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STABILIZATION OF EULER-BERNOULLI BEAM EQUATIONS WITH VARIABLE COEFFICIENTS UNDER DELAYED BOUNDARY OUTPUT FEEDBACK

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Abstract. In this article, we study the stabilization of an Euler-Bernoulli beam equation with variable coefficients where boundary observation is subject to a time delay. To resolve the mathematical complexity of variable coefficients, we design an observer-predictor based on the well-posed open-loop system: the state of system is estimated with available observation and then predicted without observation. We show that the closed-loop system is stable exponentially under estimated state feedback by a numerical simulation illustrating our results.

1. Introduction

The phenomenon of time delay is commonly observed in modern engineering and scientific research [3, 4, 5, 6, 7, 9, 21, 19]. Much attention has been devoted to the stability of control systems with time delay. Nevertheless, even a small delay may break the system's stability [3, 4, 5, 6, 7, 10]. It is indicated in [8] that for distributed parameter control systems, time delay in observation and control can cause complications. Stimulated by the work in [14], we solve the stabilization problem with delayed observation and boundary control, for the one-dimensional Euler-Bernoulli beam equation [16].

In this article, we focus on the boundary stabilization of an Euler-Bernoulli beam equation with variable coefficients where boundary observation contains a fixed time delay. This is a generalization of the similar work such as [16] for the beam equation with constant coefficients. It is obvious that variable coefficients present more mathematical challenges, making the stabilization problems of the system much more complicated since it is difficult to construct the Lyapunov functions and estimate the eigenvalues and eigenfunctions by asymptotic analysis.

Consider the following nonuniform Euler-Bernoulli beam equation with linear boundary feedback control:

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$$
\begin{gathered}
\rho(x)w_{tt}(x,t)+(EI(x)w_{xx}(x,t))_{xx}=0,\quad 0<x<1,\ t>0,\\
w(0,t)=w_x(0,t)=w_{xx}(1,t)=0,\quad t\ge 0,\\
(EI(x)w_{xx})_x(1,t)=u(t),\quad t\ge 0,\\
y(t)=w_t(1,t-\tau),\quad t>\tau,\\
w(x,0)=w_0(x),w_t(x,0)=w_1(x),\quad 0\le x\le 1,
\end{gathered}
\tag{1.1}
$$

where x stands for the position and t the time, w is the state, u is the boundary controller input, $(w_0,w_1)^T$ is the initial value, $\tau>0$ is a known constant time delay, and y is the delayed observation(or output) which suffers from a given time delay τ. $EI(x)(>0)\in C^2[0,1]$ is the flexural rigidity of the beam, and $\rho(x)(>0)\in C[0,1]$ is the mass density at x.

The system above is considered in the energy state space

$$
\mathcal{H}=H_E^2(0,1)\times L^2(0,1),\quad H_E^2(0,1)=\{f\in H^2(0,1):f(0)=f'(0)=0\}.
$$

The energy of the system is

$$
E_0(t)=\frac{1}{2}\int_0^1[EI(x)w_{xx}^2(x,t)+\rho(x)w_t^2(x,t)]dx.
$$

As noted in [4] (where $EI(x)=\rho(x)=1$), even a small amount of time delay in the stabilizing boundary output feedback schemes destabilizes the system. Therefore, it is important to design stabilizing controllers that are robust to time delay for systems described in (1.1).

The next section shows the well-posedness of the considered open-loop system. In section 3, we design the observer and predictor for the system. The asymptotic stability of the closed-loop system under the estimated state feedback control is then studied in section 4. Section 5 illustrates the simulation results and concludes the paper.

2. Well-posedness of the open-loop system

We introduce a new variable $z(x,t)=w_t(1,t-x\tau)$. Then the system (1.1) becomes

$$
\begin{gathered}
\rho(x)w_{tt}(x,t)+(EI(x)w_{xx}(x,t))_{xx}=0,\quad 0<x<1,\ t>0,\\
w(0,t)=w_x(0,t)=w_{xx}(1,t)=0,\quad t\ge 0,\\
(EI(x)w_{xx})_x(1,t)=u(t),\quad t\ge 0,\\
\tau z_t(x,t)+z_x(x,t)=0,\quad 0<x<1,t\ge 0,\\
z(0,t)=w_t(1,t),\quad t\ge 0,\\
w(x,0)=w_0(x),w_t(x,0)=w_1(x),\quad 0\le x\le 1,\\
z(x,0)=z_0(x),\quad 0\le x\le 1,\\
y(t)=z(1,t),\quad t\ge \tau,
\end{gathered}
\tag{2.1}
$$

where z_0 is the initial value of the variable z.

We consider the system (2.1) in the energy state space $\mathbb{H}=\mathcal{H}\times L^2(0,1)$, with the state variable $(w(\cdot,t),w_t(\cdot,t),z(\cdot,t))^T$ for which the inner product induced norm is defined as following:

$$
E_1(t)=\frac{1}{2}\|(w(\cdot,t),w_t(\cdot,t),z(\cdot,t))^T\|_{\mathbb{H}}^2
$$

$$= \frac{1}{2}\int_0^1 [EI(x)w_{xx}^2(x,t) + \rho(x)w_t^2(x,t) + z^2(x,t)]dx.$$

The input space and the output space are the same $U = Y = \mathbb{C}$.

Theorem 2.1. *System (2.1) is well-posed: For any $(w_0, w_1, z_0)^T \in \mathbb{H}$ and $u \in L^2_{\rm loc}(0,\infty)$, there exists a unique solution of (2.1) such that $(w(\cdot,t), w_t(\cdot,t), z(\cdot,t))^T$ belongs to $C(0,\infty;\mathbb{H})$; and for any $T>0$, there exist a constant $C_T>0$ such that*

$$\begin{aligned} &\|(w(\cdot,T), w_t(\cdot,t), z(\cdot,T))^T\|^2_{\mathbb{H}} + \int_0^T |y(t)|^2 dt \\ &\le C_T\Big[\|(w_0,w_1,z_0)^T\|^2_{\mathbb{H}} + \int_0^T |u(t)|^2 dt\Big]. \end{aligned}$$

