Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 218, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ROBUST OBSERVABILITY FOR REGULAR LINEAR SYSTEMS UNDER NONLINEAR PERTURBATION

WEISHENG JIANG, BIN LIU, ZHIBING ZHANG

Abstract. In this article, we consider the admissibility and exact observability of a class of semilinear systems obtained by nonlinear perturbation for regular linear systems. We obtain the well-posedness of the semilinear system and the admissibility of the observation operator for the nonlinear semigroup, the solution semigroup of the semilinear system. Further, we obtain the robustness of the exact observability with respect to nonlinear perturbations when the Lipschitz constant is small enough. Finally, we give two examples to illustrate the obtained results.

1. Introduction

Many control systems described by partial differential equations can be rewritten as a regular linear system (see e.g. [4, 5, 10, 11, 12, 13])

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

where A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on Hilbert X, input operator $B : U \to X$ and output operator $C : X \to Y$ are linear operator (maybe unbounded), here U and Y are other Hilbert spaces, and $D \in \mathcal{L}(U, Y)$ is the feedthrough operator. For the definition of regular linear system, we refer to [30, 31]; also we introduce the definition in Section 2. In this work we take nonlinear state-feedback for (1.1) with $D = 0$, that is, $u(t) = F(x(t))$, where $F : X \to U$ is a nonlinear continuous function. Then we obtain the following closed-loop system

$$\dot{x}(t) = Ax(t) + BF(x(t)), \quad u(0) = x_0 \in X, \quad t \geq 0 \tag{1.2}$$

with output

$$y(t) = Cx(t). \tag{1.3}$$

We first consider the well-posedness of (1.2), that is, we prove that (1.2) admits a unique mild solution $u(t, x_0)$ for all $x_0 \in X$. Moreover, by $S(t)x_0 = u(t, x_0)$ we define a nonlinear semigroup $(S(t))_{t\geq 0}$. Then we consider the admissibility and observability of C for $(S(t))_{t\geq 0}$.

2010 *Mathematics Subject Classification.* 47H20, 93B07, 93C25, 93C73.
Key words and phrases. Admissible observation operator; exact observability; nonlinear semigroup; regular linear system.

Submitted July 18, 2015. Published August 20, 2015.

The problem of admissibility of unbounded observation operator has been studied by many authors. In the case of linear systems, Salamon [26] and Weiss [29] introduce the definition of admissibility, and many authors gave the different conditions for admissibility, see e.g. [6, 7, 8, 14, 17, 18]. Moreover, many authors considered the problem of robustness of admissibility under different linear perturbations, see e.g. [15, 21, 28, 30]. In addition, the problem of observability of unbounded observation operator is well studied for linear systems, see e.g. [1, 19, 20, 22, 24, 25, 28, 35]. Recently, Baroun and Jacob [2] extended the definition of admissibility and observability of the observation operator C for semilinear systems in the case that the nonlinear function is globally Lipschitz continuous, and they obtained the conditions guaranteeing that the semilinear system is exactly observable if and only if the linearized system has this property. In addition, Baroun, Jacob et al. [3] considered the same problem in the case that the nonlinear function is locally Lipschitz continuous.

In the spirit of [2, 3], we consider the admissibility and observability of the semilinear system (1.2) and (1.3) in the case that the nonlinear function F is globally Lipschitz continuous, and obtain the admissibility of C for the nonlinear semigroup $(S(t))_{t\geq 0}$, and prove that the semilinear system (1.2) and (1.3) is exactly observable if and only if the linearized system has this property when the Lipschitz constant for F is small enough. The results in this work can be applied to some control systems with nonlinear boundary perturbations.

This article is organized as follows. In Section 2, we introduce the concepts of the regular linear system and the admissible state feedback, and their some properties. In Section 3 we obtain the well-posedness of (1.2), and introduce a nonlinear semigroup $(S(t))_{t\geq 0}$ by the solution of (1.2). In Section 4 we obtain the admissibility of C for $(S(t))_{t\geq 0}$, and prove that the semilinear system (1.2) and (1.3) is exactly observable if and only if the linearized system has this property when the Lipschitz constant for F is small enough. Finally, in Section 5, we illustrate the results in this work by two examples.

2. Regular linear system

In this section, we introduce the concepts of the regular linear system and the admissible state feedback, and their some properties in state-space framework. We refer the reader to [26, 27, 30, 31] for more details.

Throughout this paper, X, U and Y are Hilbert spaces. $A : D(A) \to X$ is the infinitesimal generator of C_0-semigroup $(T(t))_{t\geq 0}$ (with $\|T(t)\| \leq Me^{\omega t}$ for some constants $M > 0$ and ω) on X. The Hilbert space X_1 is $D(A)$ with the graph norm. The Hilbert space X_{-1} is the completion of X with respect to the norm $\|(\alpha I - A)^{-1} \cdot \|$, where $\alpha \in \rho(A)$ (the resolvent set of A) is fixed. We have

$$X_1 \subset X \subset X_{-1}$$

with continuous and dense embeddings. $(T(t))_{t\geq 0}$ restricts to a C_0-semigroup on X_1 and extends to a C_0-semigroup on X_{-1} denoted by the same symbol.

