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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR PERTURBED KIRCHHOFF TYPE ELLIPTIC PROBLEMS WITH HARDY POTENTIAL

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Abstract. In this article, by using critical point theory, we show the existence of infinitely many weak solutions for a fourth-order Kirchhoff type elliptic problems with Hardy potential.

1. Introduction

This article concerns the existence of infinitely many weak solutions for the p-biharmonic equation with Hardy potential of Kirchhoff type

$$\begin{gathered} M\Big(\int_\Omega |\Delta u|^p dx\Big)\Delta_p^2 u - \frac{a}{|x|^{2p}}|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ containing the origin and with smooth boundary $\partial\Omega$, $1 < p < \frac{N}{2}$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is an operator of fourth order, the so-called p-biharmonic operator, λ, μ are two positive parameters, $M : [0, +\infty[\to \mathbb{R}$ is a continuous function, and $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions.

Kirchhoff [16] first introduced a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. After that, many authors studied the following nonlocal elliptic boundary value problem

$$\begin{gathered} -M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u(x) = f(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

Problems like this are called the Kirchhoff type problems. In recent years, many interesting results for problem of Kirchhoff type were obtained [1, 9, 13, 14, 17, 18,

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21]. Recently, using the variational methods, Graef, Heidarkham and Kong [12] studied the existence of at least three weak solutions to the Kirchhoff-type problem

$$\begin{gathered} -K\Big(\int_\Omega |\nabla u|^2dx\Big)\Delta u(x) = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.4}$$

In [7], using variational methods and critical point theory, Ferrara, Khademloo and Heidarkhani established the multiplicity results of nontrivial and nonnegative solutions for the following perturbed fourth-order Kirchhoff type elliptic problem

$$\begin{gathered} \Delta_p^2 u - [M(\int_\Omega |\nabla u|^p dx)]^{p-1}\Delta_p u + \rho|u|^{p-2}u = \lambda f(x,u) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.5}$$

On the other hand, singular elliptic problems have been intensively studied in recent years, see for example, [11, 10, 19] and the references. Ferrara and Molica Basic [8] studied the existence of solutions for the elliptic problem with Hardy potential

$$\begin{gathered} -\Delta_p u = \mu\frac{|u|^{p-2}u}{|x|^p} + \lambda f(x,u) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.6}$$

Huang and Liu [15] studied the sign-changing solutions for p-biharmonic equations with Hardy potential

$$\begin{gathered} \Delta_p^2 u - \frac{a}{|x|^{2p}}|u|^{p-2}u = f(x,u) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.7}$$

by using the method of invariant sets of descending flow.

Motivated by the papers [7, 8, 2, 3, 4, 12, 15], in this paper, we look for the existence of infinitely many solutions of problem (1.1). Precisely, under appropriate hypotheses on the nonlinear term f, g, the existence of two intervals Λ and J such that, for each $\lambda \in \Lambda$ and $\mu \in J$, BVP (1.1) admits a sequence of pairwise distinct solutions is proved. Our analysis is mainly based on a recent critical point theorem in [5].

This article is organized as follows. In section 2, we present some necessary preliminary facts that will be needed in the paper. In section 3, we establish our main two existence results.

Remark 1.1. If $M(\cdot) \equiv 1$, then Kirchhoff type problem (1.1) reduces to the p-biharmonic equation with Hardy potential

$$\begin{gathered} \Delta_p^2 u - \frac{a}{|x|^{2p}}|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u), \quad \text{in } \Omega \\ u = \Delta u = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

2. Preliminaries

Let X be the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \Big(\int_\Omega |\Delta u|^p dx\Big)^{1/p}.$$

We recall Rellich inequality [6], which says that

$$\int_\Omega \frac{|u(x)|^p}{|x|^{2p}}dx \le \frac{1}{H}\int_\Omega |\Delta u|^p dx, \tag{2.1}$$

where the best constant is

$$H = \Big(\frac{(p-1)N(N-2p)}{p^2}\Big)^p. \tag{2.2}$$

Define the functionals $\Phi, \Psi : X \to \mathbb{R}$ by

$$\begin{gathered}\Phi(u) = \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{a}{p}\int_\Omega \frac{|u(x)|^p}{|x|^{2p}}dx,\\ \Psi(u) = \int_\Omega \Big[F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))\Big]\,dx,\end{gathered} \tag{2.3}$$

where

$$\widehat{M}(t) = \int_0^t M(s)ds, \quad t \ge 0,$$
$$F(x,t) = \int_0^t f(x,\xi)d\xi, \quad G(x,t) = \int_0^t g(x,\xi)d\xi, \quad (x,t) \in \Omega \times \mathbb{R}.$$