Proof. Firstly, we represent the system

$$\begin{gathered} \rho(x)w_{tt}(x,t) + (EI(x)w_{xx}(x,t))_{xx} = 0, \quad 0<x<1,\ t\ge 0, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \quad t\ge 0, \\ (EI(x)w_{xx})_x(1,t) = u(t), \quad t\ge 0, \\ y_w(t) = w_t(1,t), \quad t\ge 0, \end{gathered} \tag{2.2}$$

as a second-order system in $\mathcal{H}$,

$$\begin{gathered} w_{tt}(\cdot,t) + Aw(\cdot,t) + Bu(t) = 0, \quad 0<x<1,\ t\ge 0, \\ y_\omega(t) = B^* w_t(\cdot,t), \quad t\ge 0, \end{gathered} \tag{2.3}$$

where A is a self-adjoint operator in $\mathcal{H}$ and B is the input operator:

$$\begin{gathered} Af = \frac{1}{\rho(x)}(EI(x)f'')'', \\ \forall f\in D(A) = \{f\in H^4(0,1)\cap H^2_E(0,1) : f''(1) = (EIf'')'(1) = 0\}, \\ B = \delta(x-1). \end{gathered} \tag{2.4}$$

Here $\delta(\cdot)$ denote the Dirac distribution. It was shown in [13] that system (2.3) and (2.4) is well-posed in the sense of Salamon [2]: for any $u\in L^2_{\rm loc}(0,\infty)$ and $(w_0,w_1)^T\in\mathcal{H}$, there exists a unique solution $(w(\cdot,t), w_t(\cdot,t))^T \in C(0,\infty;\mathcal{H})$ to (2.3) and for any $T>0$, there exists a constant $D_T>0$ such that

$$\begin{aligned} &\|(w(\cdot,T), w_t(\cdot,T))^T\|^2_{\mathcal{H}} + \int_0^T |y_w(t)|^2 dt \\ &\le D_T\Big[\|(w_0,w_1)^T\|^2_{\mathcal{H}} + \int_0^T |u(t)|^2 dt\Big]. \end{aligned} \tag{2.5}$$

Then the following inequality can be shown similarly as those in [17]:

$$\begin{aligned} &\|(w(\cdot,T), w_t(\cdot,T), z(\cdot,T))^T\|^2_{\mathcal{H}} + \int_0^T |y(t)|^2 dt \\ &\le C_T\left[\|(w_0,w_1,z_0)^T\|^2_{\mathcal{H}} + \int_0^T |u(t)|^2 dt\right], \end{aligned}$$

for a constant $C_T>0$. The details are omitted. □

Theorem 2.1 illustrates that, for any initial value in the state space, the output belongs to $L^2_{\rm loc}(\tau,\infty)$ as long as the input u belongs to $L^2_{\rm loc}(0,\infty)$. This fact is particularly necessary to the solvability of observer shown in the next section ([13,14]).

3. Observer and predictor design

For any fixed time delay $\tau>0$, and when $t>\tau$, we propose a two-step method to estimate the state of (1.1) by designing the observer and predictor systems.

Step 1. From the known observation signal $\{y(s+\tau): s\in[0,t-\tau], t>\tau\}$, we construct an observer system to estimate the state $\{w(x,s): s\in[0,t-\tau], t>\tau\}$ which satisfies

$$\begin{gathered}
\rho(x)w_{ss}(x,s)+(EI(x)w_{xx}(x,s))_{xx}=0,\quad 0<x<1,\ 0<s<t-\tau,\ t>\tau,\\
w(0,s)=w_x(0,s)=w_{xx}(1,s)=0,\quad 0\le s\le t-\tau,\ t>\tau,\\
(EI(x)w_{xx})_x(1,s)=u(s),\quad 0\le s\le t-\tau,\ t>\tau,\\
y(s+\tau)=w_s(1,s),\quad 0\le s\le t-\tau,\ t>\tau.
\end{gathered}\tag{3.1}$$

Then a Luenberger observer naturally can be constructed for the system (3.1),

$$\begin{gathered}
\rho(x)\widehat{w}_{ss}(x,s)+(EI(x)\widehat{w}_{xx}(x,s))_{xx}=0,\quad 0<x<1,\ 0<s<t-\tau,\ t>\tau,\\
\widehat{w}(0,s)=\widehat{w}_x(0,s)=\widehat{w}_{xx}(1,s)=0,\quad 0\le s\le t-\tau,\ t>\tau,\\
(EI(x)\widehat{w}_{xx})_x(1,s)=u(s)+k_1[\widehat{w}_s(1,s)-y(s+\tau)],\quad 0\le s\le t-\tau,\ t>\tau,k_1>0,\\
\widehat{w}(x,0)=\widehat{w}_0(x),\widehat{w}_s(x,0)=\widehat{w}_1(x),\quad 0\le x\le 1,
\end{gathered}\tag{3.2}$$

where $(\widehat{w}_0,\widehat{w}_1)^T$ is an arbitrary assigned initial state of the observer.

For (3.2) to be an observer for (3.1), we have to show its convergence. To do this, we set

$$\varepsilon(x,s)=\widehat{w}(x,s)-w(x,s),\quad 0\le s\le t-\tau,t>\tau.\tag{3.3}$$

Then by (3.1) and (3.2), ε satisfies

$$\begin{gathered}
\rho(x)\varepsilon_{ss}(x,s)+(EI(x)\varepsilon_{xx}(x,s))_{xx}=0,\quad 0<x<1,\ 0<s<t-\tau,\ t>\tau,\\
\varepsilon(0,s)=\varepsilon_x(0,s)=\varepsilon_{xx}(1,s)=0,\quad 0\le s\le t-\tau,\ t>\tau,\\
(EI(x)\varepsilon_{xx})_x(1,s)=k_1\varepsilon_s(1,s),\quad 0\le s\le t-\tau,\ t>\tau,\ k_1>0,\\
\varepsilon(x,0)=\widehat{w}_0(x)-w_0(x),\quad 0\le x\le 1,\\
\varepsilon_s(x,0)=\widehat{w}_1(x)-w_1(x),\quad 0\le x\le 1.
\end{gathered}\tag{3.4}$$

The system above can be written as

$$\frac{d}{ds}\begin{pmatrix}\varepsilon(\cdot,s)\\ \varepsilon_s(\cdot,s)\end{pmatrix}=\mathbb{B}\begin{pmatrix}\varepsilon(\cdot,s)\\ \varepsilon_s(\cdot,s)\end{pmatrix},\tag{3.5}$$

where

$$\begin{gathered}
\mathbb{B}(f,g)^T=(g,-\frac{1}{\rho(x)}(EI(x)f''(x))'')^T,\\
D(\mathbb{B})=\big\{(f,g)\in(H^4(0,1)\cap H^2_E(0,1))\times H^2_E(0,1):\\
f''(1)=0,(EIf'')'(1)=k_1g(1)\big\},
\end{gathered}\tag{3.6}$$

and $\mathbb{B}$ generates an exponentially stable C_0-semigroup on $\mathcal{H}$ satisfying:

$$\|e^{\mathbb{B}s}\|\le Me^{-\omega s},\quad \forall s\ge 0,\tag{3.7}$$

for some positive constants M, ω. Hence, for any $(w_0, w_1)^T \in \mathcal{H}$ and $(\widehat{w}_0, \widehat{w}_1)^T \in \mathcal{H}$, there exists a unique solution to (3.4) such that

$$\|(\varepsilon(\cdot, s), \varepsilon_s(\cdot, s))^T\|_{\mathcal{H}} \leq Me^{-\omega s}\|(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T\|_{\mathcal{H}}, \tag{3.8}$$

for all $s \in [0, t-\tau]$ and all $t > \tau$.