$B \in \mathcal{L}(U, X_{-1})$ (the set of all bounded and linear operators from U to X_{-1}) is called an admissible control operator for $(T(t))_{t\geq 0}$ if there exist some $t > 0$ (and hence for all $t > 0$) and $\alpha_t = \alpha(t)$ such that

$$\int_0^t T(t-s)Bu(s)ds \in X,$$

and

$$\|\int_0^t T(t-s)Bu(s)ds\|_X \le \alpha_t \|u(\cdot)\|_{L^2(0,t;U)} \text{ for all } u(\cdot) \in L^2(0,t;U). \tag{2.1}$$

$C \in \mathcal{L}(X_1, Y)$ is called an admissible observation operator for $(T(t))_{t\ge 0}$ if there exist some $t > 0$ (and hence for all $t > 0$) and $\beta_t = \beta(t)$ such that

$$\|CT(\cdot)x\|_{L^2(0,t;Y)} \le \beta_t \|x\|_X, \quad \text{for all } x \in X_1. \tag{2.2}$$

We can choose $\alpha(t)$ and $\beta(t)$ such that they are nondecreasing functions. It is clear from (2.2) that $CT(\cdot)$ can be extended to a bounded linear operator from X to $L^2(0,t;Y)$, denoted by the same symbol. For the admissible observation operator C, define its Λ-extension C_Λ as follows

$$C_\Lambda x = \lim_{\lambda\to+\infty} C\lambda(\lambda I - A)^{-1}x \tag{2.3}$$

with $x \in D(C_\Lambda) = \{x \in X : \lim_{\lambda\to+\infty} C\lambda(\lambda I - A)^{-1}x \text{ exists}\}$.

The system $\Sigma(A, B, C, D)$ is called a regular linear system if A, B, C and D satisfy

(a) A generates a C_0-semigroup $(T(t))_{t\ge 0}$ on X;
(b) B is an admissible control operator for $(T(t))_{t\ge 0}$;
(c) C is an admissible observation operator for $(T(t))_{t\ge 0}$;
(d) $C_\Lambda(sI - A)^{-1}B$ makes sense for some $s \in \rho(A)$, i.e., $(sI - A)^{-1}Bu \in D(C_\Lambda)$ for all $u \in U$;
(e) The function $s \to \|C_\Lambda(sI - A)^{-1}B + D\|$ is uniformly bounded in some right half-plane, where $D \in \mathcal{L}(U, Y)$.

In [31], the definition of regular linear system is given by the time-domain way while the above definition is given by the equivalent conditions (see [31, 34] for details).

$F \in \mathcal{L}(X_1, U)$ is called an admissible state-feedback operator for the pair (A, B) if (A, B, F) is a regular linear system with state space X, input space U and output space U, and $I - F_\Lambda(sI - A)^{-1}B$ is invertible on the right half-plane $\mathbb{C}_\alpha^+ = \{s : \mathrm{Re} s > \alpha\}$, where α is some real number, and this inverse is uniformly bounded.

We summarize the results about admissible state-feedback operators as follows and refer to [32, 33, 34, 36] for details:

Theorem 2.1. *Let F be an admissible state-feedback operator for the pair (A, B). Then the following statements hold:*

(i) The operator $A_F := A + BF_\Lambda$ with domain $D(A_F) = \{x \in D(F_\Lambda) : (A + BF_\Lambda)x \in X\}$ generates a C_0-semigroup $(T_F(t))_{t\ge 0}$ on X. Moreover, $(T_F(t))_{t\ge 0}$ is described by

$$\begin{aligned} T_F(t)x_0 &= T(t)x_0 + \int_0^t T(t-\tau)BF_\Lambda T_F(\tau)x_0 d\tau \\ &= T(t)x_0 + \int_0^t T_F(t-\tau)BF_\Lambda T(\tau)x_0 d\tau, \quad x_0 \in X; \end{aligned} \tag{2.4}$$

(ii) B is an admissible control operator for $(T_F(t))_{t\ge 0}$;

(iii) F^1 defined as F_Λ restricted to $D(A_F)$ is an admissible observation operator for $(T_F(t))_{t\ge 0}$;

(iv) if F_Λ^1 denotes the Λ-extension of F^1 with respect to $(T_F(t))_{t\ge 0}$, i.e.,

$$F_\Lambda^1 x = \lim_{\lambda\to+\infty} F^1\lambda(\lambda I - A_F)^{-1}x, \quad x \in D(F_\Lambda^1),$$

then $F^1_\Lambda = F_\Lambda$*, in particular,* $D(F^1_\Lambda) = D(F_\Lambda)$*;*

(v) $\Sigma(A_F, B, F^1)$ *is a regular linear system.*

3. Well-posedness and nonlinear semigroup

In this section, we show the well-posedness of the system

$$\frac{dx(t)}{dt} = Ax(t) + BF(x(t)), \quad x(0) = x_0, \quad t \ge 0,\ x_0 \in X, \tag{3.1}$$

where A generates a C_0-semigroup $(T(t))_{t\ge 0}$ on Hilbert X, $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $(T(t))_{t\ge 0}$, and $F(\cdot) : X \to U$ is a globally Lipschitz continuous function, that is, there exists a positive constant L such that

$$\|F(x) - F(y)\| \le L\|x - y\|, \tag{3.2}$$

for all $x, y \in X$, and $F(0) = 0$.

Theorem 3.1. *Assume that B is an admissible control operator for $(T(t))_{t\ge 0}$ generated by A, and that $F(\cdot) : X \to U$ is a globally Lipschitz continuous function. Then, for any $x_0 \in X$, (3.1) has a unique mild solution given by*

$$x(t) = T(t)x_0 + \int_0^t T(t-\sigma)BF(x(\sigma))d\sigma. \tag{3.3}$$

Proof. Given $t_0 \ge 0$. Define a function G on $C(0, t_0; X)$ (the set of all continuous functions from $[0, t_0]$ to X) as follows:

$$G(x(t)) = T(t)x_0 + \int_0^t T(t-\sigma)BF(x(\sigma))d\sigma,\ x(\cdot) \in C(0, t_0; X). \tag{3.4}$$

Firstly, we show that $G(x(\cdot)) \in C(0, t_0; X)$ for all $x(\cdot) \in C(0, t_0; X)$.

