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GLOBAL SOLUTIONS FOR 2D COUPLED BURGERS-COMPLEX-GINZBURG-LANDAU EQUATIONS

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ABSTRACT. In this article, we study the periodic initial-value problem of the 2D coupled Burgers-complex-Ginzburg-Landau (Burgers-CGL) equations. Applying the Brezis-Gallout inequality which is available in 2D case and establishing some prior estimates, we obtain the existence and uniqueness of a global solution under certain conditions.

1. Introduction

In this article, we are concerned with the periodic initial-value problem for the following 2D coupled Burgers-complex-Ginzburg-Landau (Burgers-CGL) equations:

$$\begin{gathered} P_t = \xi P + (1+iu)\Delta P - (1+iv)|P|^2 P - \nabla P \cdot \nabla Q - rP\Delta Q + f(x), \\ (x,t) \in \Omega \times \mathbb{R}^+, \end{gathered} \tag{1.1}$$

$$Q_t = m\Delta Q - \frac{1}{2}|\nabla Q|^2 - \omega|P|^2 + g(x), \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.2}$$

$$P(x_i + 2L, t) = P(x_i, t), Q(x_i + 2L, t) = Q(x_i, t), \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.3}$$

$$P(x,0) = P_0(x), \quad Q(x,0) = Q_0(x), \quad x \in \Omega, \tag{1.4}$$

where $\Omega = [-L, L] \times [-L, L]$, $2L$ is the period, $f(x)$ and $g(x)$ are given real functions. The complex function $P(x,t)$ is the rescaled amplitude of the flame oscillations, the real function $Q(x,t)$ is the deformation of the first front. The Landau coefficients u, v are real and the coefficients ξ, m are positive, while the coupling coefficients $\omega > 0$ and $r = r_1 + ir_2$.

The coupled Burgers-CGL equations was derived from the nonlinear evolution of the coupled long-scale oscillatory and monotonic instabilities of a uniformly propagating combustion wave governed by a sequential chemical reaction, having two flame fronts corresponding to two reaction zones with a finite separation distance between them (see [2, 9, 12, 16, 17, 18]). It describes the interaction of the excited oscillatory mode and the damped monotonic mode, that is $\xi > 0, m > 0$.

Let us recall some previous works which are related to our results. If there are no terms involved Q in (1.1), then (1.1) reduces to the well-known complex Ginzburg-Landau equation (CGL) that describes the weakly nonlinear evolution of

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a long-scale instability (see [15]). Many results on the well-posedness and global attractors for the CGL equation have been established, one can find the details in [6, 7, 8, 13, 14, 19]. And if taking the coupling coefficient $\omega = 0$, (1.1) would be the well-known Burgers equation (see [3]). There are also many works concerning the Burgers equation, one can see [4, 20] and so on. So far, the mathematical analysis and physical study about the coupled Burgers-CGL have been done by a few researchers. For the periodic initial-value problem of the 1-D Burgers-Ginzburg-Landau equation, well-posedness and global attractors are obtained in [10]. By some prior estimates and so-called continuity method, the authors in [11] firstly showed the global existence of solutions and attractors to 1-D coupled Burgers-CGL equations for flames governed by sequential reaction, and then obtained the existence of the global attractor $\mathcal{A} \subset H^2(\Omega) \times H^3(\Omega)$ for the problem. In addition, they established the estimates of the upper bounds of Hausdorff and fractal dimensions for the attractors. However, to our knowledge, there is few work on the 2D coupled Burgers-CGL equations. We will prove global well-posedness of problem (1.1)-(1.4) as conjectured in [11]. The method used here is well established but need to be applied skillfully to get the desired result.

The purpose of this article is to study the global solutions of (1.1)-(1.4). The existence and uniqueness of the global solutions to (1.1)-(1.4) is obtained by virtue of the Brezis-Gallout inequality, which is available in the 2D case. The main result in our paper is as follows.

Theorem 1.1. *Suppose that $P_0(x) \in H^2(\Omega)$, $Q_0(x) \in H^3(\Omega)$, $f(x) \in H^1(\Omega)$ and $g(x) \in H^2(\Omega)$, for any given $T > 0$, then* (1.1)-(1.4) *has the unique solution $P(x,t)$ and $Q(x,t)$ satisfying*

$$P(x,t) \in L^\infty\big(0,T; H^2(\Omega)\big), \quad P_t(x,t) \in L^\infty\big(0,T; L^2(\Omega)\big),$$
$$Q(x,t) \in L^\infty\big(0,T; H^3(\Omega)\big), \quad Q_t(x,t) \in L^\infty\big(0,T; H^1(\Omega)\big).$$

The rest of this article is organized as follows. In section 2, we briefly give some preliminaries. Some prior estimates for the solutions to (1.1)-(1.2) are established in section 3. Then, by the standard method, we can extend the local solutions to global solutions and subsequently the proof of Theorem 1.1 is presented.

2. Preliminaries

In this section, We firstly recall some concepts and conventional notation. Let $W^{k,p}(\Omega)$ denote the usual kth order Sobolev space with its norm

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \Big(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p \,\mathrm{d}x \Big)^{1/p}, & 1 \le p < \infty, \\ \sum_{|\alpha|\le k} \operatorname{ess\,sup}_\Omega |D^\alpha u|, & p = \infty. \end{cases}$$

When $p = 2$, we denote $\|u\|_{H^k(\Omega)} = \|u\|_{W^{k,2}(\Omega)}$. The inner product in $L^2(\Omega)$ is defined by

$$(u,v) = \mathrm{Re} \int_\Omega u(x)\bar{v}(x) dx.$$

For simplicity, we write $\|u\| = \|u\|_{L^2}$. Without any ambiguity, we denote a generic positive constant by C which may vary from line to line. The following inequalities play an important role in the proof of the main results in $\mathbb{R}^2$.