In this article, we assume that the following condition holds,

(H1) $M : [0,+\infty[\to \mathbb{R}$ is a continuous function. And there are two positive constants m_0, m_1 such that

$$m_0 \le M(t) \le m_1, \quad \forall t \ge 0. \tag{2.4}$$

It is easy to show that the functionals Φ and Ψ are well defined and continuously Gateaux differentiable and whose derivative are

$$\begin{aligned}\Phi'(u)(v) = M\Big(&\int_\Omega |\Delta u(x)|^p dx\Big)\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)dx\\ &- a\int_\Omega \frac{|u(x)|^{p-2}}{|x|^{2p}}u(x)v(x)dx,\end{aligned} \tag{2.5}$$

and

$$\Psi'(u)(v) = \int_\Omega [f(x,u(x)) + \frac{\mu}{\lambda}g(x,u(x))]v(x)dx, \tag{2.6}$$

for every $u, v \in X$.

Set $p^* = \frac{pN}{N-p}$. By the Sobolev embedding theorem there exist a positive constant c such that

$$\|u\|_{L^{p^*}(\Omega)} \le c\|u\|, \quad \forall u \in X,$$

where

$$c := \pi^{-\frac{1}{2}}N^{-\frac{1}{p}}\Big(\frac{p-1}{N-p}\Big)^{1-\frac{1}{p}}\Big[\frac{\Gamma(1+\frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(N+1-\frac{N}{p})}\Big]^{1/N}, \tag{2.7}$$

see, for instance, [20]. Fixing $q \in [1, p^*)$, again from the Sobolev embedding theorem, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \le c_q\|u\|, \quad \forall u \in X. \tag{2.8}$$

Thus, the embedding $X \hookrightarrow L^q(\Omega)$ is compact. By (2.7), as a simple consequence of Hölder's inequality, one has the upper bound

$$c_q \leq \pi^{-\frac{1}{2}} N^{-1/p} \Big(\frac{p-1}{N-p}\Big)^{1-\frac{1}{p}} \Big[\frac{\Gamma(1+\frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(N+1-\frac{N}{p})}\Big]^{1/N} |\Omega|^{\frac{p^*-q}{p^* q}}, \tag{2.9}$$

where $|\Omega|$ denotes the Lebesgue measure of the open set Ω.

Our main tools is an infinitely many critical points theorem [5] which is recalled below.

Theorem 2.1. *Let X be a reflexive real Banach space; $\Phi, \Psi : X \to \mathbf{R}$ be two Gateaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$
$$\gamma = \liminf_{r \to +\infty} \varphi(r), \quad \delta = \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then, one has

- (i) *If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either the functional $\Phi - \lambda\Psi$ has a global minimum, or there exists a sequence $\{u_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*
- (ii) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either there exists a global minimum of Φ which is a local minimum of $\Phi - \lambda\Psi$, or there exists a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $\Phi - \lambda\Psi$, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.*

3. Main results

Pick $s > 0$ such that $B(0, s) \subset \Omega$, where $B(0, s)$ denotes the ball with center at 0 and radius of s. Let

$$L = \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_{\frac{s}{2}}^{s} \Big|\frac{12(N+1)}{s^3} r - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{r}\Big|^p r^{N-1} dr. \tag{3.1}$$

Theorem 3.1. *Suppose that* (H1) *and $0 < a < m_0 H$ hold (with H is as in* (2.2)*). Also assume*

- (H2) *$f \in C(\overline{\Omega} \times \mathbb{R})$, and $F(x, t) \geq 0$ for every $(x, t) \in \Omega \times [0, +\infty[$;*
- (H3) *There exists $s > 0$ as considered in* (3.1) *such that, if we put*

$$\alpha := \liminf_{t\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)} \leq t} \int_\Omega F(x, \xi) dx}{t^p}, \quad \beta := \limsup_{t\to+\infty} \frac{\int_{B(0,s/2)} F(x, t/h) dx}{t^p},$$

one has

$$\alpha < R\beta, \tag{3.2}$$

where $R = \frac{(m_0 H - a)h^p}{m_1 H L c_q^p}$ (constants $h > 1$, c_q and L are as in (2.8) *and* (3.1)*, respectively).*

Then, for every $\lambda \in \Lambda := \frac{m_0 H - a}{p H c_q^p} \Big] \frac{1}{R\beta}, \frac{1}{\alpha} \Big[$ and for every $g \in C(\overline{\Omega} \times \mathbb{R})$ such that