For $t \in [0, t_0]$ and h small enough such that $t + h \in [0, t_0]$. Without loss of generality, we assume that $h > 0$ (the case of $h < 0$ can be proved by the same method). It follows from (3.4) that

$$\begin{aligned} G(x(t+h)) - G(x(t)) = T(t+h)x_0 &+ \int_0^{t+h} T(t+h-\sigma)BF(x(\sigma))d\sigma \\ &- T(t)x_0 - \int_0^t T(t-\sigma)BF(x(\sigma))d\sigma. \end{aligned} \tag{3.5}$$

Changing σ into $\sigma + h$, we have

$$\begin{aligned} &\int_0^{t+h} T(t+h-\sigma)BF(x(\sigma))d\sigma \\ &= \int_{-h}^0 T(t-\sigma)BF(x(\sigma+h))d\sigma + \int_0^t T(t-\sigma)BF(x(\sigma+h))d\sigma. \end{aligned} \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\begin{aligned} G(x(t+h)) - G(x(t)) &= (T(h) - I)T(t)x_0 \\ &\quad + \int_0^t T(t-\sigma)B(F(x(\sigma+h)) - F(x(\sigma)))d\sigma \\ &\quad + \int_{-h}^0 T(t-\sigma)BF(x(\sigma+h))d\sigma \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.7}$$

For I_1, using the strong continuity of C_0-semigroup $(T(t))_{t\geq 0}$, we have

$$\|I_1\| = \|(T(h) - I)T(t)x_0\| \to 0, \quad \text{as } h \to 0. \tag{3.8}$$

For I_2, it follows from (2.1) and (3.2) that

$$\begin{aligned}\|I_2\| &\leq \alpha(t)(\int_0^t \|F(x(\sigma + h)) - F(x(\sigma))\|^2 d\sigma)^{1/2} \\ &\leq L\alpha(t)(\int_0^t \|x(\sigma + h) - x(\sigma)\|^2 d\sigma)^{1/2}.\end{aligned} \tag{3.9}$$

In addition, $x(\cdot)$ is uniformly continuous in $[0, t]$ since $x(\cdot)$ is continuous. Then

$$\|I_2\| \to 0, \quad \text{as } h \to 0. \tag{3.10}$$

For I_3, changing $\sigma + h$ into σ and using (2.1), and that $\alpha(t)$ is nondecreasing, we have

$$\begin{aligned}\|I_3\| &= \|\int_0^h T(t + h - \sigma)BF(x(\sigma))d\sigma\| \\ &\leq \|T(t)\|\|\int_0^h T(h - \sigma)BF(x(\sigma))d\sigma\| \\ &\leq \alpha(t_0)\|T(t)\|(\int_0^h \|F(x(\sigma))\|^2 d\sigma)^{1/2}.\end{aligned} \tag{3.11}$$

It follows from (3.11) and the continuity of $F(x(\cdot))$ that

$$\|I_3\| \to 0, \quad \text{as } h \to 0. \tag{3.12}$$

It follows from (3.7), (3.8), (3.10) and (3.12) that

$$\|G(x(t + h)) - G(x(t))\| \to 0 \quad \text{as } h \to 0,$$

and consequently, $G : C(0, t_0; X) \to C(0, t_0; X)$.

Secondly, we show the existence of mild solution of (3.1). For any $x_1(\cdot), x_2(\cdot) \in C(0, t_0; X)$, note that $\alpha(t)$ is a nondecreasing function, it follows from (2.1) and (3.2) that

$$\begin{aligned}\|G(x_1(t)) - G(x_2(t))\| &= \|\int_0^t T(t - \sigma)B(F(x_1(\sigma)) - F(x_2(\sigma)))d\sigma\| \\ &\leq \alpha(t_0)(\int_0^t \|F(x_1(\sigma)) - F(x_2(\sigma))\|^2 d\sigma\|)^{1/2} \\ &\leq \alpha(t_0)L(\int_0^t \|x_1(\sigma) - x_2(\sigma)\|^2 d\sigma\|)^{1/2} \\ &\leq \alpha(t_0)Lt^{1/2}\|x_1 - x_2\|_{C(0,t_0;X)},\end{aligned}$$

By induction on n, we have

$$\|G^n(x_1(t)) - G^n(x_2(t))\| \leq \alpha^n(t_0)L^n(\frac{t^n}{n!})^{1/2}\|x_1 - x_2\|_{C(0,t_0;X)},$$

where G^n represents the n-time iteration of G, that is, $G^n = G(G(\cdots G))$. So

$$\|G^n(x_1) - G^n(x_2)\|_{C(0,t_0;X)} \leq \alpha^n(t_0)L^n(\frac{t_0^n}{n!})^{1/2}\|x_1 - x_2\|_{C(0,t_0;X)}.$$

It is clear that $\alpha^n(t_0)L^n(\frac{t_0^n}{n!})^{1/2} \to 0$ as $n \to \infty$. Then it follows from a well known existence of the contraction principle that G has a unique fixed point $x(\cdot)$ in $C(0, t_0; X)$. The fixed point is the desired mild solution of (3.1).