Step 2. Predict $\{(w(x,s), w_s(x,s))^T, s \in (t-\tau, t], t > \tau\}$ by

$$\{(\widehat{w}(x,s), \widehat{w}_s(x,s))^T, s \in [0, t-\tau], t > \tau\}.$$

This is done by solving (1.1) with estimated initial value $(\widehat{w}(x, t-\tau), \widehat{w}_s(x, t-\tau))^T$ obtained from (3.2):

$$\begin{gathered}
\rho(x)\widehat{w}_{ss}(x,s,t) + (EI(x)\widehat{w}_{xx}(x,s,t))_{xx} = 0, \quad 0 < x < 1,\ t-\tau < s < t,\ t > \tau, \\
\widehat{w}(0,s,t) = \widehat{w}_x(0,s,t) = \widehat{w}_{xx}(1,s,t) = 0, \quad t-\tau \leq s \leq t,\ t > \tau, \\
(EI(x)\hat{w}_{xx})_x(1,s,t) = u(s), \quad t-\tau \leq s \leq t,\ t > \tau, \\
\widehat{w}(x, t-\tau, t) = \widehat{w}(x, t-\tau), \widehat{w}_s(x, t-\tau, t) = \widehat{w}_s(x, t-\tau), \\
0 \leq x \leq 1,\ t-\tau \leq s \leq t,\ t > \tau.
\end{gathered} \tag{3.9}$$

We finally get the estimated state variable by

$$\widetilde{w}(x,t) = \widehat{w}(x,t,t), \quad \forall t > \tau, \tag{3.10}$$

which is assured by Theorem 3.1 below.

Theorem 3.1. *For all $t > \tau$, we have*

$$\|(w(\cdot,t) - \tilde{w}_t(\cdot,t), w_t(\cdot,t) - \tilde{w}_t(\cdot,t))^T\|_{\mathcal{H}} \leq Me^{-\omega(t-\tau)}\|(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T\|_{\mathcal{H}}, \tag{3.11}$$

where $(\widehat{w}_0, \widehat{w}_1)^T$ is the initial state of observer (3.2), $(w_0, w_1)^T$ is the initial state of original system (1.1), M, ω are constants in (3.7).

Proof. Let

$$\varepsilon(x,s,t) = \widehat{w}(x,s,t) - w(x,s), \quad t-\tau \leq s \leq t,\ t > \tau. \tag{3.12}$$

Then $\varepsilon(x,s,t)$ satisfies

$$\begin{gathered}
\rho(x)\varepsilon_{ss}(x,s,t) + (EI(x)\varepsilon_{xx}(x,s,t))_{xx} = 0, \\
0 < x < 1,\ t-\tau < s < t,\ t > \tau; \\
\varepsilon(0,s,t) = \varepsilon_x(0,s,t) = \varepsilon_{xx}(1,s,t) = (EI(x)\varepsilon_{xx})_x(1,s,t) = 0, \\
t-\tau \leq s \leq t,\ t > \tau; \\
\varepsilon(x, t-\tau, t) = \varepsilon(x, t-\tau), \varepsilon_s(x, t-\tau, t) = \varepsilon_s(x, t-\tau), \\
0 \leq x \leq 1,\ t-\tau \leq s \leq t,\ t > \tau;
\end{gathered} \tag{3.13}$$

which is a conservative system

$$\|(\varepsilon(\cdot,s,t), \varepsilon_s(\cdot,s,t))^T\|_{\mathcal{H}} = \|(\varepsilon(\cdot, t-\tau), \varepsilon_s(\cdot, t-\tau))^T\|_{\mathcal{H}}. \tag{3.14}$$

Collecting (3.8), (3.10) and (3.14) gives (3.11). □

4. Stabilization by the estimated state feedback

Since the feedback $u(t) = k_2\tilde{w}_t(1,t) = k_2\widehat{w}_s(1,t,t)$ ($k_2 > 0$) stabilizes exponentially the system (1.1), and we have the estimation $\tilde{w}_t(1,t)$ of $w_t(1,t)$, it is natural to design the estimated state feedback control law of the following:

$$u^*(t) = \begin{cases} k_2\tilde{w}_t(1,t) = k_2\widehat{w}_s(1,t,t), & t > \tau,\ k_2 > 0, \\ 0, & t \in [0,\tau]. \end{cases} \tag{4.1}$$

The closed-loop system becomes a system of partial differential equations (4.2)-(4.3) via applying the control law above:

$$\begin{gathered} \rho(x)w_{tt}(x,t) + (EI(x)w_{xx}(x,t))_{xx} = 0, \quad 0 < x < 1,\ t > 0, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \quad t \geq 0, \\ (EI(x)w_{xx})_x(1,t) = u^*(t), \quad t \geq 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad 0 \leq x \leq 1, \end{gathered} \tag{4.2}$$

and

$$\begin{gathered} \rho(x)\widehat{w}_{ss}(x,s) + (EI(x)\widehat{w}_{xx}(x,s))_{xx} = 0, \quad 0 < x < 1,\ 0 < s < t-\tau,\ t > \tau, \\ \widehat{w}(0,s) = \widehat{w}_x(0,s) = \widehat{w}_{xx}(1,s) = 0, \quad 0 \leq s \leq t-\tau,\ t > \tau, \\ (EI(x)\widehat{w}_{xx})_x(1,s) = u^*(s) + k_1[\widehat{w}_s(1,s) - w_s(1,s)], \quad 0 \leq s \leq t-\tau,\ t > \tau,\ k_1 > 0, \\ \widehat{w}(x,0) = \widehat{w}_0(x), \quad \widehat{w}_s(x,0) = \widehat{w}_1(x),\ 0 \leq x \leq 1, \end{gathered}$$

and

$$\begin{gathered} \rho(x)\widehat{w}_{ss}(x,s,t) + (EI(x)\widehat{w}_{xx}(x,s,t))_{xx} = 0, \quad 0 < x < 1,\ t-\tau < s < t,\ t > \tau, \\ \widehat{w}(0,s,t) = \widehat{w}_x(0,s,t) = \widehat{w}_{xx}(1,s,t) = 0, \quad t-\tau \leq s \leq t,\ t > \tau, \\ (EI(x)\widehat{w}_{xx})_x(1,s,t) = u^*(s),\ t-\tau \leq s \leq t, \quad t > \tau, \\ \widehat{w}(x,t-\tau,t) = \widehat{w}(x,t-\tau), \widehat{w}_s(x,t-\tau,t) = \widehat{w}_s(x,t-\tau), \quad 0 \leq x \leq 1,\ t > \tau. \end{gathered} \tag{4.3}$$