Lemma 2.1 (Gagliardo-Nirenberg inequality [5]). *Let Ω be a bounded domain with $\partial\Omega$ in C^m, and let u be any function in $W^{m,r}(\Omega)\cap L^q(\Omega)$, $1\le q,r\le\infty$. For any integer j, $0\le j\le m$, and for any number a satisfying $j/m\le a\le 1$, set*

$$\frac{1}{p}=\frac{j}{n}+a\Big(\frac{1}{r}-\frac{m}{n}\Big)+(1-a)\frac{1}{q}.$$

If $m-j-n/r$ is not a non-negative integer, then

$$\|D^j u\|_{L^p}\le C\|u\|^a_{W^{m,r}}\|u\|^{1-a}_{L^q}. \tag{2.1}$$

If $m-j-n/r$ is a non-negative integer, then (2.1) *holds for $a=j/m$. The constant C depends only on Ω,r,q,j and a.*

Lemma 2.2 (Brezis-Gallout inequality [1]). *Let Ω be a bounded domain in $\mathbb{R}^2$, and let u be any function in $H^2(\Omega)$. If $\|u\|_{H^1}\le 1$, then there holds*

$$\|u\|_{L^\infty}\le C\big(1+\sqrt{\log(1+\|u\|_{H^2})}\big). \tag{2.2}$$

3. Existence and uniqueness of solutions

In this section, we firstly establish some prior estimates for the solutions to (1.1)-(1.4) which guarantee the existence of the global solutions.

Lemma 3.1. *Assume that $g(x)\in L^2(\Omega)$, then for the problem* (1.1)-(1.4)*, it holds that*

$$\frac{d}{\mathrm{d}t}(\|Q\|^2)\le E_0\big(1+\log(1+\|P\|_{H^2}+\|Q\|_{H^3})\big)\big(\|P\|^2_{H^2}+\|Q\|^2_{H^3}\big)+E_1,$$

where E_0,E_1 are constants.

Proof. Multiplying (1.2) by Q and integrating with respect to x over Ω gives

$$\frac{d}{\mathrm{d}t}\|Q\|^2=-2m\|\nabla Q\|^2-\int_\Omega|\nabla Q|^2Q\,\mathrm{d}x-2\omega\int_\Omega|P|^2Q\,\mathrm{d}x+2\int_\Omega gQ\,\mathrm{d}x. \tag{3.1}$$

Applying the Brezis-Gallout inequality (Lemma 2.2) and Hölder inequality, we deduce that

$$\begin{aligned}\big|-\int_\Omega|\nabla Q|^2Q\,\mathrm{d}x\big|&\le\|Q\|_{L^\infty}\|\nabla Q\|^2\\&\le C\Big(1+\sqrt{\log(1+\|Q\|_{H^2})}\Big)\|\nabla Q\|^2\\&\le C\left(1+\log(1+\|Q\|_{H^2})\right)\|Q\|^2_{H^1}\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\big|-2\omega\int_\Omega|P|^2Q\,\mathrm{d}x\big|&\le 2\omega\|Q\|_{L^\infty}\|P\|^2\\&\le C\left(1+\sqrt{\log(1+\|Q\|_{H^2})}\right)\|P\|^2\\&\le C\left(1+\log(1+\|Q\|_{H^2})\right)\|P\|^2.\end{aligned} \tag{3.3}$$

Since $g(x)\in L^2(\Omega)$, obviously

$$\big|2\int_\Omega gQ\,\mathrm{d}x\big|\le\|Q\|^2+\|g\|^2\le\|Q\|^2+C_1. \tag{3.4}$$

Substituting (3.2)-(3.4) back into (3.1), we infer that

$$\begin{aligned}\frac{d}{\mathrm{d}t}\|Q\|^2 &\le -2m\|\nabla Q\|^2 + C\left(1+\log(1+\|Q\|_{H^2})\right)\|Q\|_{H^1}^2 \\ &\quad + C\left(1+\log(1+\|Q\|_{H^2})\right)\|P\|^2 + \|Q\|^2 + C_1 \\ &\le C(1+\log(1+\|P\|_{H^2}+\|Q\|_{H^3}))\left(\|P\|_{H^2}^2+\|Q\|_{H^3}^2\right) + C_1 \\ &= E_0\left(1+\log(1+\|P\|_{H^2}+\|Q\|_{H^3})\right)\left(\|P\|_{H^2}^2+\|Q\|_{H^3}^2\right) + E_1.\end{aligned} \tag{3.5}$$

Thus, the proof is complete. □

Lemma 3.2. *Assume that $f(x) \in L^2(\Omega)$ and $g(x) \in L^2(\Omega)$, then for (1.1)-(1.4), we have*

$$\frac{d}{\mathrm{d}t}\left(\|P\|^2+\|\nabla Q\|^2\right) \le E_2\left(1+\log(1+\|P\|_{H^2}+\|Q\|_{H^3})\right)\left(\|P\|_{H^2}^2+\|Q\|_{H^3}^2\right)+E_3,$$

where E_2, E_3 are constants.

Proof. By differentiating (1.2) with respect to x and setting

$$W = \nabla Q, \tag{3.6}$$

Equations (1.1)-(1.2) become

$$P_t = \xi P + (1+iu)\Delta P - (1+iv)\mid P\mid^2 P - \nabla P\cdot W - rP\operatorname{div}W + f, \tag{3.7}$$

$$W_t = m\nabla(\operatorname{div}W) - \frac{1}{2}\nabla(W^2) - \omega\nabla(|P|^2) + \nabla g. \tag{3.8}$$

Multiplying (3.7) by $\overline{P}$, integrating with respect to x over Ω and taking the real part gives

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\|P\|^2 &= \xi\|P\|^2 - \|\nabla P\|^2 - \|P\|_{L^4}^4 - \operatorname{Re}\int_\Omega \nabla P\cdot W\overline{P}\,\mathrm{d}x \\ &\quad - \operatorname{Re}\int_\Omega r|P|^2\operatorname{div}W\,\mathrm{d}x + \operatorname{Re}\int_\Omega f\overline{P}\,\mathrm{d}x,\end{aligned} \tag{3.9}$$

where

$$-\operatorname{Re}\int_\Omega \nabla P\cdot W\overline{P}\,\mathrm{d}x = -\frac{1}{2}\int_\Omega W\cdot\nabla(|P|^2)\,\mathrm{d}x = \frac{1}{2}\int_\Omega |P|^2\operatorname{div}W\,\mathrm{d}x. \tag{3.10}$$