Finally, we show the uniqueness of mild solution of (3.1), and the Lipschitz continuity of the map $x_0 \to x(\cdot)$. Let $y(\cdot)$ be a mild solution of (3.1) with the initial value y_0. Then

$$
\begin{aligned}
\|x(t) - y(t)\| &\le \|T(t)(x_0 - y_0)\| + \| \int_0^t T(t-\sigma)B(F(x(\sigma)) - F(y(\sigma)))d\sigma\| \\
&\le Me^{\omega t}\|x_0 - y_0\| + \alpha(t)(\int_0^t \|F(x(\sigma)) - F(y(\sigma))\|^2 d\sigma)^{1/2} \\
&\le Me^{\omega t}\|x_0 - y_0\| + \alpha(t_0)L(\int_0^t \|x(\sigma) - y(\sigma)\|^2 d\sigma)^{1/2},
\end{aligned}
$$

and consequently,

$$
\|x(t) - y(t)\|^2 \le 2M^2 e^{2\omega t}\|x_0 - y_0\|^2 + 2\alpha^2(t_0)L^2 \int_0^t \|x(\sigma) - y(\sigma)\|^2 d\sigma,
$$

which implies, by Gronwall's inequality, that

$$
\|x(t) - y(t)\|^2 \le 2M^2 e^{2\omega t} e^{2\alpha^2(t_0)L^2 t}\|x_0 - y_0\|^2.
$$

That is,

$$
\|x(t) - y(t)\| \le \sqrt{2}Me^{\omega t}e^{\alpha^2(t_0)L^2 t}\|x_0 - y_0\|.
$$

Then

$$
\|x(t) - y(t)\|_{C(0,t_0;X)} \le \sqrt{2}Me^{|\omega|t_0}e^{\alpha^2(t_0)L^2 t_0}\|x_0 - y_0\|,
$$

which yields both the uniqueness of mild solution of (3.1), and the Lipschitz continuity of the map $x_0 \to x(\cdot)$. □

Let $(S(t))_{t\ge 0}$ be the family of nonlinear operators defined in X by

$$
S(t)x_0 = x(t), \quad t \ge 0, \tag{3.13}
$$

where $x_0 \in X$ and $x(t)$ is the mild solution of (3.1) with the initial value x_0.

Proposition 3.2. *Let $(S(t))_{t\ge 0}$ be defined by (3.13). Then $(S(t))_{t\ge 0}$ is a nonlinear semigroup on X.*

Proof. It is sufficient to prove that the following two properties hold:

(P1) $S(0)x_0 = x_0$ and $S(s+t)x_0 = S(t)S(s)x_0$ for $s, t \ge 0$ and $x_0 \in X$;
(P2) $S(\cdot)x_0$ is continuous over $[0, +\infty)$ for each $x_0 \in X$.

Firstly, we prove that the property (P1) holds. It is clear that $S(0)x_0 = x_0$ for all $x_0 \in X$. In addition, using the definition of $S(t)$ and changing σ into $s+\sigma$, we have

$$\begin{aligned} S(t+s)x_0 &= T(t+s)x_0 + \int_0^{t+s} T(t+s-\sigma)BF(x(\sigma))d\sigma \\ &= T(t+s)x_0 + \int_0^{t+s} T(t+s-\sigma)BF(S(\sigma)x_0)d\sigma \\ &= T(t)T(s)x_0 + \int_0^{s} T(t+s-\sigma)BF(S(\sigma)x_0)d\sigma \\ &\quad + \int_s^{t+s} T(t+s-\sigma)BF(S(\sigma)x_0)d\sigma \\ &= T(t)T(s)x_0 + \int_0^{s} T(t+s-\sigma)BF(S(\sigma)x_0)d\sigma \\ &\quad + \int_0^{t} T(t-\sigma)BF(S(s+\sigma)x_0)d\sigma. \end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned} S(t)S(s)x_0 &= T(t)S(s)x_0 + \int_0^{t} T(t-\sigma)BF(S(\sigma)S(s)x_0)d\sigma \\ &= T(t)(T(s)x_0 + \int_0^{s} T(s-\sigma)BF(S(\sigma)x_0)d\sigma) \\ &\quad + \int_0^{t} T(t-\sigma)BF(S(\sigma)S(s)x_0)d\sigma \\ &= T(t)T(s)x_0 + \int_0^{s} T(t+s-\sigma)BF(S(\sigma)x_0)d\sigma \\ &\quad + \int_0^{t} T(t-\sigma)BF(S(\sigma)S(s)x_0)d\sigma. \end{aligned} \tag{3.15}$$

Then it follows from (3.14) and (3.15) that

$$S(t+s)x_0 - S(t)S(s)x_0 = \int_0^t T(t-\sigma)B(F(S(s+\sigma)x_0) - F(S(\sigma)S(s)x_0))d\sigma,$$

and consequently, by (2.1) and (3.2), we have

$$\begin{aligned} &\|S(t+s)x_0 - S(t)S(s)x_0\|^2 \\ &= \|\int_0^t T(t-\sigma)B(F(S(s+\sigma)x_0) - F(S(\sigma)S(s)x_0))d\sigma\|^2 \\ &\le \alpha^2(t)\int_0^t \|F(S(s+\sigma)x_0) - F(S(\sigma)S(s)x_0)\|^2 d\sigma \\ &\le \alpha^2(t_0)L^2\int_0^t \|S(s+\sigma)x_0 - S(\sigma)S(s)x_0\|^2 d\sigma. \end{aligned}$$

By Gronwall's inequality, we have

$$\|S(t+s)x_0 - S(t)S(s)x_0\|^2 \le 0,$$

and consequently, $S(t+s)x_0 = S(t)S(s)x_0$.

Property (P2) follows from the fact that the solution $x(\cdot)$ is continuous. □

