (3.22) writes as

$$\begin{aligned} |A_0(\eta)| &\le \beta_1 d^{\lambda/p+\mu/p'-\lambda_0}|\eta|^{q/p} \\ |B_0(\eta)| &\le \beta_2 d^{\mu-\lambda_0}|\eta|^{q/q'} \\ |B_1(\eta)| &\le \beta_3 d^{\lambda/p+\mu/q+\mu/r-\lambda_0}|\eta|^{q/r}, \end{aligned} \tag{3.23}$$

and the Hardy inequality reads

$$\Big(\int_\Omega |u(x)|^q\, d^\mu(x)\,dx\Big)^{1/q} \le c\Big(\sum_{i\neq i_0}\int_\Omega |\frac{\partial u}{\partial x_i}|^p\, d^\lambda(x)\,dx\Big)^{1/p}, \tag{3.24}$$

and the corresponding imbedding (1.7) is compact for $1 \le p \le q < \infty$ (resp. $1 \le q < p < \infty$,), if and only if $\lambda \neq p-1$, $\frac{N}{q} - \frac{N}{p} + 1 \ge 0$, $\frac{\mu}{q} - \frac{\lambda}{p} + \frac{N}{q} - \frac{N}{p} + 1 > 0$, (resp. $\lambda \in \mathbb{R}$, $\frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} > 0$) (see [8]). Moreover, the two monotonicity conditions (2.4) and (2.5) are satisfied:

$$\begin{aligned} &\sum_{i\in I}\big(b_i(x,\eta,\zeta_I) - b_i(x,\eta,\bar\zeta_I)\big)\big(\zeta_i - \bar\zeta_i\big) \\ &\qquad = \omega(x)\sum_{i\in I}\big(|\zeta_i|^{p-1}\operatorname{sgn}\zeta_i - |\bar\zeta_i|^{p-1}\operatorname{sgn}\bar\zeta_i\big)\big(\zeta_i - \bar\zeta_i\big) > 0 \end{aligned}$$

for almost all $x \in \Omega$ and for all $\zeta, \bar\zeta \in \mathbb{R}^N$ with $\zeta_I \neq \bar\zeta_I$, since $\omega > 0$ *a.e.* in Ω; and

$$\begin{aligned} &\sum_{i\in I^c}\big(b_i(x,\eta,\zeta_{I^c}) - b_i(x,\eta,\bar\zeta_{I^c})\big)\big(\zeta_i - \bar\zeta_i\big) \\ &\qquad = \omega(x)\sum_{\substack{i\in I^c\\ i\neq i_0}}\big(|\zeta_i|^{p-1}sgn\zeta_i - |\bar\zeta_i|^{p-1}sgn\bar\zeta_i\big)\big(\zeta_i - \bar\zeta_i\big) \ge 0 \end{aligned}$$

for almost all $x \in \Omega$ and for all $\zeta, \bar\zeta \in \mathbb{R}^N$. This last inequality can not be strict, since for $\zeta_{I^c} \neq \bar\zeta_{I^c}$ with $\zeta_{i_0} \neq \bar\zeta_{i_0}$ but $\zeta_i = \bar\zeta_i$ for all $i \in I^c$ and $i \neq i_0$, the corresponding expression is zero. Finally, the hypotheses of theorem 3.1 are verify, then the mapping T defined as (3.1) corresponding to (3.21) is pseudo-monotone.

4 Specific case

Let Ω be a bounded open subset of $\mathbb{R}^N$ satisfying the cone condition. In this section we assume in addition that the collection of weight functions $\omega = \{\omega_i(x)\ i = 0,\dots,N\}$ satisfy $\omega_0(x) = 1$ and the integrability condition: There exists $\nu \in]\frac{N}{p},\infty[\cap[\frac{1}{p-1},\infty[$ such that

$$\omega_i^{-\nu} \in L^1(\Omega) \quad \forall i = 1,\dots,N. \tag{4.1}$$

Note that (4.1) is stronger than (2.2).