While multiplying (3.8) by W and integrating over Ω yields

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\|W\|^2 &= -m\|\operatorname{div}W\|^2 - \frac{1}{2}\int_\Omega \nabla(W^2)\cdot W\,\mathrm{d}x + \int_\Omega \nabla g\cdot W\,\mathrm{d}x \\ &\quad - \omega\int_\Omega \nabla(|P|^2)\cdot W\,\mathrm{d}x,\end{aligned} \tag{3.11}$$

where

$$-\omega\int_\Omega \nabla(|P|^2)\cdot W\,\mathrm{d}x = \omega\int_\Omega |P|^2\operatorname{div}W\,\mathrm{d}x. \tag{3.12}$$

Combining (3.9) with (3.12), we arrive at

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}(\|P\|^2+\|W\|^2)\\
&=2\xi\|P\|^2-2\|\nabla P\|^2-2m\|\operatorname{div}W\|^2-2\|P\|_{L^4}^4\,\mathrm{d}x\\
&\quad+(2\omega+1-2r_1)\int_\Omega |P|^2\operatorname{div}W\,\mathrm{d}x-\int_\Omega \nabla(W^2)\cdot W\,\mathrm{d}x\\
&\quad+2\operatorname{Re}\int_\Omega f\overline{P}\,\mathrm{d}x+2\int_\Omega \nabla g\cdot W\,\mathrm{d}x=:\sum_{i=1}^5 I_i,
\end{aligned}\tag{3.13}$$

where

$$\begin{gathered}
I_1=2\xi\,\|P\|^2-2\|\nabla P\|^2-2m\|\operatorname{div}W\|^2-2\|P\|_{L^4}^4\,,\\
I_2=(2\omega+1-2r_1)\int_\Omega |P|^2\operatorname{div}W\,\mathrm{d}x,\quad I_3=-\int_\Omega \nabla(W^2)\cdot W\,\mathrm{d}x,\\
I_4=2\operatorname{Re}\int_\Omega f\overline{P}\,\mathrm{d}x,\quad I_5=2\int_\Omega \nabla g\cdot W\,\mathrm{d}x.
\end{gathered}\tag{3.14}$$

We now estimate each term on the right-hand side of (3.13). By the Brezis-Gallout inequality and Hölder inequality, I_2 and I_3 can be estimated as follows

$$\begin{aligned}
|I_2|&\le |2\omega+1-2r_1|\|P\|_{L^\infty}^2|\Omega|^{1/2}\|\operatorname{div}W\|\\
&\le \frac{m}{2}\|\operatorname{div}W\|^2+C\|P\|_{L^\infty}^4\\
&\le \frac{m}{2}\|\operatorname{div}W\|^2+C(1+\sqrt{\log(1+\|P\|_{H^2})})^4\\
&\le \frac{m}{2}\|\operatorname{div}W\|^2+C(1+\log^2(1+\|P\|_{H^2}))\\
&\le \frac{m}{2}\|\operatorname{div}W\|^2+C\|P\|_{H^2}^2+C\\
&\le C(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2}))(\|P\|_{H^2}^2+\|W\|_{H^2}^2)\\
&\quad+\frac{m}{2}\|\operatorname{div}W\|^2+C
\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}
|I_3|&\le \|W\|_{L^\infty}\|W\|\|\operatorname{div}W\|\le \frac{m}{2}\|\operatorname{div}W\|^2+C\|W\|_{L^\infty}^2\|W\|^2\\
&\le \frac{m}{2}\|\operatorname{div}W\|^2+C(1+\log(1+\|W\|_{H^2}))\|W\|^2\\
&\le C(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2}))(\|P\|_{H^2}^2+\|W\|_{H^2}^2)+\frac{m}{2}\|\operatorname{div}W\|^2.
\end{aligned}\tag{3.16}$$

For I_4 and I_5, one can deduce that

$$\begin{aligned}
|I_4|&\le \|P\|^2+\|f\|^2\le \|P\|_{H^2}+C_2\\
&\le (1+\log(1+\|P\|_{H^2}+\|W\|_{H^2}))\big(\|P\|_{H^2}^2+\|W\|_{H^2}^2\big)+C_2
\end{aligned}\tag{3.17}$$

and

$$|I_5|\le 2\|g\|\|\operatorname{div}W\|\le m\|\operatorname{div}W\|^2+C_3,\tag{3.18}$$

provided $f(x) \in L^2(\Omega)$ and $g(x) \in L^2(\Omega)$. Substituting the above estimates (3.15)–(3.18) back into (3.13) leads to

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\left(\|P\|^2+\|W\|^2\right)\\
&\le 2\xi\|P\|^2-2\|\nabla P\|^2-2\int_\Omega |P|^4\,\mathrm{d}x+C\big(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\big)\\
&\quad\times\left(\|P\|_{H^2}^2+\|W\|_{H^2}^2\right)+C+C_2+C_3\\
&=E_2\left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right)\left(\|P\|_{H^2}^2+\|W\|_{H^2}^2\right)+E_3.
\end{aligned} \tag{3.19}$$

Noting (3.6), we finally obtain

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\left(\|P\|^2+\|\nabla Q\|^2\right)\\
&\le E_2\left(1+\log(1+\|P\|_{H^2}+\|Q\|_{H^3})\right)\left(\|P\|_{H^2}^2+\|Q\|_{H^3}^2\right)+E_3,
\end{aligned} \tag{3.20}$$

which is the desired result. □

To obtain the higher order estimates with the complex amplitude of the flame oscillations $P(x,t)$ and the deformation of the first front $Q(x,t)$, we can use the transformed equations (3.7)-(3.8) and have the following lemmas.

Lemma 3.3. *Assume that $f(x) \in L^2(\Omega$ and $g(x) \in H^1(\Omega)$, then for the problem* (1.1)-(1.4)*, we have*

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\left(\|\nabla P\|^2+\|\Delta Q\|^2\right)\\
&\le E_4\left(1+\log(1+\|P\|_{H^2}+\|Q\|_{H^3})\right)\left(\|P\|_{H^2}^2+\|Q\|_{H^3}^2\right)+E_5,
\end{aligned}$$

where E_4, E_5 are constants.