Proposition 3.3. *Let $(S(t))_{t\geq 0}$ be defined by (3.13). Then, for every $x_0, y_0 \in X$ and $t \geq 0$, we have*

$$\|S(t)x_0 - S(t)y_0\| \leq \sqrt{2}Me^{(\omega+\alpha^2(t)L^2)t}\|x_0 - y_0\|, \tag{3.16}$$

$$\|S(t)x_0\| \leq \sqrt{2}Me^{(\omega+\alpha^2(t)L^2)t}\|x_0\|. \tag{3.17}$$

Proof. Let $x_0, y_0 \in X$. It follows from (2.1) and (3.2) that

$$\begin{aligned}
&\|S(t)x_0 - S(t)y_0\| \\
&\leq \|T(t)x_0 - T(t)y_0\| + \|\int_0^t T(t-\sigma)B(F(S(\sigma)x_0) - F(S(\sigma)y_0))d\sigma\| \\
&\leq Me^{\omega t}\|x_0 - y_0\| + \alpha(t)\int_0^t \|F(S(\sigma)x_0) - F(S(\sigma)y_0)\|d\sigma \\
&\leq Me^{\omega t}\|x_0 - y_0\| + \alpha(t)L(\int_0^t \|S(\sigma)x_0 - S(\sigma)y_0\|^2 d\sigma)^{1/2},
\end{aligned}$$

and consequently,

$$\|S(t)x_0 - S(t)y_0\|^2 \leq 2M^2e^{2\omega t}\|x_0 - y_0\|^2 + 2\alpha^2(t)L^2\int_0^t \|S(\sigma)x_0 - S(\sigma)y_0\|^2 d\sigma,$$

By Gronwall's inequality, we have

$$\|S(t)x_0 - S(t)y_0\| \leq \sqrt{2}Me^{(\omega+\alpha^2(t)L^2)t}\|x_0 - y_0\|.$$

Writing $y_0 = 0$ in (3.16), we get the assertion (3.17). □

Remark 3.4. If $(T(t))_{t\geq 0}$ is exponentially stable, then $\alpha(t)$ can be chosen a constant $\alpha > 0$. So $(S(t))_{t\geq 0}$ is also exponentially stable if $\omega < -\alpha^2L^2$.

Remark 3.5. By the definition of $(S(t))_{t\geq 0}$, we have, for any $x_0 \in X$,

$$S(t)x_0 = T(t)x_0 + \int_0^t T(t-\sigma)BF(S(\sigma)x_0)d\sigma. \tag{3.18}$$

Note that $\Sigma(A, B, C)$ is a regular linear system, it follows from [31, Theorem 2.3] that $S(t)x_0 \in D(C_\Lambda)$ for any $x_0 \in D(A)$ and almost every $t \geq 0$. In addition, it follows from the bounbedness of input/output operator of regular linear system $\Sigma(A, B, C)$ that there exists a constant $M_1 > 0$ such that, for all $x \in X$,

$$\int_0^{t_0} \|C\int_0^t T(t-\sigma)BF(S(\sigma)x)d\sigma\|^2 dt \leq M_1\int_0^{t_0} \|F(S(\sigma)x)\|^2 d\sigma, \tag{3.19}$$

and consequently, $CS(\cdot)x \in L^2(0, t_0; Y)$ for all $x \in X$.

4. Admissibility and robust observability

We start this section with the definition of admissibility of output operator C for nonlinear semigroup $(S(t))_{t\geq 0}$ given by (3.13). The reader is referred to see [2] for more details on this definition.

Definition 4.1. Let $\Sigma(A, B, C)$ be a regular linear system, $(S(t))_{t\geq 0}$ nonlinear semigroup given by (3.13). We say that C is a finite-time admissible observation operator for $(S(t))_{t\geq 0}$, if there exist some $t>0$ (and hence for all $t>0$), and $\gamma(t)>0$ such that

$$\int_0^t \|CS(\sigma)x-CS(\sigma)y\|^2 d\sigma \leq \gamma(t)\|x-y\|^2, \text{ for all } x,y\in D(A). \tag{4.1}$$

Definition 4.2. Let $\Sigma(A,B,C)$ be a regular linear system, $(S(t))_{t\geq 0}$ nonlinear semigroup given by (3.13). We say that C is an infinite-time admissible observation operator for $(S(t))_{t\geq 0}$, if there is some $\gamma>0$ such that

$$\int_0^\infty \|CS(\sigma)x-CS(\sigma)y\|^2 d\sigma \leq \gamma\|x-y\|^2, \text{ for all } x,y\in D(A). \tag{4.2}$$

Remark 4.3. (i) For a linear operator semigroup, equation (4.1) is equivalent to equation (2.2).

(ii) It follows from (4.1) (resp. (4.2)) that the mapping $x\mapsto CS(\cdot)x$ has a continuous extension from X to $L^2(0,t;Y)$ for every $t>0$ (resp. $L^2(0,\infty;Y)$).

(iii) If $(S(t))_{t\geq 0}$ is exponentially stable, then the notion of finite-time admissibility and infinite-time admissibility are equivalent.

The following theorem is one of main results of this article.

Theorem 4.4. *Assume that $\Sigma(A,B,C)$ is a regular linear system and that $F(\cdot): X\to U$ is a globally Lipschitz continuous function. Then C is a finite-time admissible observation operator for $(S(t))_{t\geq 0}$ given by* (3.13).

Proof. Because $\Sigma(A,B,C)$ is a regular linear system, C is a finite-time admissible observation operator for $(T(t))_{t\geq 0}$. That is, there exist some $t_0>0$ and K_{t_0} such that

$$\int_0^{t_0} \|CT(\sigma)x\|^2 d\sigma \leq K_{t_0}\|x\|^2, \text{ for all } x\in D(A). \tag{4.3}$$

In addition, for $x,y\in D(A)$, it follows from (3.18) that

$$\begin{aligned} &\|CS(t)x-CS(t)y\| \\ &\leq \|CT(t)x-CT(t)y\| + \|C\int_0^t T(t-\sigma)B(F(S(\sigma)x)-F(S(\sigma)y))d\sigma\|. \end{aligned} \tag{4.4}$$

It follows from (3.2), (3.16), (3.19), (4.3) and (4.4) that

$$\begin{aligned} &\int_0^{t_0} \|CS(t)x-CS(t)y\|^2 dt \\ &\leq 2\int_0^{t_0} \|CT(t)x-CT(t)y\|^2 dt \\ &\quad + 2\int_0^{t_0} \|C\int_0^t T(t-\sigma)B(F(S(\sigma)x)-F(S(\sigma)y))d\sigma\|^2 dt \\ &\leq 2K_{t_0}\|x-y\|^2 + 2M_1L^2\int_0^{t_0} \|S(\sigma)x-S(\sigma)y\|^2 d\sigma \\ &\leq 2K_{t_0}\|x-y\|^2 + 4M_1L^2M^2\int_0^{t_0} e^{2(\omega+\alpha(t)L^2)t}\|x-y\|^2 d\sigma \\ &\leq 2(K_{t_0}+2M_1L^2M^2e^{2(\omega+\alpha(t_0)L^2)t_0}t_0)\|x-y\|^2, \end{aligned}$$

and consequently, C is finite-time admissible for $(S(t))_{t\geq 0}$. □

From Remark 4.3, we have the following result.