(H4) *$G(t,u) \geq 0$, for all $(t,u) \in \overline{\Omega} \times [0,+\infty[$, and*

$$G_\infty := \limsup_{t\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\leq t} \int_\Omega G(x,\xi)dx}{t^p},$$

if we put

$$\mu^* = \begin{cases} \frac{m_0H-a-pHc_q^p\alpha\lambda}{pHc_q^pG_\infty}, & G_\infty > 0, \\ +\infty, & G_\infty = 0, \end{cases} \tag{3.3}$$

then (1.1) *possesses an unbounded sequence of weak solutions in X for every $\mu \in J := [0, \mu_*[$.*

Proof. Our aim is to apply part (i) of Theorem 2.1. Let Φ, Ψ be the functionals defined in (2.3). From the above, we know that the Gateaux derivative of Φ and Ψ are given by (2.5) and (2.6), respectively. By (2.1), it follows that

$$\frac{m_0H-a}{pH}\|u\|^p \leq \Phi(u) \leq \frac{m_1}{p}\|u\|^p, \quad u \in X, \tag{3.4}$$

which implies that Φ is coercive. Moreover, from the weakly lower semicontinuity of norm, and the monotonicity and continuity of $\widehat{M}$, we known that Φ is sequentially weakly lower semicontinuous. The functional Ψ has compact derivative, hence it is sequentially weakly upper semicontinuous.

By (2.8) and (3.4), we obtain

$$\begin{aligned} \Phi^{-1}(]-\infty,r[) &= \{u \in X : \Phi(u) < r\} \\ &\subset \big\{u \in X : \frac{m_0H-a}{pH}\|u\|^p < r\big\} \\ &\subset \big\{u \in X : \|u\|_{L^q(\Omega)} < c_q\big(\frac{pHr}{m_0H-a}\big)^{1/p}\big\}. \end{aligned} \tag{3.5}$$

Note that $\Phi(0) = 0$ and $\Psi(0) = 0$. For every $r > 0$, we obtain by (3.5) that

$$\begin{aligned} \varphi(r) &= \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)} \\ &\leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)}{r} \\ &\leq \frac{\sup_{\|\xi\|_{L^q(\Omega)}\leq l} \int_\Omega F(x,\xi)dx}{r} + \frac{\mu}{\lambda}\frac{\sup_{\|\xi\|_{L^q(\Omega)}\leq l} \int_\Omega G(x,\xi)dx}{r}, \end{aligned}$$

where $l = c_q\big(\frac{pHr}{m_0H-a}\big)^{1/p}$.

Let $\{\sigma_n\}$ be a sequence of positive numbers such that $\sigma_n \to +\infty$ and

$$\begin{aligned} &\lim_{n\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\leq\sigma_n} \int_\Omega F(x,\xi)dx}{\sigma_n^p} \\ &= \liminf_{t\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\leq t} \int_\Omega F(x,\xi)dx}{t^p}. \end{aligned} \tag{3.6}$$

Let $r_n = \frac{m_0H-a}{pHc_q^p}\sigma_n^p$ for all $n \in \mathbb{N}$. From (H3), (H4) and (3.6), we obtain

$$\begin{aligned} \gamma &= \liminf_{r\to+\infty} \varphi(r) \le \liminf_{n\to+\infty} \varphi(r_n) \\ &\le \frac{pHc_q^p}{m_0H-a} \lim_{n\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\le\sigma_n} \int_\Omega F(x,\xi)dx}{\sigma_n^p} \\ &\quad + \frac{\mu}{\lambda}\frac{pHc_q^p}{m_0H-a} \limsup_{n\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\le\sigma_n} \int_\Omega G(x,\xi)dx}{\sigma_n^p} \\ &\le \frac{pHc_q^p}{m_0H-a}\Big(\alpha + \frac{\mu}{\lambda}G_\infty\Big) < +\infty. \end{aligned} \tag{3.7}$$

By (3.3) and (3.7), we easily check that

$$\gamma < \begin{cases} \frac{pHc_q^p}{m_0H-a}\big(\alpha + \frac{\mu^*}{\lambda}G_\infty\big) = \frac{1}{\lambda}, & G_\infty > 0, \\ \frac{pHc_q^p}{m_0H-a}\alpha < \frac{1}{\lambda}, & G_\infty = 0 \end{cases} \tag{3.8}$$

From the definition of Λ and (3.2), we have that $\Lambda \subset]0, \frac{1}{\gamma}[$.