Proof. Multiplying (3.7) by $(-\overline{\Delta P})$, integrating with respect to x over Ω and taking the real part, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{\mathrm{d}t}\|\nabla P\|^2 &= \xi\|\nabla P\|^2-\|\Delta P\|^2+\mathrm{Re}\int_\Omega (1+iv)|P|^2P\,\overline{\Delta P}\,\mathrm{d}x\\
&\quad+\mathrm{Re}\int_\Omega \nabla P\cdot W\overline{\Delta P}\,\mathrm{d}x+\mathrm{Re}\int_\Omega rP\operatorname{div}W\overline{\Delta P}\,\mathrm{d}x\\
&\quad-\mathrm{Re}\int_\Omega f\overline{\Delta P}\,\mathrm{d}x.
\end{aligned} \tag{3.21}$$

On the other hand, multiplying (3.8) by $(-\nabla(\operatorname{div}W))$ and integrating with respect to x over Ω yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{\mathrm{d}t}\|\operatorname{div}W\|^2 &= -m\|\nabla(\operatorname{div}W)\|^2+\frac{1}{2}\int_\Omega \nabla(W^2)\cdot\nabla(\operatorname{div}W)\,\mathrm{d}x\\
&\quad+\omega\int_\Omega \nabla(|P|^2)\cdot\nabla(\operatorname{div}W)\,\mathrm{d}x-\int_\Omega \nabla g\cdot\nabla(\operatorname{div}W)\,\mathrm{d}x.
\end{aligned} \tag{3.22}$$

Adding (3.22) and (3.21), one has

$$
\begin{aligned}
&\frac{d}{\mathrm{d}t}\left(\|\nabla P\|^2+\|\operatorname{div} W\|^2\right)\\
&=2\xi\|\nabla P\|^2-2\|\Delta P\|^2-2m\|\nabla(\operatorname{div} W)\|^2\\
&\quad+2\operatorname{Re}\int_\Omega(1+iv)|P|^2P\overline{\Delta P}\,\mathrm{d}x+2\operatorname{Re}\int_\Omega\nabla P\cdot W\overline{\Delta P}\,\mathrm{d}x\\
&\quad+2\operatorname{Re}\int_\Omega rP\operatorname{div} W\overline{\Delta P}\,\mathrm{d}x+\int_\Omega\nabla(W^2)\cdot\nabla(\operatorname{div} W)\,\mathrm{d}x\\
&\quad+2\omega\int_\Omega\nabla(|P|^2)\cdot\nabla(\operatorname{div} W)\,\mathrm{d}x-2\operatorname{Re}\int_\Omega f\overline{\Delta P}\,\mathrm{d}x\\
&\quad-2\int_\Omega\nabla g\cdot\nabla(\operatorname{div} W)\,\mathrm{d}x=\sum_{i=1}^{8}J_i,
\end{aligned}
\tag{3.23}
$$

where

$$
\begin{gathered}
J_1=2\xi\|\nabla P\|^2-2\|\Delta P\|^2-2m\|\nabla(\operatorname{div} W)\|^2,\\
J_2=2\operatorname{Re}\int_\Omega(1+iv)|P|^2P\overline{\Delta P}\,\mathrm{d}x,\quad J_3=\operatorname{Re}\int_\Omega\nabla P\cdot W\overline{\Delta P}\,\mathrm{d}x,\\
J_4=2\operatorname{Re}\int_\Omega rP\operatorname{div} W\overline{\Delta P}\,\mathrm{d}x,\quad J_5=\int_\Omega\nabla(W^2)\cdot\nabla(\operatorname{div} W)\,\mathrm{d}x,\\
J_6=\omega\int_\Omega\nabla(|P|^2)\cdot\nabla(\operatorname{div} W)\,\mathrm{d}x,\quad J_7=-2\operatorname{Re}\int_\Omega f\overline{\Delta P}\,\mathrm{d}x,\\
J_8=-2\int_\Omega\nabla g\cdot\nabla(\operatorname{div} W)\,\mathrm{d}x.
\end{gathered}
\tag{3.24}
$$

We now estimate J_i $(i=2,\dots,8)$ in (3.23). From Lemma 2.2, it follows that

$$
\begin{aligned}
|J_2|&\le 2|1+iv|\|P\|_{L^\infty}^2\|P\|\|\Delta P\|\le C\|P\|_{L^\infty}^2\|P\|_{H^2}^2\\
&\le C\left(1+\log(1+\|P\|_{H^2})\right)\|P\|_{H^2}^2\\
&\le C\left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right)\left(\|P\|_{H^2}^2+\|W\|_{H^2}^2\right),
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
|J_3|&\le 2\|W\|_{L^\infty}\|\nabla P\|\|\Delta P\|\le 2\|W\|_{L^\infty}\|P\|_{H^2}^2\\
&\le C(1+\log(1+\|W\|_{H^2}))\|P\|_{H^2}^2\\
&\le C(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2}))(\|P\|_{H^2}^2+\|W\|_{H^2}^2)
\end{aligned}
\tag{3.26}
$$

and

$$
\begin{aligned}
|J_4|&\le 2|r|\|P\|_{L^\infty}\|\operatorname{div} W\|\|\Delta P\|\le C\|P\|_{L^\infty}^2\|\Delta P\|^2+\|\operatorname{div} W\|^2\\
&\le C\left(1+\sqrt{\log(1+\|P\|_{H^2})}\right)^2\|\Delta P\|^2+\|\operatorname{div} W\|^2\\
&\le C\left(1+\log(1+\|P\|_{H^2})\right)\|P\|_{H^2}+C\|W\|_{H^2}\\
&\le C\left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right)\left(\|P\|_{H^2}^2+\|W\|_{H^2}^2\right),
\end{aligned}
\tag{3.27}
$$

where we used the Brezis-Gallout and Hölder inequalities. By direct computations, we have

$$
\begin{aligned}
&|J_5| \\
&\le 2\|W\|_{L^\infty}\|\nabla W\|\|\nabla(\operatorname{div} W)\| \le \frac{1}{m}\|W\|_{L^\infty}^2\|\nabla W\|^2 + m\|\nabla(\operatorname{div} W)\|^2 \\
&\le C\left(1+\log(1+\|W\|_{H^2})\right)\|W\|_{H^2}^2 + m\|\nabla(\operatorname{div} W)\|^2 \\
&\le C\left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right)\left(\|P\|_{H^2}^2+\|W\|_{H^2}^2\right) + m\|\nabla(\operatorname{div} W)\|^2
\end{aligned} \tag{3.28}
$$

and

$$
\begin{aligned}
|J_6| &\le 4\omega\|P\|_{L^\infty}\|\nabla P\|\|\nabla(\operatorname{div} W)\| \le \frac{8\omega^2}{m}\|P\|_{L^\infty}^2\|\nabla P\|^2 \\
&\quad + \frac{m}{2}\|\nabla(\operatorname{div} W)\|^2 \\
&\le C\left(1+\log(1+\|P\|_{H^2})\right)\|P\|_{H^2}^2 + \frac{m}{2}\|\nabla(\operatorname{div} W)\|^2 \\
&\le C\left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right)\left(\|P\|_{H^2}^2+\|W\|_{H^2}^2\right) \\
&\quad + \frac{m}{2}\|\nabla(\operatorname{div} W)\|^2.
\end{aligned} \tag{3.29}
$$