We consider the closed-loop system (4.2)-(4.3) in the state space $X = \mathcal{H}^3$. Obviously the system (4.2)-(4.3) is equivalent to the system (4.4)-(4.6) for $t > \tau$:

$$\begin{gathered} \rho(x)w_{tt}(x,t) + (EI(x)w_{xx}(x,t))_{xx} = 0, \quad 0 < x < 1,\ t > \tau, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \quad t > \tau, \\ (EI(x)w_{xx})_x(1,t) = k_2[w_t(1,t) + \varepsilon_s(1,t,t)], \quad t > \tau,\ k_2 > 0, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), \quad 0 \leq x \leq 1, \end{gathered} \tag{4.4}$$

and

$$\begin{gathered} \rho(x)\varepsilon_{ss}(x,s) + (EI(x)\varepsilon_{xx}(x,s))_{xx} = 0, \quad 0 < x < 1,\ 0 < s < t-\tau,\ t > \tau, \\ \varepsilon(0,s) = \varepsilon_x(0,s) = \varepsilon_{xx}(1,s) = 0, \quad 0 \leq s \leq t-\tau,\ t > \tau, \\ (EI(x)\varepsilon_{xx})_x(1,s) = k_1\varepsilon_s(1,s), \quad 0 \leq s \leq t-\tau,\ t > \tau,\ k_1 > 0, \\ \varepsilon(x,0) = \widehat{w}_0(x) - w_0(x), \quad \varepsilon_s(x,0) = \widehat{w}_1(x) - w_1(x), \quad 0 \leq x \leq 1, \end{gathered} \tag{4.5}$$

and

$$
\begin{gathered}
\rho(x)\varepsilon_{ss}(x,s,t)+(EI(x)\varepsilon_{xx}(x,s,t))_{xx}=0,\\
0<x<1,\ t-\tau<s<t,\ t>\tau;\\
\varepsilon(0,s,t)=\varepsilon_x(0,s,t)=\varepsilon_{xx}(1,s,t)=(EI(x)\varepsilon_{xx})_x(1,s,t)=0,\\
t-\tau\le s\le t,\ t>\tau;\\
\varepsilon(x,t-\tau,t)=\varepsilon(x,t-\tau,t),\quad \varepsilon_s(x,t-\tau,t)=\varepsilon_s(x,t-\tau),\\
0\le x\le 1,\ t>\tau;
\end{gathered}
\tag{4.6}
$$

where $\varepsilon(x,s)$ and $\varepsilon(x,s,t)$ are given by (3.3) and (3.12) respectively.

Theorem 4.1. *Let $t>\tau$, for any $(w_0,w_1)^T\in\mathcal{H}$, $(\widehat{w}_0,\widehat{w}_1)^T\in\mathcal{H}$, there exists a unique solution of systems (4.4)-(4.6) such that $(w(\cdot,t),w_t(\cdot,t))^T\in\mathcal{C}(\tau,\infty;\mathcal{H})$, $(\varepsilon(\cdot,s),\varepsilon_s(\cdot,s))^T\in\mathcal{C}(0,t-\tau;\mathcal{H})$, $(\varepsilon(\cdot,s,t),\varepsilon_s(\cdot,s,t))^T\in\mathcal{C}([t-\tau,t]\times[\tau,\infty);\mathcal{H})$ for any $(\widehat{w}_0-w_0,\widehat{w}_1-w_1)^T\in D(\mathbb{B})$, where $\mathbb{B}$ is defined by (3.6), system (4.4) decays exponentially in the sense that*

$$
\begin{aligned}
&\|(w(\cdot,t),w_t(\cdot,t))^T\|_{\mathcal{H}}\\
&\le M_0e^{-\omega_0(t-\tau)}\|(w_0,w_1)^T\|_{\mathcal{H}}\\
&\quad+\frac{L_0CMM_0e^{\omega_0\tau}}{\sqrt{2\omega}}\Big(e^{-\frac{\omega_0t}{2}}+e^{\omega\tau}\cdot e^{-\frac{\omega t}{2}}\Big)\|\mathbb{B}\big(\varepsilon(\cdot,0),\varepsilon_s(\cdot,0)\big)^T\|_{\mathcal{H}}.
\end{aligned}
\tag{4.7}
$$

Proof. For any $(w_0,w_1)^T\in\mathcal{H}$, $(\widehat{w}_0,\widehat{w}_1)^T\in\mathcal{H}$, since $\mathbb{B}$ defined by (3.6) generates an exponentially stable C_0-semigroup on $\mathcal{H}$, there is a unique solution $(\varepsilon(\cdot,s),\varepsilon_s(\cdot,s))^T\in\mathcal{C}(0,t-\tau;\mathcal{H})$ to (4.5) such that (3.8) holds.