In the following, we claim that the functional $\Phi - \lambda\Psi$ for $\lambda \in \Lambda$ is unbounded from below. Indeed, since $\frac{1}{\lambda} < \frac{pHc_q^p}{m_0H-a}R\beta = \frac{ph^p}{m_1L}\beta$, there exists a sequence $\{\tau_n\}$ of positive numbers and $\eta > 0$ such that $\tau_n \to +\infty$ and

$$\frac{1}{\lambda} < \eta < \frac{ph^p}{m_1L}\frac{\int_{B(0,s/2)} F(x,\tau_n/h)dx}{\tau_n^p}, \tag{3.9}$$

for n large enough.

Let $h > 1$ be as in R ((3.2)), we consider a sequence $\{w_n\}$ in X defined by setting

$$w_n(x) = \begin{cases} 0, & x \in \overline{\Omega} \setminus B(0,s), \\ \frac{\tau_n}{h}\left(\frac{4}{s^3}\rho^3 - \frac{12}{s^2}\rho^2 + \frac{9}{s}\rho - 1\right), & x \in B(0,s) \setminus B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}, & x \in B(0,\frac{s}{2}) \end{cases} \tag{3.10}$$

with $\rho = \operatorname{dist}(x,0) = \sqrt{\sum_{i=1}^N x_i^2}$. Clearly $w_n \in X$. A direct calculation shows

$$\frac{\partial w_n(x)}{\partial x_i} = \begin{cases} 0, & x \in (\overline{\Omega} \setminus B(0,s)) \cap B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}\left(\frac{12\rho x_i}{s^3} - \frac{24x_i}{s^2} + \frac{9x_i}{s\rho}\right), & x \in B(0,s) \setminus B(0,\frac{s}{2}) \end{cases}$$

and

$$\frac{\partial^2 w_n(x)}{\partial x_i^2} = \begin{cases} 0, & x \in (\overline{\Omega} \setminus B(0,s)) \cap B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}\left(\frac{12(x_i^2+\rho^2)}{s^3\rho} - \frac{24}{s^2} + \frac{9(\rho^2-x_i^2)}{s\rho^3}\right), & x \in B(0,s) \setminus B(0,\frac{s}{2}). \end{cases} \tag{3.11}$$

By (3.11) and (3.1) we have

$$\sum_{i=1}^N \frac{\partial^2 w_n(x)}{\partial x_i^2} = \begin{cases} 0, & x \in (\overline{\Omega} \setminus B(0,s)) \cap B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}\left(\frac{12\rho(N+1)}{s^3} - \frac{24N}{s^2} + \frac{9(N-1)}{s\rho}\right), & x \in B(0,s) \setminus B(0,\frac{s}{2}), \end{cases}$$

and

$$\begin{aligned} &\int_\Omega |\Delta w_n(x)|^p dx \\ &= \Big(\frac{\tau_n}{h}\Big)^p \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_{\frac{s}{2}}^s \Big|\frac{12(N+1)}{s^3}r - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{r}\Big|^p r^{N-1}dr = \frac{L}{h^p}\tau_n^p. \end{aligned} \tag{3.12}$$

Thus, we have by (2.4) and (3.12) that

$$\begin{aligned}\Phi(w_n) &= \frac{1}{p}\widehat{M}(\|w_n\|^p) - \frac{a}{p}\int_\Omega \frac{|w_n(x)|^p}{|x|^{2p}}dx \le \frac{1}{p}\widehat{M}\Big(\int_\Omega |\Delta w_n(x)|^p dx\Big) \\ &\le \frac{m_1 L}{ph^p}\tau_n^p. \end{aligned} \tag{3.13}$$

On the other hand, by (H4), one has

$$\Psi(w_n) = \int_\Omega \Big[F(x, w_n(x) + \frac{\mu}{\lambda}G(x, w_n(x))\Big]\,dx \ge \int_{B(0,s/2)} F(x, \tau_n/h)dx. \tag{3.14}$$

Hence, it follows from (3.13), (3.14) and (3.9) that

$$\Phi(w_n) - \lambda\Psi(w_n) \le \frac{m_1 L}{ph^p}\tau_n^p - \lambda\int_{B(0,s/2)} F(x,\tau_n)dx < \frac{m_1 L}{ph^p}(1-\lambda\eta)\tau_n^p$$

for every $n \in \mathbb{N}$ large enough, which leads to $\lim_{n\to+\infty}(\Phi(w_n) - \lambda\Psi(w_n)) = -\infty$.