In addition, J_7 and J_8 can be estimated as follows

$$
\begin{aligned}
|J_7| &\le \|\Delta P\|^2 + \|f\|^2 \le \|P\|_{H^2}^2 + C_4 \\
&\le \left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right)\left(\|P\|_{H^2}^2+\|W\|_{H^2}^2\right) + C_4
\end{aligned} \tag{3.30}
$$

and

$$
\begin{aligned}
|J_8| &\le 2\|\nabla g\|\|\nabla(\operatorname{div} W)\| \le \frac{m}{2}\|\nabla(\operatorname{div} W)\|^2 + \frac{2}{m}\|\nabla g\|^2 \\
&\le \frac{m}{2}\|\nabla(\operatorname{div} W)\|^2 + C_5.
\end{aligned} \tag{3.31}
$$

Substituting the above estimate (3.25)-(3.31) into (3.23) leads to

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla P\|^2 + \|\operatorname{div} W\|^2) \\
&\le 2\xi\|\nabla P\|^2 - 2\|\Delta P\|^2 + (C+1)\left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right) \\
&\quad \times (\|P\|_{H^2}^2+\|W\|_{H^2}^2) + C_4 + C_5 \\
&\le E_4(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2}))(\|P\|_{H^2}^2+\|W\|_{H^2}^2) + E_5.
\end{aligned} \tag{3.32}
$$

From the transformation (3.6), we finally obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla P\|^2 + \|\Delta Q\|^2) \\
&\le E_4(1+\log(1+\|P\|_{H^2}+\|Q\|_{H^3}))(\|P\|_{H^2}^2+\|Q\|_{H^3}^2) + E_5,
\end{aligned} \tag{3.33}
$$

which completes the proof of this lemma. □

Lemma 3.4. *Assume that $f(x) \in H^1(\Omega)$ and $g(x) \in H^2(\Omega)$, then for the problem (1.1)-(1.4), we have*

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\Delta P\|^2 + \|\nabla\Delta Q\|^2\right) \\
&\le E_6\left(1+\log(1+\|P\|_{H^2}+\|Q\|_{H^3})\right)\left(\|P\|_{H^2}^2+\|Q\|_{H^3}^2\right) + E_7,
\end{aligned}
$$

where E_6 and E_7 are constants.

Proof. Multiplying (3.7) by $\overline{\Delta^2 P}$, integrating with respect to x over Ω and taking the real part gives

$$\begin{aligned} \frac{1}{2}\frac{d}{\mathrm{d}t}\|\Delta P\|^2 &= \xi\|\Delta P\|^2 - \|\nabla\Delta P\|^2 - \operatorname{Re}\int_\Omega (1+iv)|P|^2 P\overline{\Delta^2 P}\,\mathrm{d}x \\ &\quad - \operatorname{Re}\int_\Omega \nabla P\cdot W\overline{\Delta^2 P}\,\mathrm{d}x - \operatorname{Re}\int_\Omega rP\operatorname{div}W\overline{\Delta^2 P}\,\mathrm{d}x \\ &\quad + \operatorname{Re}\int_\Omega f\overline{\Delta^2 P}\,\mathrm{d}x. \end{aligned} \tag{3.34}$$

Taking the inner product of (3.8) with $\nabla\Delta(\operatorname{div}W)$ over Ω, one has

$$\begin{aligned} &\frac{1}{2}\frac{d}{\mathrm{d}t}\|\nabla(\operatorname{div}W)\|^2 \\ &= -m\|\Delta(\operatorname{div}W)\|^2 - \frac{1}{2}\int_\Omega \nabla(W^2)\cdot\nabla\Delta(\operatorname{div}W)\,\mathrm{d}x \\ &\quad - \omega\int_\Omega \nabla(|P|^2)\cdot\nabla\Delta(\operatorname{div}W)\,\mathrm{d}x + \int_\Omega \nabla g\cdot\nabla\Delta(\operatorname{div}W)\,\mathrm{d}x. \end{aligned} \tag{3.35}$$

Adding (3.35) and (3.34), we have

$$\begin{aligned} &\frac{d}{\mathrm{d}t}(\|\Delta P\|^2 + \|\nabla(\operatorname{div}W)\|^2) \\ &= 2\xi\|\Delta P\|^2 - 2\|\nabla\Delta P\|^2 - 2m\|\Delta(\operatorname{div}W)\|^2 \\ &\quad - 2\operatorname{Re}\int_\Omega (1+iv)|P|^2P\overline{\Delta^2 P}\,\mathrm{d}x - 2\operatorname{Re}\int_\Omega \nabla P\cdot W\overline{\Delta^2 P}\,\mathrm{d} \\ &\quad - 2\operatorname{Re}\int_\Omega rP\operatorname{div}W\overline{\Delta^2 P}\,\mathrm{d}x - \int_\Omega \nabla(W^2)\cdot\nabla\Delta(\operatorname{div}W)\,\mathrm{d}x \\ &\quad - 2\omega\int_\Omega \nabla(|P|^2)\cdot\nabla\Delta(\operatorname{div}W)\,\mathrm{d}x + 2\operatorname{Re}\int_\Omega f\overline{\Delta^2 P}\,\mathrm{d}x \\ &\quad + 2\int_\Omega \nabla g\cdot\nabla\Delta(\operatorname{div}W)\,\mathrm{d}x =: \sum_{i=1}^{8} K_i, \end{aligned} \tag{3.36}$$