Now, for any given time $t>\tau$, write (4.6) as

$$
\frac{d}{ds}\begin{pmatrix}\varepsilon(\cdot,s,t)\\ \varepsilon_s(\cdot,s,t)\end{pmatrix}=\mathbb{A}\begin{pmatrix}\varepsilon(\cdot,s,t)\\ \varepsilon_s(\cdot,s,t)\end{pmatrix},
\tag{4.8}
$$

where $\mathbb{A}$ is defined by

$$
\begin{gathered}
\mathbb{A}(f,g)^T=(g,-\frac{1}{\rho(x)}(EI(x)f'')'')^T,\\
D(\mathbb{A})=\{(f,g)^T\in(H^4(0,1)\cap H_E^2(0,1))\times H_E^2(0,1):f''(1)=(EIf'')'(1)=0\}.
\end{gathered}
\tag{4.9}
$$

Then $\mathbb{A}$ is skew-adjoint in $\mathcal{H}$ and hence generates a conservative C_0-semigroup on $\mathcal{H}$. For any $(\varepsilon(\cdot,t-\tau),\varepsilon_s(\cdot,t-\tau))^T\in\mathcal{H}$ that is determined by (4.5), there exists a unique solution to (4.6) such that

$$
\|(\varepsilon(\cdot,s,t),\varepsilon_s(\cdot,s,t))^T\|_{\mathcal{H}}=\|(\varepsilon(\cdot,t-\tau),\varepsilon_s(\cdot,t-\tau))^T\|_{\mathcal{H}},
\tag{4.10}
$$

for all $s\in[t-\tau,t]$. So, $(\varepsilon(\cdot,s,t),\varepsilon_s(\cdot,s,t))\in\mathcal{C}([t-\tau,t]\times[\tau,\infty);\mathcal{H})$. Moreover, since $\mathbb{A}$ is skew-adjoint with compact resolvent, the solution of (4.6) can be, in terms of s, represented as

$$
\begin{pmatrix}\varepsilon(x,s,t)\\ \varepsilon_s(x,s,t)\end{pmatrix}=\sum_{n=0}^{\infty}a_n(t)e^{\lambda_ns}\begin{pmatrix}\frac{1}{\lambda_n}\phi_n(x)\\ \phi_n(x)\end{pmatrix}+\sum_{n=0}^{\infty}b_n(t)e^{-\lambda_ns}\begin{pmatrix}-\frac{1}{\lambda_n}\phi_n(x)\\ \phi_n(x)\end{pmatrix}
\tag{4.11}
$$

where $(\pm\frac{1}{\lambda}\phi(x),\phi(x))$ is a sequence of all ω-linearly independent approximated normalized orthogonal eigenfunctions of $\mathbb{A}$ corresponding to eigenvalues $\pm\lambda$ satisfies:

$$
\begin{gathered}
\phi^{(4)}(x)+\frac{2EI'(x)}{EI(x)}\phi'''(x)+\frac{EI''(x)}{EI(x)}\phi''(x)+\lambda^2\frac{\rho(x)}{EI(x)}\phi(x)=0,\\
\phi(0)=\phi'(0)=\phi''(1)=\phi'''(1)=0.
\end{gathered}
\tag{4.12}
$$

Set $h=\int_0^1(\frac{\rho(\tau)}{EI(\tau)})^{1/4}d\tau$ and $\lambda_n=\beta_n^2/h^2$, then from the reference [12], when n is large enough the solutions of the equations above can be represented as

$$\begin{aligned}\beta_n&=\frac{1}{\sqrt{2}}(n+\frac12)\pi(1+i)+\mathcal{O}(\frac1n),\\ \phi_n(x)&=e^{-\frac14\int_0^z a(\tau)d\tau}\sqrt{2}(i-1)\Big[\sin\Big((n+\frac12)\pi z\Big)-\cos\Big((n+\frac12)\pi z\Big)\\ &\quad+e^{-(n+\frac12)\pi z}+(-1)^n e^{-(n+\frac12)\pi(1-z)}\Big]+\mathcal{O}(\frac1n),\\ \beta_n^{-2}\phi_n''(x)&=\frac{1}{h^2}\Big(\frac{\rho(x)}{EI(x)}\Big)^{1/2}e^{-\frac14\int_0^z a(\tau)d\tau}\sqrt{2}(1+i)\Big[\cos\Big((n+\frac12)\pi z\Big)\\ &\quad-\sin\Big((n+\frac12)\pi z\Big)+e^{-(n+\frac12)\pi z}+(-1)^n e^{-(n+\frac12)\pi(1-z)}\Big]\\ &\quad+\mathcal{O}(\frac1n).\end{aligned}\tag{4.13}$$

From (4.11),

$$\varepsilon_s(1,t,t)=\sum_{n=0}^{\infty}[a_n(t)e^{\lambda_n t}+b_n(t)e^{-\lambda_n t}]\phi_n(1).\tag{4.14}$$

For (4.6) we have

$$\begin{aligned}&l_n a_n(t)e^{\lambda_n(t-\tau)}\\ &=\Big\langle\begin{pmatrix}\varepsilon(\cdot,t-\tau)\\ \varepsilon_s(\cdot,t-\tau)\end{pmatrix},\begin{pmatrix}\frac{1}{\lambda_n}\phi_n(\cdot)\\ \phi_n(\cdot)\end{pmatrix}\Big\rangle_{\mathcal{H}}\\ &=\frac{1}{\lambda_n}\Big\langle\begin{pmatrix}\varepsilon(\cdot,t-\tau)\\ \varepsilon_s(\cdot,t-\tau)\end{pmatrix},\mathbb{A}\begin{pmatrix}\frac{1}{\lambda_n}\phi_n(\cdot)\\ \phi_n(\cdot)\end{pmatrix}\Big\rangle_{\mathcal{H}}\\ &=\frac{1}{\lambda_n}\Big\langle\begin{pmatrix}\varepsilon(\cdot,t-\tau)\\ \varepsilon_s(\cdot,t-\tau)\end{pmatrix},\begin{pmatrix}\phi_n(\cdot)\\ -\frac{1}{\lambda_n\rho(\cdot)}(EI(\cdot)\phi_n''(\cdot))''\end{pmatrix}\Big\rangle_{\mathcal{H}}\\ &=\frac{1}{\lambda_n}\Big[\int_0^1 EI(x)\varepsilon_{xx}(x,t-\tau)\phi_n''(x)dx-\frac{1}{\lambda_n}\int_0^1\varepsilon_s(x,t-\tau)(EI(x)\phi_n''(x))''dx\Big]\\ &=\frac{1}{\lambda_n}\Big[-\int_0^1(EI(x)\varepsilon_{xx}(x,t-\tau))_x\phi_n'(x)dx+\frac{1}{\lambda_n}\int_0^1\varepsilon_{sx}(x,t-\tau)(EI(x)\phi_n''(x))'dx\Big]\\ &=\frac{1}{\lambda_n}\Big[-(EI(x)\varepsilon_{xx})_x(1,t-\tau)\phi_n(1)+\int_0^1(EI(x)\varepsilon_{xx}(x,t-\tau))_{xx}\phi_n(x)dx\\ &\quad-\frac{1}{\lambda_n}\int_0^1\varepsilon_{sxx}(x,t-\tau)EI(x)\phi_n''(x)dx\Big]\\ &=\frac{1}{\lambda_n}\Big[-k_1\varepsilon_s(1,t-\tau)\phi_n(1)+\int_0^1(EI(x)\varepsilon_{xx}(x,t-\tau))_{xx}\phi_n(x)dx\\ &\quad-\frac{1}{\lambda_n}\int_0^1\varepsilon_{sxx}(x,t-\tau)EI(x)\phi_n''(x)dx\Big]\\ &=\frac{1}{\lambda_n}\Big\{-k_1\varepsilon_s(1,t-\tau)\phi_n(1)+\int_0^1(EI(x)\varepsilon_{xx}(x,t-\tau))_{xx}\Big[e^{-\frac14\int_0^z a(\tau)d\tau}\sqrt{2}(i-1)\\ &\quad\times\Big(\sin\big((n+\frac12)\pi z\big)-\cos\big((n+\frac12)\pi z\big)+e^{-(n+\frac12)\pi z}+(-1)^n e^{-(n+\frac12)\pi(1-z)}\Big)\Big]dx\end{aligned}$$