Corollary 4.5. *Suppose that the assumptions of Theorem 4.4 are satisfied. If $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are exponentially stable, then C is infinite-time admissible for $(S(t))_{t\geq 0}$.*

We consider the exact observability of C for the nonlinear semigroup $(S(t))_{t\geq 0}$. We start by giving the definition of exact observability.

Let (A, C) denote the linear system

$$\begin{gathered} \dot{x}(t) = Ax(t), \quad t > 0,\ x(0) = x_0, \\ y(t) = Cx(t). \end{gathered} \tag{4.5}$$

Definition 4.6. Let $C \in \mathcal{L}(D(A), Y)$ be an admissible observation operator for $(T(t))_{t\geq 0}$. We call (A, C) is exactly observable if there is some constant $K > 0$ such that

$$\Big(\int_0^{+\infty} \|CT(t)x\|^2 dt\Big)^{1/2} \geq K\|x\|, \quad x \in D(A), \tag{4.6}$$

and (A, C) is τ-exactly observable if there is some $K_\tau > 0$ such that

$$\Big(\int_0^{\tau} \|CT(t)x\|^2 dt\Big)^{1/2} \geq K_\tau\|x\|, \quad x \in D(A). \tag{4.7}$$

Definition 4.7. Suppose that the assumptions of Theorem 4.4 are satisfied. We call $(S(t), C)$ is exactly observable if there is some constant $K > 0$ such that

$$\Big(\int_0^{+\infty} \|CS(t)x - CS(t)y\|^2 dt\Big)^{1/2} \geq K\|x - y\|, \quad x, y \in D(A), \tag{4.8}$$

and $(S(t), C)$ is τ-exactly observable if there is some $K_\tau > 0$ such that

$$\Big(\int_0^{\tau} \|CS(t)x - CS(t)y\|^2 dt\Big)^{1/2} \geq K_\tau\|x - y\|, \quad x, y \in D(A). \tag{4.9}$$

Next, we state the main result of this section.

Theorem 4.8. *Suppose that the assumptions of Theorem 4.4 are satisfied and that $\tau > 0$.*

(i) If (A, C) given by (4.5) is τ-exactly observable, then there exists a constant $L_0 > 0$ such that $(S(t), C)$ is also τ-exactly observable when the Lipschitz constant L in (3.2) satisfies $L < L_0$.

(ii) If $(S(t), C)$ is τ-exactly observable, then there exists a constant $L_1 > 0$ such that (A, C) is also τ-exactly observable when the Lipschitz constant L in (3.2) satisfies $L < L_1$.

Proof. (i) It follows from (3.18) that, for all $x, y \in D(A)$ and almost every $t \geq 0$,

$$CS(t)x - CS(t)y = CT(t)(x-y) + C\int_0^t T(t-\sigma)B(F(S(\sigma)x) - F(S(\sigma)y))d\sigma. \tag{4.10}$$

We may rewrite (4.10) as

$$CT(t)(x-y) = CS(t)x - CS(y) - C\int_0^t T(t-\sigma)B(F(S(\sigma)x) - F(S(\sigma)y))d\sigma. \tag{4.11}$$

Therefore,

$$
\begin{aligned}
&\|CT(t)(x-y)\|^2 \\
&\leq 2\|CS(t)x - CS(y)\|^2 + 2\|C\int_0^t T(t-\sigma)B(F(S(\sigma)x) - F(S(\sigma)y))d\sigma\|^2.
\end{aligned}
\tag{4.12}
$$

It follows from (3.2), (3.19), (4.7) and (4.12) that

$$
\begin{aligned}
&\int_0^\tau \|CS(t)x - CS(t)y\|^2 dt \\
&\geq \frac{1}{2}\int_0^\tau \|CT(t)(x-y)\|^2 dt \\
&\quad - \int_0^\tau \|C\int_0^t T(t-\sigma)B(F(S(\sigma)x) - F(S(\sigma)y))d\sigma\|^2 dt \\
&\geq \frac{1}{2}K_\tau\|x-y\|^2 - M_1^2\int_0^\tau \|F(S(t)x) - F(S(t)y)\|^2 dt \\
&\geq \frac{1}{2}K_\tau\|x-y\|^2 - M_1^2L^2\int_0^\tau \|S(t)x - S(t)y\|^2 dt \\
&\geq \frac{1}{2}K_\tau\|x-y\|^2 - M_1^2L^2\int_0^\tau 2M^2e^{2(\omega+\alpha^2(t)L^2)t}\|x-y\|^2 dt \\
&\geq \frac{1}{2}K_\tau\|x-y\|^2 - 2M_1^2L^2M^2\tau e^{2(\omega+\alpha^2(\tau)L^2)\tau}\|x-y\|^2 \\
&= J_\tau\|x-y\|^2,
\end{aligned}
\tag{4.13}
$$

where $J_\tau = \frac{1}{2}K_\tau - 2M_1^2L^2M^2\tau e^{2(\omega+\alpha^2(\tau)L^2)\tau}$.