where

$$\begin{gathered} K_1 = 2\xi\|\Delta P\|^2 - 2\|\nabla\Delta P\|^2 - 2m\|\Delta(\operatorname{div}W)\|^2 \\ K_2 = -2\operatorname{Re}\int_\Omega (1+iv)|P|^2P\overline{\Delta^2 P}\,\mathrm{d}x, \quad K_3 = -2\operatorname{Re}\int_\Omega \nabla P\cdot W\overline{\Delta^2 P}\,\mathrm{d}, \\ K_4 = -2\operatorname{Re}\int_\Omega rP\operatorname{div}W\overline{\Delta^2 P}\,\mathrm{d}x, \quad K_5 = -\int_\Omega \nabla(W^2)\cdot\nabla\Delta(\operatorname{div}W)\,\mathrm{d}x \\ K_6 = -2\omega\int_\Omega \nabla(|P|^2)\cdot\nabla\Delta(\operatorname{div}W)\,\mathrm{d}x, \quad K_7 = 2\operatorname{Re}\int_\Omega f\overline{\Delta^2 P}\,\mathrm{d}x, \\ K_8 = 2\int_\Omega \nabla g\cdot\nabla\Delta(\operatorname{div}W)\,\mathrm{d}x. \end{gathered} \tag{3.37}$$

Next we analyze the each integrand in (3.36). It follows from Lemmas 2.1 2.2 that

$$\begin{aligned} |K_2| &\le 6|1+iv|\|P\|_{L^\infty}^2\|\nabla P\|\|\nabla\Delta P\| \le \frac{1}{2}\|\nabla\Delta P\|^2 + C\|P\|_{L^\infty}^4\|\nabla P\|^2 \\ &\le \frac{1}{2}\|\nabla\Delta P\|^2 + C\left(1+\log^2(1+\|P\|_{H^2})\right)\|P\|\|\Delta P\| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\|\nabla\Delta P\|^2 + C(1+\|P\|_{H^2})\|P\|\|P\|_{H^2}\\
&\le \frac{1}{2}\|\nabla\Delta P\|^2 + C\|P\|_{H^2}^2 + C|\Omega|^{\frac{1}{2}}\|P\|_{L^\infty}\|P\|_{H^2}^2\\
&\le \frac{1}{2}\|\nabla\Delta P\|^2 + C\left(1+\log(1+\|P\|_{H^2})\right)\|P\|_{H^2}^2\\
&\le C\left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right)\left(\|P\|_{H^2}^2+\|W\|_{H^2}^2\right)\\
&\quad + \frac{1}{2}\|\nabla\Delta P\|^2.
\end{aligned}
\tag{3.38}
$$

With respect to the term K_3, by direct computation, one has

$$
\begin{aligned}
|K_3| &= \Big|2\,\mathrm{Re}\int_\Omega (D^2P\cdot W\cdot\overline{\nabla\Delta P} + \nabla P\cdot\nabla W\cdot\overline{\nabla\Delta P})\,\mathrm{d}x\Big|\\
&\le 2\|W\|_{L^\infty}\|D^2P\|\|\nabla\Delta P\| + \|\nabla P\|_{L^4}\|\nabla W\|_{L^4}\|\nabla\Delta P\|\\
&\le \frac{1}{2}\|\nabla\Delta P\|^2 + 4\|W\|_{L^\infty}^2\|D^2P\|^2 + \|\nabla P\|_{L^4}^2\|\nabla W\|_{L^4}^2\\
&\le \frac{1}{2}\|\nabla\Delta P\|^2 + 4\|W\|_{L^\infty}^2\|D^2P\|^2 + \frac{1}{2}\|\nabla P\|_{L^4}^4 + \frac{1}{2}\|\nabla W\|_{L^4}^4.
\end{aligned}
\tag{3.39}
$$

Applying the Gagliardo-Nirenberg inequality, it is sufficient to see that

$$
\|\nabla P\|_{L^4}^4 \le \|P\|^2\|\Delta P\|^2,
\tag{3.40}
$$

which enable us to estimate K_3 as

$$
\begin{aligned}
&|K_3|\\
&\le \frac{1}{2}\|\nabla\Delta P\|^2 + 4\|W\|_{L^\infty}^2\|D^2P\|^2 + \frac{1}{2}\|P\|_{H^2}^2\|P\|_{L^\infty}^2 + \frac{1}{2}\|W\|_{H^2}^2\|W\|_{L^\infty}^2\\
&\le \frac{1}{2}\|\nabla\Delta P\|^2 + C\left(1+\log(1+\|W\|_{H^2})\right)\|P\|_{H^2}^2\\
&\quad + C\left(1+\log(1+\|P\|_{H^2})\right)\|P\|_{H^2}^2 + C\left(1+\log(1+\|W\|_{H^2})\right)\|W\|_{H^2}^2\\
&\le C\left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right)\left(\|P\|_{H^2}^2+\|W\|_{H^2}^2\right) + \frac{1}{2}\|\nabla\Delta P\|^2.
\end{aligned}
\tag{3.41}
$$

Similarly to (3.41), we have

$$
\begin{aligned}
&|K_4|\\
&= \big|2\,\mathrm{Re}\int_\Omega r(\operatorname{div}W\nabla P\cdot\overline{\nabla\Delta P} + P\nabla(\operatorname{div}W)\cdot\overline{\nabla\Delta P})\,\mathrm{d}x\big|\\
&\le 2|r|\|\operatorname{div}W\|_{L^4}\|\nabla P\|_{L^4}\|\nabla\Delta P\| + 2|r|\|P\|_{L^\infty}\|\nabla(\operatorname{div}W)\|\|\nabla\Delta P\|\\
&\le \frac{1}{2}\|\nabla\Delta P\|^2 + C\|\nabla W\|_{L^4}^2\|\nabla P\|_{L^4}^2 + C\|P\|_{L^\infty}^2\|\nabla(\operatorname{div}W)\|^2\\
&\le \frac{1}{2}\|\nabla\Delta P\|^2 + C\|\nabla W\|_{L^4}^4 + C\|\nabla P\|_{L^4}^4 + C\|P\|_{L^\infty}^2\|W\|_{H^2}^2\\
&\le \frac{1}{2}\|\nabla\Delta P\|^2 + C\|W\|_{H^2}^2\|W\|_{L^\infty}^2 + C\|P\|_{H^2}^2\|P\|_{L^\infty}^2 + C\|P\|_{L^\infty}^2\|W\|_{H^2}^2\\
&\le C\left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right)\left(\|P\|_{H^2}^2+\|W\|_{H^2}^2\right) + \frac{1}{2}\|\nabla\Delta P\|^2.
\end{aligned}
\tag{3.42}
$$