$$
\begin{aligned}
&-\int_0^1 \varepsilon_{sxx}(x,t-\tau)\sqrt{EI(x)}\sqrt{\rho(x)}e^{-\frac{1}{4}\int_0^z a(\tau)d\tau}\sqrt{2}(1+i)\Big[\cos\big((n+\frac{1}{2})\pi z\big)\\
&-\sin\big((n+\frac{1}{2})\pi z\big)+e^{-(n+\frac{1}{2})\pi z}+(-1)^n e^{-(n+\frac{1}{2})\pi(1-z)}\Big]dx+\mathcal{O}(\frac{1}{n})\Big\}.
\end{aligned}
$$

By the expression of $\phi_n(x)$, there exists a constant $c_0>0$ such that

$$
|\phi_n(1)|\le c_0. \tag{4.15}
$$

Notice that

$$
\begin{aligned}
|\varepsilon_s(1,t-\tau)| &= |\int_0^1 \varepsilon_{sx}(x,t-\tau)dx| = |\int_0^1\int_0^x \varepsilon_{sxx}(y,t-\tau)dydx|\\
&\le \int_0^1 \Big[\int_0^x \varepsilon_{sxx}^2(y,t-\tau)dy\Big]^{1/2}dx \le \Big[\int_0^1 \varepsilon_{sxx}^2(x,t-\tau)dx\Big]^{1/2}\\
&\le \frac{1}{m}\Big[\int_0^1 EI(x)\varepsilon_{sxx}^2(x,t-\tau)dx\Big]^{1/2}
\end{aligned} \tag{4.16}
$$

where $m=min_{(0\le x\le 1)}\{EI(x)\}$. Then

$$
\begin{aligned}
|l_n a_n(t)| &\le \frac{1}{|\lambda_n|}\Big\{c_0k_1|\varepsilon_s(1,t-\tau)|+8\Big[\int_0^1 \rho(x)(EI(x)\varepsilon_{xx}(x,t-\tau))_{xx}^2dx\Big]^{1/2}\\
&\quad\times\Big(\int_0^1 \frac{1}{\rho(x)}dx\Big)^{1/2}+8\Big[\int_0^1 EI(x)\varepsilon_{sxx}^2(x,t-\tau)dx\Big]^{1/2}\int_0^1\rho(x)dx\Big\}\\
&\le \frac{1}{|\lambda_n|}\Big[\frac{c_0k_1}{m}+8\Big(\int_0^1\frac{1}{\rho(x)}dx\Big)^{1/2}+8\int_0^1\rho(x)dx\Big]\\
&\quad\times\|\mathbb{B}(\varepsilon(\cdot,t-\tau),\varepsilon_s(\cdot,t-\tau))^T\|_{\mathcal{H}}.
\end{aligned} \tag{4.17}
$$

Similarly,

$$
\begin{aligned}
&l_nb_n(t)e^{-\lambda_n(t-\tau)}\\
&=\Big\langle\begin{pmatrix}\varepsilon(\cdot,t-\tau)\\ \varepsilon_s(\cdot,t-\tau)\end{pmatrix},\begin{pmatrix}-\frac{1}{\lambda_n}\phi_n(\cdot)\\ \phi_n(\cdot)\end{pmatrix}\Big\rangle_{\mathcal{H}}\\
&=\frac{1}{\lambda_n}\Big\langle\begin{pmatrix}\varepsilon(\cdot,t-\tau)\\ \varepsilon_s(\cdot,t-\tau)\end{pmatrix},\mathbb{A}\begin{pmatrix}-\frac{1}{\lambda_n}\phi_n(\cdot)\\ \phi_n(\cdot)\end{pmatrix}\Big\rangle_{\mathcal{H}}\\
&=\frac{1}{\lambda_n}\Big\langle\begin{pmatrix}\varepsilon(\cdot,t-\tau)\\ \varepsilon_s(\cdot,t-\tau)\end{pmatrix},\begin{pmatrix}\phi_n(\cdot)\\ \frac{1}{\lambda_n\rho(\cdot)}(EI(\cdot)\phi_n''(\cdot))''\end{pmatrix}\Big\rangle_{\mathcal{H}}\\
&=\frac{1}{\lambda_n}\Big[\int_0^1 EI(x)\varepsilon_{xx}(x,t-\tau)\phi_n''(x)dx+\frac{1}{\lambda_n}\int_0^1\varepsilon_s(x,t-\tau)(EI(x)\phi_n''(x))''dx\Big]\\
&=\frac{1}{\lambda_n}\Big[-\int_0^1(EI(x)\varepsilon_{xx}(x,t-\tau))d\phi_n(x)+\frac{1}{\lambda_n}\int_0^1\varepsilon_{sxx}(x,t-\tau)EI(x)\phi_n''(x)dx\Big]\\
&=\frac{1}{\lambda_n}\Big[-(EI(x)\varepsilon_{xx})_x(1,t-\tau)\phi_n(1)+\int_0^1(EI(x)\varepsilon_{xx}(x,t-\tau))_{xx}\phi_n(x)dx\\
&\quad+\frac{1}{\lambda_n}\int_0^1\varepsilon_{sxx}(x,t-\tau)EI(x)\phi_n''(x)dx\Big],\\
&=\frac{1}{\lambda_n}\Big\{-k_1\varepsilon_s(1,t-\tau)\phi_n(1)+\int_0^1(EI(x)\varepsilon_{xx}(x,t-\tau))_{xx}\Big[e^{-\frac{1}{4}\int_0^z a(\tau)d\tau}\sqrt{2}(i-1)
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big(\sin\big((n+\frac{1}{2})\pi z\big) - \cos\big((n+\frac{1}{2})\pi z\big) + e^{-(n+\frac{1}{2})\pi z} + (-1)^n e^{-(n+\frac{1}{2})\pi(1-z)}\Big)\Big] dx \\
&+ \int_0^1 \varepsilon_{sxx}(x,t-\tau)\sqrt{EI(x)}\sqrt{\rho(x)} e^{-\frac{1}{4}\int_0^z a(\tau)d\tau}\sqrt{2}(1+i)\Big[\cos\big((n+\frac{1}{2})\pi z\big) \\
&- \sin\big((n+\frac{1}{2})\pi z\big) + e^{-(n+\frac{1}{2})\pi z} + (-1)^n e^{-(n+\frac{1}{2})\pi(1-z)}\Big] dx + \mathcal{O}\big(\frac{1}{n}\big)\Big\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
|l_n b_n(t)| &\le \frac{1}{|\lambda_n|}\Big\{c_0 k_1 |\varepsilon_s(1,t-\tau)| + 8\Big[\int_0^1 \rho(x)(EI(x)\varepsilon_{xx}(x,t-\tau))_{xx}^2 dx\Big]^{1/2} \\
&\quad \times \Big(\int_0^1 \frac{1}{\rho(x)} dx\Big)^{1/2} + 8\Big[\int_0^1 EI(x)\varepsilon_{sxx}^2(x,t-\tau)dx\Big]^{1/2}\int_0^1 \rho(x)dx\Big\} \\
&\le \frac{1}{|\lambda_n|}\Big[\frac{c_0 k_1}{m} + 8\Big(\int_0^1 \frac{1}{\rho(x)}dx\Big)^{1/2} + 8\int_0^1 \rho(x)dx\Big] \\
&\quad \times \|\mathbb{B}(\varepsilon(\cdot,t-\tau),\varepsilon_s(\cdot,t-\tau))^T\|_{\mathcal{H}}.
\end{aligned}
\tag{4.18}
$$