Let $L \leq 1$. Then

$$
J_\tau = \frac{1}{2}K_\tau - 2M_1^2L^2M^2\tau e^{2(\omega+\alpha^2(\tau)L^2)\tau} \geq \frac{1}{2}K_\tau - 2M_1^2L^2M^2\tau e^{2(\omega+\alpha^2(\tau))\tau}.
$$

Take

$$
L_0 = \min\{1, \frac{\sqrt{\tau K_\tau}}{2M_1M\tau e^{2(\omega+\alpha^2(\tau))\tau}}\},
$$

and therefore, $J_\tau > 0$ when $L < L_0$. So $(S(t), C)$ is also τ-exactly observable.

Statement (ii) can be proved by the same method as above. □

Corollary 4.9. *Suppose that the assumptions of Theorem 4.4 are satisfied, and that $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are exponentially stable.*

(i) If (A, C) given by (4.5) is exactly observable, then there exists a constant $L_0 > 0$ such that $(S(t), C)$ is also exactly observable when the Lipschitz constant L in (3.2) satisfies $L < L_0$.

(ii) If $(S(t), C)$ is exactly observable, then there exists a constant $L_1 > 0$ such that (A, C) is also exactly observable when $L < L_1$.

5. Examples

Example 5.1. Consider the beam equation with boundary control

$$
\begin{gathered}
w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \\
w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \\
w_{xxx}(1,t) = u(t),
\end{gathered}
\tag{5.1}
$$

with the output function

$$y(t) = w_t(1,t). \tag{5.2}$$

Guo and Luo [9] proved that the system (5.2) can be rewritten as a regular linear system $\Sigma(\mathcal{A},\mathcal{B},\mathcal{C})$ with well-defined operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ on (H,U,U), and system state $x(t) = (w, w_t)$, where $H = D(A^{1/2}) \times L^2(0,1)$ and $U = \mathbb{C}$. In addition, in the same paper, they also proved that the observation system $(\mathcal{A},\mathcal{C})$ is exactly observable on some $[0,T], T > 0$.

System (5.1) and (5.2) with $u = f(w_t(1,t))$, where $f(\cdot)$ is a globally Lipschitz continuous function with Lipschitz constant L, can be rewritten as the abstract form (1.2) and (1.3) with $F(x(t)) = f(w_t(1,t))$. It is clear that F is a globally Lipschitz continuous function with Lipschitz constant L. Therefore, by Theorems 4.4 and 4.8, $\mathcal{C}$ is an admissible observation operator for nonlinear Semigroup $(S(t))_{t\geq 0}$, the solution semigroup of (5.1) with $u = f(w_t(1,t))$, and the semilinear problem (5.1) and (5.2) with $u = f(w_t(1,t))$ is exactly observable in time $T > 0$ when the Lipschitz constant L is small enough.

Example 5.2. Consider the Schrödinger equation with nonlinear boundary perturbation described by

$$\begin{gathered}
w_t(x,t) + i\triangle w(x,t) = 0, \quad x \in \Omega, t > 0,\\
w(x,t) = 0, \quad x \in \Gamma_1, t \geq 0,\\
w(x,t) = u(x,t), \quad x \in \Gamma_0, t \geq 0,\\
y(x,t) = i\frac{\partial(\triangle^{-1}w)}{\partial\nu} \quad x \in \Gamma_0, t \geq 0,
\end{gathered} \tag{5.3}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is an open bounded region with smooth C^3-boundary $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$. Γ_0 and Γ_1 are disjoint parts of the boundary relatively open in $\partial\Omega$ and $\operatorname{int}(\Gamma_0) \neq \emptyset$. ν is the unit normal vector of Γ_0 pointing towards the exterior of Γ. u is the input function and y is the output function. Let $H = H^{-1}(\Omega)$ be the state space and $U = L^2(\Gamma_0)$ the input or output space. Guo and Shao [12] proved that the system (5.3) can be rewritten as a regular linear system $\Sigma(A,B,C)$ with well-defined operators A, B and C on (H,U,U). In addition, Lasiecka and Triggiani [16] proved that the system (5.3) with $u = 0$ is exactly observable at some $\tau > 0$.

System (5.3) with $u = F(w(x,t))$, where $F(\cdot)$ is a globally Lipschitz continuous function with Lipschitz constant L, can be rewritten as the abstract form (1.2) and (1.3). Therefore, by Theorems 4.4 and 4.8, C is an admissible observation operator for nonlinear semigroup $(S(t))_{t\geq 0}$, where $(S(t))_{t\geq 0}$ is the solution semigroup of (5.3) with $u = F(w(x,t))$, and the semilinear problem (5.3) with $u = F(w(x,t))$ is exactly observable in some time $\tau > 0$ when the Lipschitz constant L is small enough.

Acknowledgements. This research was supported by the Natural Science Foundation Project of CQ CSTC (No. 2011BB2069) and by the Scientific and Technological Research Program of Chongqing Municipal Education Commission (No. KJ130733).

References

[1] F. El Alaoui, H. Zwart, A. Boutoulout; *Spectral conditions implied by observability.* SIAM J. Control Optim., 49 (2011), 672-685.