To estimate K_5, we denote $W=(W^1,W^2)$ for simplicity. Note that

$$
\begin{aligned}
&\big|\int_\Omega \Delta(W^2)\Delta(\operatorname{div} W)\,\mathrm{d}x\big| \\
&= \big|\int_\Omega (2W^1_{x_1}W^1_{x_1}+2W^1W^1_{x_1x_1}+2W^2_{x_1}W^2_{x_1}+2W^2W^2_{x_1x_1} \\
&\quad +2W^1_{x_2}W^1_{x_2}+2W^1W^1_{x_2x_2}+2W^2_{x_2}W^2_{x_2}+2W^2W^2_{x_2x_2})\Delta(\operatorname{div} W)\,\mathrm{d}x\big| \\
&\le \big|\int_\Omega (2W^1_{x_1}W^1_{x_1}+2W^2_{x_1}W^2_{x_1}+2W^1_{x_2}W^1_{x_2}+2W^2_{x_2}W^2_{x_2})\Delta(\operatorname{div} W)\,\mathrm{d}x\big| \\
&\quad +\big|\int_\Omega (2W^1W^1_{x_1x_1}+2W^2W^2_{x_1x_1}+2W^1W^1_{x_2x_2} \\
&\quad +2W^2W^2_{x_2x_2})\Delta(\operatorname{div} W)\,\mathrm{d}x\big| \\
&\le 8\|\nabla W\|^2_{L^4}\|\Delta(\operatorname{div} W)\|+8\|W\|_{L^\infty}\|D^2W\|\|\Delta(\operatorname{div} W)\|.
\end{aligned}
\tag{3.43}
$$

By (3.43), we find that

$$
\begin{aligned}
|K_5| &= \big|\int_\Omega \Delta(W^2)\Delta(\operatorname{div} W)\,\mathrm{d}x\big| \\
&\le 8\|\nabla W\|^2_{L^4}\|\Delta(\operatorname{div} W)\|+8\|W\|_{L^\infty}\|D^2W\|\|\Delta(\operatorname{div} W)\| \\
&\le \frac{m}{3}\|\Delta(\operatorname{div} W)\|^2+\frac{96}{m}\|W\|^2_{H^2}\|W\|^2_{L^\infty}+\frac{96}{m}\|W\|^2_{L^\infty}\|W\|^2_{H^2} \\
&\le C\left(1+\log(1+\|W\|_{H^2})\right)\|W\|^2_{H^2}+\frac{m}{3}\|\Delta(\operatorname{div} W)\|^2 \\
&\le C\left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right)\left(\|P\|^2_{H^2}+\|W\|^2_{H^2}\right) \\
&\quad +\frac{m}{3}\|\Delta(\operatorname{div} W)\|^2.
\end{aligned}
\tag{3.44}
$$

For the last three terms of (3.36), we have

$$
\begin{aligned}
|K_6| &= \big|2\omega\int_\Omega \left(\Delta P\overline{P}+2\nabla P\cdot\nabla\overline{P}+P\Delta\overline{P}\right)\Delta(\operatorname{div} W)\,\mathrm{d}x\big| \\
&\le 4\omega\|P\|_{L^\infty}\|\Delta P\|\|\Delta(\operatorname{div} W)\|+4\omega\|\nabla P\|^2_{L^4}\|\Delta(\operatorname{div} W)\| \\
&\le \frac{m}{3}\|\Delta(\operatorname{div} W)\|^2+C\|P\|^2_{L^\infty}\|P\|^2_{H^2}+C\|P\|^2_{H^2}\|P\|^2_{L^\infty} \\
&\le C\left(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2})\right)\left(\|P\|^2_{H^2}+\|W\|^2_{H^2}\right) \\
&\quad +\frac{m}{3}\|\Delta(\operatorname{div} W)\|^2,
\end{aligned}
\tag{3.45}
$$

$$
|K_7| \le 2\|\nabla f\|\|\nabla\Delta P\| \le \frac{1}{2}\|\nabla\Delta P\|^2+C_6 \tag{3.46}
$$

and

$$
|K_8| \le 2\|\Delta g\|\|\Delta(\operatorname{div} W)\| \le \frac{m}{3}\|\Delta(\operatorname{div} W)\|^2+C_7 \tag{3.47}
$$

because $f(x) \in H^1(\Omega)$ and $g(x) \in H^2(\Omega)$. Substituting (3.38)-(3.47) into (3.36), we obtain the following estimates

$$\begin{aligned}
&\frac{d}{dt}\big(\|\Delta P\|^2 + \|\nabla(\operatorname{div} W)\|^2\big) \\
&\le C\,(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2}))\big(\|P\|_{H^2}^2+\|W\|_{H^2}^2\big) + 2\xi\|\Delta P\|^2 \\
&\quad + C_6 + C_7 \\
&\le E_6\big(1+\log(1+\|P\|_{H^2}+\|W\|_{H^2}))(\|P\|_{H^2}^2+\|W\|_{H^2}^2\big) + E_7,
\end{aligned} \tag{3.48}$$

where $\|\Delta P\| \le \|P\|_{H^2}$ and $E_6 = 2\xi + C, E_7 = C_6 + C_7$. □

Under the conditions of Lemmas 3.1–3.4, we have the following result.