Collecting (4.14), (4.17), (4.18), and the expression of λ_n gives

$$
|\varepsilon_s(1,t,t)| \le C\|\mathbb{B}(\varepsilon(\cdot,t-\tau),\varepsilon_s(\cdot,t-\tau))^T\|_{\mathcal{H}} \tag{4.19}
$$

for some constant $C > 0$ independent of t. Now by (3.8) and C_0-semigroup theory, we have

$$
\|\mathbb{B}(\varepsilon(\cdot,t-\tau),\varepsilon_s(\cdot,t-\tau))\|_{\mathcal{H}} \le Me^{-\omega(t-\tau)}\|\mathbb{B}(\varepsilon(\cdot,0),\varepsilon_s(\cdot,0))^T\|_{\mathcal{H}} \tag{4.20}
$$

for any $t \in [\tau,+\infty)$, where M,ω are given by (3.7). We finally get

$$
|\varepsilon_s(1,t,t)| \le CMe^{-\omega(t-\tau)}\|\mathbb{B}(\varepsilon(\cdot,0),\varepsilon_s(\cdot,0))^T\|_{\mathcal{H}}. \tag{4.21}
$$

Furthermore, the equation (4.4) can be written as

$$
\frac{d}{dt}\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t)\end{pmatrix} = \mathcal{A}_0\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t)\end{pmatrix} + \mathcal{B}_0\varepsilon_s(1,t,t) \tag{4.22}
$$

where

$$
\begin{gathered}
\mathcal{A}_0(f,g)^T = (g, -\frac{1}{\rho(x)}(EI(x)f'')'')^T, \\
\forall (f,g)^T \in D(\mathcal{A}_0) = \big\{(f,g)^T \in (H^4(0,1)\cap H_E^2(0,1)) \times H_E^2(0,1) : \\
f''(1) = 0, (EIf'')'(1) = k_2 g(1)\big\}, \\
\mathcal{B}_0 = \begin{pmatrix} 0 \\ \delta(x-1)\end{pmatrix}.
\end{gathered}
\tag{4.23}
$$

A direct computation shows that

$$
\mathcal{B}_0\mathcal{A}_0^{-1}(f,g)^T = f(1), \quad \forall (f,g)^T \in \mathcal{H}, \tag{4.24}
$$

which means $\mathcal{B}_0{\mathcal{A}_0}^{-1}$ is bounded.

For the energy $E_0(t)$ of the system (4.4), simple computations tells us that

$$
\dot{E}_0(t) = -k_2 w_t^2(1,t), \tag{4.25}
$$

which shows that

$$
k_2\int_0^T |w_t(1,t)|^2 dt \le E_0(0), \tag{4.26}
$$

for any $T > 0$. This inequality together with (4.24) illustrates that $\mathcal{B}_0$ is admissible for $e^{\mathcal{A}_0 t}$. Therefore, there exists a unique solution to (4.22) such that $(w(\cdot,t), w_t(\cdot,t))^T \in \mathcal{C}(\tau,\infty;\mathcal{H})$. Since $\mathcal{A}_0$ generates an exponentially stable C_0-semigroup, it follows from [24, Proposition 2.5] and (4.21) that

$$\begin{aligned}\|\int_\tau^{t/2} e^{\mathcal{A}_0(t/2-s)}\mathcal{B}_0\varepsilon_s(1,s,s)ds\|_{\mathcal{H}} &\le L_0\|\varepsilon_s(1,\cdot,\cdot)\|_{L^2(\tau,t/2)} \\ &\le \frac{L_0 CM}{\sqrt{2\omega}}\|\mathbb{B}(\varepsilon(\cdot,0),\varepsilon_s(\cdot,0))^T\|_{\mathcal{H}},\end{aligned}$$

and

$$\begin{aligned}\|\int_{t/2}^{t} e^{\mathcal{A}_0(t-s)}\mathcal{B}_0\varepsilon_s(1,s,s)ds\|_{\mathcal{H}} &\le \|\int_0^{t} e^{\mathcal{A}_0(t-s)}\mathcal{B}_0(0 \underset{t/2}{\diamondsuit} \varepsilon_s(1,s,s))ds\|_{\mathcal{H}} \\ &\le L_0\|\varepsilon_s(1,\cdot,\cdot)\|_{L^2(t/2,t)} \\ &\le \frac{L_0 CM e^{\omega\tau}e^{-\frac{\omega t}{2}}}{\sqrt{2\omega}}\|\mathbb{B}(\varepsilon(\cdot,0),\varepsilon_s(\cdot,0))^T\|_{\mathcal{H}}, \quad \forall t\ge 0,\end{aligned}$$

for some constant $L_0 > 0$ that is independent of $\varepsilon_s(1,t,t)$, and

$$(u \underset{\tau}{\diamondsuit} v)(t) = \begin{cases} u(t), & 0\le t\le \tau,\\ v(t), & t>\tau.\end{cases}$$