The alternative of Theorem 2.1 case (i) assures the existence of unbounded sequence $\{u_n\}$ of critical points of the functional $\Phi - \lambda\Psi$. This completes the proof in view of the relation between the critical points of $\Phi - \lambda\Psi$ and the weak solutions of problem (1.1). □

Remark 3.2. If $\alpha < \infty$, $\beta > 0$, and $h > 1$ large enough, then (3.2) holds.

In the following, arguing in a similar way, but applying case (ii) of Theorem 2.1, we can establishes the existence of infinitely many solutions to (1.1) converging at zero.

Theorem 3.3. *Suppose that* (H1) *and* $0 < a < m_0 H$ *hold (with* H *is as in* (2.2)*). Also assume*

- (H5) $f \in C(\overline{\Omega} \times \mathbb{R})$, *and there exists* $c > 0$ *such that* $F(x,t) \ge 0$ *for every* $(x,t) \in \Omega \times [0,c]$*;*
- (H6) *There exists* $s > 0$ *as considered in* (3.1) *such that, if we put*

$$\alpha^0 := \liminf_{t\to 0^+} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\le t}\int_\Omega F(x,\xi)dx}{t^p}, \quad \beta^0 := \limsup_{t\to 0^+} \frac{\int_{B(0,s/2)} F(x,t/h)dx}{t^p},$$

one has

$$\alpha^0 < R\beta^0, \tag{3.15}$$

where $R = \frac{(m_0 H - a)h^p}{m_1 H L c_q^p}$ *(constants* $h > 1$, c_q *and* L *are as in* (2.8) *and* (3.1)*, respectively).*

Then, for every $\lambda \in \Lambda^0 := \frac{m_0 H - a}{pHc_q^p}\,]\frac{1}{R\beta^0}, \frac{1}{\alpha^0}[$ *and for every* $g \in C(\overline{\Omega}\times\mathbb{R})$ *such that*

- (H7) $G(t,u) \ge 0$, *for all* $(t,u) \in \overline{\Omega} \times [0,c]$ *and*

$$G_0 := \limsup_{t\to 0^+} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\le t}\int_\Omega G(x,\xi)dx}{t^p},$$

if we put

$$\mu_* = \begin{cases} \frac{m_0 H - a - pHc_q^p\alpha\lambda}{pHc_q^p G_0}, & G_0 > 0, \\ +\infty, & G_0 = 0, \end{cases} \tag{3.16}$$

then (1.1) *admits a sequence* $\{u_n\}$ *of weak solutions such that* $u_n \to 0$ *strongly in* X *for every* $\mu \in J := [0, \mu_*[$.

Proof. We take Φ and Ψ be as in (2.3). First, note that $\min_X \Phi = \Phi(0) = 0$. Let $\{\sigma_n\}$ be a sequence of positive numbers such that $\sigma_n \to 0^+$, and putting $r_n = \frac{m_0 H - a}{pHc_q^p}\sigma_n^p$. Similarly as above, we get

$$\begin{aligned}\delta &:= \liminf_{r\to 0^+} \varphi(r) \le \liminf_{n\to+\infty} \varphi(r_n) \\ &\le \frac{pHc_q^p}{m_0 H - a}\Big(\alpha^0 + \frac{\mu}{\lambda}G_0\Big) < +\infty.\end{aligned} \tag{3.17}$$

From (3.16) and (3.17), we have that $\Lambda^0 \subset]0, \frac{1}{\delta}[$. Now, for $\lambda \in \Lambda^0$, we claim that $\Phi - \lambda\Psi$ does not have a local minimum at zero. Indeed, let $\{\tau_n\}$ be a sequence of positive numbers in $]0, \tau[$ and $\eta > 0$ such that $\tau_n \to 0^+$ and

$$\frac{1}{\lambda} < \eta < \frac{ph^p}{m_1 L}\frac{\int_{B(0,s/2)} F(x, \tau_n/h)dx}{\tau_n^p},$$

for n large enough. Let $\{w_n\}$ be the sequence in X defined in (3.10). By (H7), one has that (3.14) holds. Thus, from (3.13), (3.14) and (3.9) we obtain that

$$\Phi(w_n) - \lambda\Psi(w_n) < \frac{m_1 L}{ph^p}(1 - \lambda\eta)\tau_n^p < 0 = \Phi(0) - \lambda\Psi(0)$$

for every $n \in \mathbb{N}$ large enough. This together with the fact that $\|w_n\| \to 0$ shows that $\Phi - \lambda\Psi$ has not a local minimum at zero. The conclusion follows from the alternative of Theorem 2.1 case (ii). □

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