Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 182, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

A LIOUVILLE TYPE THEOREM FOR p-LAPLACE EQUATIONS

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Abstract. In this note we study solutions defined on the whole space $\mathbb{R}^N$ for the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u) = 0.$$

Under an appropriate condition on the growth of f, which is weaker than conditions previously considered in McCoy [3] and Cuccu-Mhammed-Porru [1], we prove the non-existence of non-trivial positive solutions.

1. Introduction

In this note we improve some Liouville type results previously obtained in McCoy [3] and Cuccu-Mohammed-Porru [1] for solutions to the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u) = 0 \quad \text{in } \mathbb{R}^N,\ p > 1, \tag{1.1}$$

where the nonlinearity f is a real differentiable function.

The classical Liouville Theorem states that *any harmonic function on the whole Euclidian space* $\mathbb{R}^N$, $N \geq 2$, *which is bounded from one side, must be identically constant.* Nowadays it is already known that this property is not anymore a prerogative of harmonic functions, since it is also shared by bounded (from below and/or above) entire solutions to many other differential equations (we refer the reader to the survey paper of Farina [2] for an overview on Liouville type theorems in PDEs). For instance, when $p = 2$ in (1.1), McCoy [3] has proved that *if f is differentiable and satisfies*

$$f'(t) \leq \frac{N+1}{N-1}\frac{f(t)}{t} \quad \text{for all } t > 0, \tag{1.2}$$

then any positive solution of (1.1) *must be a constant.* Later, this result was extended to the more general case $p > 1$ by Cuccu, Mohammed and Porru [1], as follows: *if f is differentiable and satisfies*

$$f'(t) \leq (p-1)\frac{N+1}{N-1}\frac{f(t)}{t} \quad \text{for all } t > 0, \tag{1.3}$$

then any positive solution of (1.1) *must be a constant.*

Adapting the main idea from the above mentioned works, we are going to show that, under a weaker condition on the growth of f, the above Liouville type results still hold. More precisely, we have:

2010 *Mathematics Subject Classification.* 70H25.
Key words and phrases. p-Laplace equation; Liouville theorem; entire solutions.

Submitted October 18, 2014. Published July 1, 2015.

Theorem 1.1. *Assume that $u(\mathbf{x}) > 0$ satisfies (1.1). If f is differentiable and satisfies*

$$f'(t) \leq \beta(p-1)\frac{N+1}{N-1}\frac{f(t)}{t} \quad \text{for all } t > 0, \tag{1.4}$$

where

$$\beta \in \begin{cases} [1, \frac{N-1}{N-p}), & \text{when } 1 < p < N, \\ [1, \infty), & \text{when } p \geq N, \end{cases} \tag{1.5}$$

then $u(\mathbf{x})$ must be a constant. As a consequence, if $f(t) = 0$ has no positive roots, then (1.1) has no positive weak solutions.

2. Proof of Theorem 1.1

We first state a lemma which plays some role in the proof of Theorem 1.1.

Lemma 2.1 (Cuccu-Mohammed-Porru [1]). *Let $p > 1$ be a real number and $N \geq 2$. If $u(x)$ is a C^2 function and u_i denotes partial differentiation with respect to x_i, then*

$$(p-1)u_{11}^2 + \sum_{i=2}^{N} u_{ii}^2 \geq \frac{(p-1)(N-1)+1}{N-1}u_{11}^2 - \frac{2}{N-1}\Delta u u_{11} + \frac{1}{N-1}(\Delta u)^2. \tag{2.1}$$

Let us now introduce the auxiliary function

$$P(u; \mathbf{x}) := \frac{|\nabla u(\mathbf{x})|^2}{u^{2\beta}(\mathbf{x})}. \tag{2.2}$$