On the other hand, the solutions of the systems (4.22) can be represented as

$$\begin{aligned}&(w(\cdot,t), w_t(\cdot,t))^T \\ &= e^{\mathcal{A}_0(t-\tau)}(w(\cdot,\tau),w_t(\cdot,\tau))^T + \int_\tau^t e^{\mathcal{A}_0(t-s-\tau)}\mathcal{B}_0\varepsilon_s(1,s,s)ds \\ &= e^{\mathcal{A}_0(t-\tau)}(w(\cdot,\tau),w_t(\cdot,\tau))^T + e^{\mathcal{A}_0(t/2-\tau)}\int_\tau^{t/2} e^{\mathcal{A}_0(t/2-s)}\mathcal{B}_0\varepsilon_s(1,s,s)ds \\ &\quad + e^{-\mathcal{A}_0\tau}\int_{t/2}^{t} e^{\mathcal{A}_0(t-s)}\mathcal{B}_0\varepsilon_s(1,s,s)ds.\end{aligned} \tag{4.27}$$

Since $\mathcal{A}_0$ generates an exponentially stable C_0-semigroup, there exists two positive constants M_0,ω_0 such that $\|e^{\mathcal{A}_0 t}\| \le M_0 e^{-\omega_0 t}$, which together with (4.27) and the conservative property of the system (4.2) for $u^*(t) = 0$ lead to

$$\begin{aligned}&\|(w(\cdot,t),w_t(\cdot,t))^T\|_{\mathcal{H}} \\ &\le M_0 e^{-\omega_0(t-\tau)}\|(w(\cdot,\tau),w_t(\cdot,\tau))^T\|_{\mathcal{H}} \\ &\quad + \frac{L_0 CMM_0 e^{\omega_0\tau}}{\sqrt{2\omega}}\big(e^{-\frac{\omega_0 t}{2}} + e^{\omega\tau}e^{-\frac{\omega t}{2}}\big)\|\mathbb{B}(\varepsilon(\cdot,0),\varepsilon_s(\cdot,0))^T\|_{\mathcal{H}} \\ &= M_0 e^{-\omega_0(t-\tau)}\|(w(\cdot,0),w_t(\cdot,0))^T\|_{\mathcal{H}} \\ &\quad + \frac{L_0 CMM_0 e^{\omega_0\tau}}{\sqrt{2\omega}}\big(e^{-\frac{\omega_0 t}{2}} + e^{\omega\tau}\cdot e^{-\frac{\omega t}{2}}\big)\|\mathbb{B}\big(\varepsilon(\cdot,0),\varepsilon_s(\cdot,0)\big)^T\|_{\mathcal{H}}.\end{aligned}$$

□

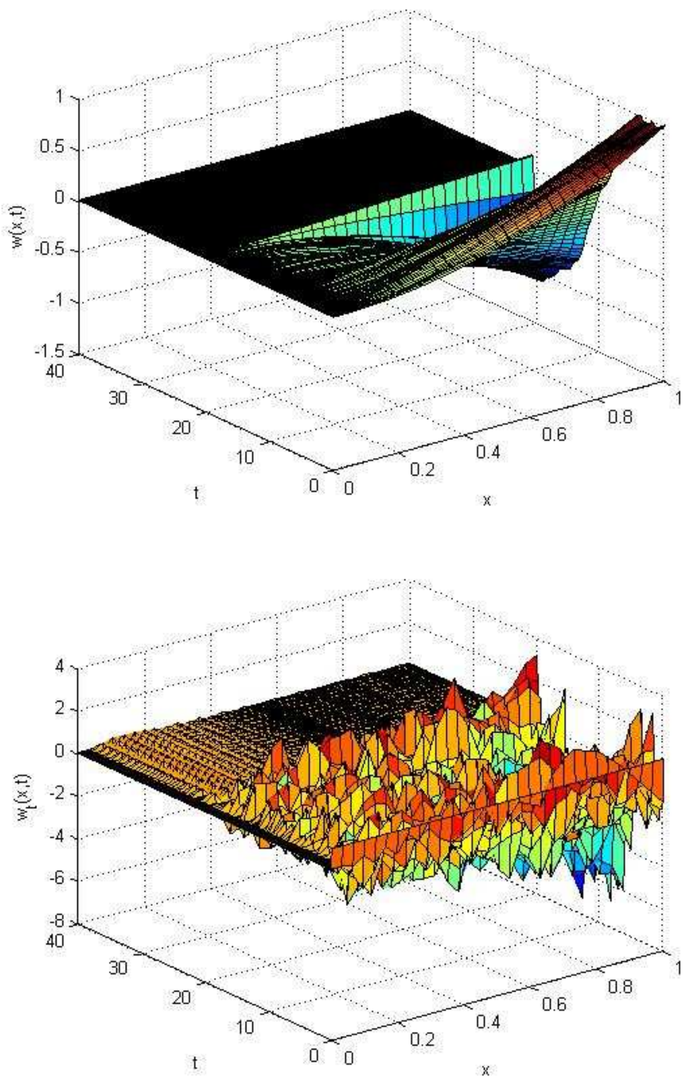

FIGURE 1. Displacement $w(x,t)$ (top), and velocity $w_t(x,t)$ (bottom) of the solution

5. Simulation results

In this section, using the finite difference method we present the numerical simulation for the closed-loop system (4.4)-(4.6). Here we choose the space grid size $N = 30$, time step $dt = 0.0003$ and time span $[0, 40]$. Parameters and coefficients respectively are chosen to be $\tau = k_1 = k_2 = 1, \rho(x) = 1 + 0.2\sin(x), EI(x) = 1 + 0.2\cos(x)$. For the initial values:

$$w_0(x) = x^2, \quad w_1(x) = 1,$$

$$\varepsilon(x,0) = x^2, \quad \varepsilon_s(x,0) = 1, \quad \forall x \in [0,1],$$

the displacement $w(x,t)$ and velocity $w_t(x,t)$ are plotted in Figure 1. It shows clearly that the system is very stable with small displacement under time-variable coefficients. This simple simulation illustrates that the observer-predictor based scheme is useful to make the unstable system exponentially stable for the Euler-Bernoulli beam equation with variable coefficients.

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