Remark 4.1 ([3]) 1. Assumptions (2.1) and (4.1) imply that,

$$|||u|||_X = \Big(\sum_{i=1}^{N}\int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\, dx\Big)^{1/p}$$

is a norm defined on $W_0^{1,p}(\Omega,\omega)$ and it's equivalent to (2.3), and that

$$W_0^{1,p}(\Omega,\omega) \hookrightarrow\hookrightarrow L^q(\Omega) \qquad (4.2)$$

for all $1 \le q < p_1^*$ if $p\nu < N(\nu+1)$ and $q \ge 1$ is arbitrary if $p\nu \ge N(\nu+1)$ where $p_1 = \frac{p\nu}{\nu+1}$ and $p_1^* = \frac{Np_1}{N-p_1} = \frac{Np\nu}{N(\nu+1)-p\nu}$ is the Sobolev conjugate of p_1.

2. Hypotheses (H1) holds for all q such that $1 < q < p_1^*$ and $\bar{\omega} \equiv 1$.

In the sequel, we replace (4.1) by the hypothesis

($\tilde{\text{H}}1$) If $\frac{2N}{N+1} < p < N$ there exists $\nu \in]\frac{N}{p},\infty[\cap]\frac{1}{(p-1)-\frac{p}{p^*}},\infty[$ such that $\omega_i^{-\nu} \in L^1(\Omega)$, for all $i = 1,\dots,N$. If $p = N$ there exists $\nu \in]1,\infty[$ such that $\omega_i^{-\nu} \in L^1(\Omega)$ for all $i = 1,\dots,N$. If $p > N$ there exist $\nu \in]\frac{N}{p-N},\infty[\cap[\frac{1}{(p-1)},\infty[$ such that $\omega_i^{-\nu} \in L^1(\Omega)$ for all $i = 1,\dots,N$.

Remark 4.2 1. Hypothesis ($\tilde{\text{H}}1$) guarantees the existence of r satisfying $\frac{1}{r}+\frac{1}{p}+\frac{1}{p_1^*} < 1$, where p_1^* is the Sobolev conjugate of p_1 in the case $\frac{2N}{N+1} < p \le N$ and where $p_1^* = \infty$ in the case $p > N$ (since $p_1 > N$ due to $\nu > \frac{N}{p-N}$).

2. If $1 < p \le \frac{2N}{N+1}$ we can't find a real $r > 1$ such that $\frac{1}{r}+\frac{1}{p}+\frac{1}{p_1^*} < 1$, since $\frac{1}{p}+\frac{1}{p_1^*} > \frac{1}{p}+\frac{1}{p^*} \ge 1$.

3. Note that ($\tilde{\text{H}}1$) is stronger than (4.1), then the compact imbedding (4.2) is satisfied whenever ($\tilde{\text{H}}1$) is assumed.

Theorem 4.3 *Let Ω be a bounded open subset of $\mathbb{R}^N$. And assume that (2.1), ($\tilde{H}1$), (H2'), (2.4) and (2.5) are satisfied. Then the operator T defined in* (3.1) *is pseudo-monotone in $X = W_0^{1,p}(\Omega,\omega)$. Moreover, assume the degenerate ellipticity condition*

$$\sum_{i=0}^{N} a_i(x,\xi)\xi_i \ge c_0 \sum_{i=1}^{N}\omega_i(x)|\xi_i|^p$$

for a.e. $x \in \Omega$, some $c_0 > 0$ and all $\xi \in \mathbb{R}^{N+1}$. Then for any $f \in X^*$ the Dirichlet associated problem

$$\langle Tu, v\rangle = \langle f, v\rangle \quad \textit{for all } v \in X,$$

has at least one solution $u \in X$.

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Youssef Akdim (e-mail y.akdim1@caramail.com)
Elhoussine Azroul (e-mail elazroul@caramail.com)
Département de Mathématiques et Informatique
Faculté des Sciences Dhar-Mahraz
B.P 1796 Atlas Fès, Maroc