[2] M. Baroun, B. Jacob; *Admissibility and observability of observation operators for semilinear problems*. Integral Equations Operator Theory, 64 (2009), 1-20.
[3] M. Baroun, B. Jacob, L. Maniar, R. Schnaubelt; *Semilinear observation system*. Systems Control Letters, 62 (2013), 924-929.
[4] S. G. Chai, B. Z. Guo; *Well-posedness and regularity of Naghdi's shell equation under boundary control*. Journal of Differential Equations, 249 (2010), 3174-3214.
[5] S. G. Chai, B. Z. Guo; *Well-posedness and regularity of weakly coupled wave-plate equation with boundary control and observation*. Journal of Dynamical and Control Systems, 15 (2009), 331-358.
[6] R. F. Curtain, H. Logemann, S. Townley, and H. Zwart; *well-posedness, stabilizability and admissibility for Pritchard-Salamon systems*. J. Math. system, Estimation Control, 7 (1997), 439-476.
[7] K. J. Engel; *On the characterization of admissible control and observation operators*. Systems Control Letters, 34 (1998), 225-227.
[8] P. Grabowski, F. Callier; *Admissible observation operators: Semigroup criteria of admissibility*. Integral Equations Operator Theory, 25 (1996), 182-198.
[9] B. Z. Guo, Y. H. Luo; *Controllability and stability of a second order hyperbolic system with collocated sensor/actuator*. Systems Control Letters, 46 (2002), 45-65.
[10] B. Z. Guo, Z. C. Shao; *Well-posedness and regularity for non-uniform Schrodinger and Euler-Bernoulli equations with boundary control and observation*. Quarterly of Applied Mathematics, 70 (2012), 111-132.
[11] B. Z. Guo, Z. C. Shao; *Regularity of an Euler-Bernoulli equation with Neumann control and collocated observation*. Journal of Dynamical and Control Systems, 12 (2006), 405-418.
[12] B. Z. Guo, Z. C. Shao; *Regularity of a Schrodinger equation with Dirichlet control and collocated observation*. Systems and Control Letters, 54 (2005), 1135-1142.
[13] B. Z. Guo, X. Zhang; *The regularity of the wave equation with partial Dirichlet control and colocated observation*. SIAM Journal on Control and Optimization, 44 (2005), 1598-1613.
[14] B. Haak, P. Kunstmann; *Admissibility of unbounded operators and wellposedness of linear systems in Banach spaces*. Integral Equations Operator Theory, 55 (2006), 497-533.
[15] S. Hadd, A. Idrissi; *On the admissibility of observation for perturbed C_0-semigroups on Banach spaces*. Systems Control Letters, 55 (2006), 1-7.
[16] I. Lasiecka, R. Triggiani; *Optimal regularity, exact controllability and uniform stabilization of Shrödinger equations with Dirichlet control*. Differential Integral Equations, 5 (1992), 521-535.
[17] B. Jacob, J. Partington; *A resolvent test for admissibility of Volterra observation operators*. J. Math. Anal. Appl., 332 (2007), 346-355.
[18] B. Jacob, J. R. Partington; *Admissibility of control and observation operators for semigroups:a survey*. in:J.A. Ball, J. W. Helton, M. Klaus, L. Rodman(Eds.), Current Trends in Operator Theory and its Applications, Proceedings of IWOTA 2002, Operator Theory:Advances and Applications, vol. 149, Birkhauser, Basel, 199-221.
[19] B. Jacob, H. Zwart; *Exact observability of diagonal systems with a one-dimensional output operator*. Infinite-dimensional systems theory and operator theory (Perpignan, 2000). Int. J. Appl. Math. Comput. Sci., 11 (2001), 1277-1283.
[20] B. Jacob, H. Zwart; *Exact observability of diagonal systems with a finite-dimensional output operator*. Systems Control Letters, 43 (2001), 101-109.
[21] Z. D. Mei, J. G. Peng; *On invariance of p-admissibility of control and observation operators to q-type of perturbations of generator of C_0-semigroup*. Systems Control Letter, 59 (2010), 470-475.
[22] Z. D. Mei, J. G. Peng; *On robustness of exact controllability and exact observability under cross perturbations of the generator in Banach spaces*. Proc. Amer. Math. Soc., 138 (2010), 4455-4468.
[23] A. Pazy; *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983.
[24] J. R. Partington, S. Pott; *Admissibility and exact observability of observation operators for semigroups*. Irish Math. Soc. Bull., 55 (2005), 19-39.
[25] D. L. Russell, G. Weiss; *A general necessary condition for exact observability*. SIAM J. Control Optimization, 32 (1994), 1-23.

[26] D. Salamon; *Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach*. Trans. Amer. Math. Soc., 300 (1987), 383-431.
[27] Olof J. Staffans; *Well-posed linear systems*. Cambridge Univ. Press, Cambridge, 2005.
[28] M. Tucsnak, G. Weiss; *Observation and Control for Operator Semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2009.
[29] G. Weiss; *Admissibility of unbounded control operators*. SIAM J Control Optim., 27 (1989), 527-545.
[30] G. Weiss; *Regular linear systems with feedback*. Math. Control Signals Systems, 7 (1994), 23-57.
[31] G. Weiss; *Transfer functions of regular linear systems. Part I:Characterization of regularity*. Trans. Amer. Math. Soc., 342 (1994), 827-854.
[32] G. Weiss, R. Rebarber; *Optimizability and estimatability for infinite-dimensional linear systems*. SIAM J. Control Optim., 39 (2000), 1204-1232.
[33] M. Weiss, G. Weiss; *Optimal control of stable weakly regular linear systems*. Math. Control Signals Systems, 10(1997), 287-330.
[34] G.Weiss, H. J. Zwart; *An example in linear quadratic optimal control*. Systems Control Letters, 33 (1998), 339-349.
[35] G. Q. Xu, C. Liu, S. P. Yung; *Necessary conditions for the exact observability of systems on Hilbert spaces*. Systems Control Letters, 57 (2008), 222-227.
[36] H. J. Zwart; *Linear quadratic optimal control for abstract linear systems*. Modelling and optimization of distributed parameter systems (Warsaw, 1995), 175-182, Chapman Hall, New York, 1996.

Weisheng Jiang
College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China
E-mail address: wsjiang@cqu.edu.cn

Bin Liu
College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China
E-mail address: liubin@ctbu.edu.cn

Zhibing Zhang
College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China
E-mail address: zhibing0719@163.com