Lemma 3.5. *Assume that $f(x) \in H^1(\Omega)$ and $g(x) \in H^2(\Omega)$, then for the problem (1.1)-(1.4), there exist positive constants E, E', E'', E''' such that for any $T > 0$,*

$$\begin{gathered}
\|P\|_{H^2} \le E'' \exp\big(\frac{1}{2}e^{ET+E'(\|P_0\|_{H^2}+\|Q_0\|_{H^3})}\big), \\
\|Q\|_{H^3} \le E''' \exp\big(\frac{1}{2}e^{ET+E'(\|P_0\|_{H^2}+\|Q_0\|_{H^3})}\big).
\end{gathered}$$

Proof. From Lemmas 3.1–3.4, it is easy to show that

$$\begin{aligned}
&\frac{d}{dt}\big(\|Q\|^2+\|P\|^2+\|\nabla Q\|^2+\|\nabla P\|^2+\|\Delta Q\|^2+\|\Delta P\|^2+\|\nabla\Delta Q\|^2\big) \\
&\le (E_0+E_2+E_4+E_6)\,(1+\log(1+\|P\|_{H^2}+\|Q\|_{H^3}))\big(\|P\|_{H^2}^2+\|Q\|_{H^3}^2\big) \\
&\quad + E_3+E_5+E_7 \\
&= E_8\,(1+\log(1+\|P\|_{H^2}+\|Q\|_{H^3}))\big(\|P\|_{H^2}^2+\|Q\|_{H^3}^2\big) + E_9.
\end{aligned} \tag{3.49}$$

Denote

$$\begin{gathered}
\|Q\|^2+\|\nabla Q\|^2+\|\Delta Q\|^2+\|\nabla\Delta Q\|^2 = [Q]_{H^3}^2, \\
\|P\|^2+\|\nabla P\|^2+\|\Delta P\|^2 = [P]_{H^2}^2.
\end{gathered}$$

Inequality (3.49) becomes

$$\begin{aligned}
&\frac{d}{dt}([P]_{H^2}^2+[Q]_{H^3}^2) \\
&\le E_8(1+\log(1+\|P\|_{H^2}+\|Q\|_{H^3}))(\|P\|_{H^2}^2+\|Q\|_{H^3}^2) + E_9.
\end{aligned} \tag{3.50}$$

We always use the notation $\|\cdot\|_{H^s} = \big(\sum_{|\alpha|\le s}|\partial^\alpha|_{L^2}\big)^{1/2}$, thus $[\cdot]_{H^2}$ and $[\cdot]_{H^3}$ are equivalent norms to $\|\cdot\|_{H^2}$ and $\|\cdot\|_{H^3}$ respectively. Furthermore, there exists a positive constant C_8 such that $\|\cdot\|_{H^s} \le C_8[\cdot]_{H^s}$, $s = 2, 3$. Applying Cauchy inequality, from (3.50) it follows that

$$\begin{aligned}
&\frac{d}{dt}\big([P]_{H^2}^2+[Q]_{H^3}^2\big) \\
&\le E_8\,(1+\log(1+C_8[P]_{H^2}+C_8[Q]_{H^3}))\big(C_8^2[P]_{H^2}^2+C_8^2[Q]_{H^3}^2\big) + E_9 \\
&\le C_8^2E_8\big(1+\log(1+C_8^2+[P]_{H^2}^2+C_8^2+[Q]_{H^3}^2)\big)\big([P]_{H^2}^2+[Q]_{H^3}^2\big) + E_9 \\
&\le C_8^2E_8\big(1+\log(1+2C_8^2)\big)\big(1+\log(1+[P]_{H^2}^2+[Q]_{H^3}^2)\big)\big([P]_{H^2}^2+[Q]_{H^3}^2\big) \\
&\quad + E_9 \\
&= E_{10}\big(1+\log(1+[P]_{H^2}^2+[Q]_{H^3}^2)\big)\big([P]_{H^2}^2+[Q]_{H^3}^2\big) + E_9.
\end{aligned} \tag{3.51}$$

If there exists a positive constant N such that $[P]_{H^2}^2 + [Q]_{H^3}^2 \leq N$ for all $T > 0$, then Lemma 3.5 holds. Otherwise, there exists a positive constant E such that

$$\frac{d}{dt}([P]_{H^2}^2 + [Q]_{H^3}^2) \leq E([P]_{H^2}^2 + [Q]_{H^3}^2)\log([P]_{H^2}^2 + [Q]_{H^3}^2), \tag{3.52}$$

which implies that

$$\begin{aligned} \log\log([P]_{H^2}^2 + [Q]_{H^3}^2) &\leq ET + \log\log([P]_{H^2}^2(0) + [Q]_{H^3}^2(0)) \\ &\leq ET + E'(\|P_0\|_{H^2} + \|Q_0\|_{H^3}). \end{aligned} \tag{3.53}$$

Therefore,

$$\begin{gathered} [P]_{H^2} \leq \exp\Big(\frac{1}{2}e^{ET+E'(\|P_0\|_{H^2}+\|Q_0\|_{H^3})}\Big), \\ [Q]_{H^3} \leq \exp\Big(\frac{1}{2}e^{ET+E'(\|P_0\|_{H^2}+\|Q_0\|_{H^3})}\Big). \end{gathered}$$

Namely,

$$\begin{gathered} \|P\|_{H^2} \leq E'' \exp\Big(\frac{1}{2}e^{ET+E'(\|P_0\|_{H^2}+\|Q_0\|_{H^3})}\Big), \\ \|Q\|_{H^3} \leq E''' \exp\Big(\frac{1}{2}e^{ET+E'(\|P_0\|_{H^2}+\|Q_0\|_{H^3})}\Big). \end{gathered}$$

This proof is complete. □

Based on the previous lemmas, we are ready to prove the main result.

Proof of Theorem 1.1. From Lemmas 3.1–3.4, when $P_0(x) \in H^2(\Omega)$, $Q_0(x) \in H^3(\Omega)$, $f(x) \in H^1(\Omega)$, $g(x) \in H^2(\Omega)$ and for any $T > 0$, we obtain that

$$\begin{gathered} \|P\|_{H^2} \leq E'' \exp\Big(\frac{1}{2}e^{ET+E'(\|P_0\|_{H^2}+\|Q_0\|_{H^3})}\Big), \\ \|Q\|_{H^3} \leq E''' \exp\Big(\frac{1}{2}e^{ET+E'(\|P_0\|_{H^2}+\|Q_0\|_{H^3})}\Big), \end{gathered}$$

where the positive constant C depends only on $\|P_0\|_{H^2}$, $\|Q_0\|_{H^3}$ and T. Furthermore, by the standard method, we can extend the local solutions $P(x,t)$ and $Q(x,t)$ of the periodic initial value problem (1.1)-(1.4) to global solutions. Thus, the proof is complete. □

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