Let us also consider a point $\mathbf{x}^*$ where $|\nabla u| > 0$. From a seminal work of Tolksdorf [4] we know that $u(\mathbf{x})$ is smooth in $\omega := \{\mathbf{x} \in \Omega : |\nabla u|(\mathbf{x}) > 0\}$. Therefore, we may compute in ω, successively, the following derivatives:

$$P_k = \frac{2}{u^{2\beta}}u_{ik}u_i - \frac{2\beta}{u^{2\beta+1}}|\nabla u|^2 u_k \tag{2.3}$$

$$\begin{aligned} P_{kl} = &\frac{2}{u^{2\beta}}u_{ik}u_{il} + \frac{2}{u^{2\beta}}u_{ikl}u_i - \frac{4\beta}{u^{2\beta+1}}u_{ik}u_i u_l - \frac{4\beta}{u^{2\beta+1}}u_{il}u_i u_k \\ &+ \frac{2\beta(2\beta+1)}{u^{2\beta+2}}|\nabla u|^2 u_k u_l - \frac{2\beta}{u^{2\beta+1}}|\nabla u|^2 u_{kl}. \end{aligned} \tag{2.4}$$

Now, performing eventually a translation and/or rotation if necessary, we choose the coordinate axes such that at $\mathbf{x}^*$ we have

$$|\nabla u| = u_1 \quad u_i = 0 \quad \text{for } i = 2, \dots, N. \tag{2.5}$$

Using (2.5) in (2.4) we find that

$$\begin{aligned} P_{11} = &\frac{2}{u^{2\beta}}u_{1i}u_{1i} + \frac{2}{u^{2\beta}}u_{111}u_1 - \frac{8\beta}{u^{2\beta+1}}u_{11}u_1^2 \\ &+ \frac{2\beta(2\beta+1)}{u^{2\beta+2}}u_1^4 - \frac{2\beta}{u^{2\beta+1}}u_{11}u_1^2, \end{aligned} \tag{2.6}$$

respectively

$$\begin{aligned} \Delta P = &\frac{2}{u^{2\beta}}u_{ik}u_{ik} + \frac{2}{u^{2\beta}}(\Delta u)_1 u_1 - \frac{8\beta}{u^{2\beta+1}}u_{11}u_1^2 \\ &+ \frac{2\beta(2\beta+1)}{u^{2\beta+2}}u_1^4 - \frac{2\beta}{u^{2\beta+1}}u_1^2\Delta u. \end{aligned} \tag{2.7}$$

It then it follows that

$$\begin{aligned}
&\Delta P+(p-2)P_{11}\\
&=\frac{2}{u^{2\beta}}[(\Delta u)_1+(p-2)u_{111}]u_1+\frac{2}{u^{2\beta}}[u_{ik}u_{ik}+(p-2)u_{1j}u_{1j}]\\
&\quad-\frac{2\beta}{u^{2\beta+1}}[\Delta u+(5p-6)u_{11}]u_1^2+\frac{2\beta(2\beta+1)}{u^{2\beta+2}}(p-1)u_1^4.
\end{aligned}\tag{2.8}$$

On the other hand, differentiating (1.1) with respect to x_1 and evaluating the result making use of (2.5), we obtain that at $\mathbf{x}^*$ we have

$$\begin{aligned}
&u_1^{p-1}[(\Delta u)_1+(p-2)u_{111}+2(p-2)\sum_{j=2}^N u_{1j}^2u_1^{-1}]\\
&\quad+(p-2)[\Delta u+(p-2)u_{11}]u_{11}u_1^{p-3}+f'u_1=0.
\end{aligned}\tag{2.9}$$

Also, evaluating equation (1.1) at $\mathbf{x}^*$ we have

$$\Delta u+(p-2)u_{11}=-f(u)u_1^{2-p}.\tag{2.10}$$

Inserting (2.10) in (2.9) we obtain

$$(\Delta u)_1+(p-2)u_{111}=2(2-p)\sum_{j=2}^N u_{1j}^2u_1^{-1}+(p-2)u_{11}fu_1^{1-p}-f'u_1^{3-p}.\tag{2.11}$$

Making now use of Lemma 2.1, we also have

$$\begin{aligned}
&u_{ij}u_{ij}+(p-2)u_{1j}u_{1j}\\
&\ge(p-1)u_{11}^2+\sum_{i=2}^N u_{ii}^2+p\sum_{j=2}^N u_{1j}^2\\
&\ge\frac{(p-1)(N-1)+1}{N-1}u_{11}^2-\frac{2}{N-1}\Delta u u_{11}+\frac{1}{N-1}(\Delta u)^2+p\sum_{j=2}^N u_{1j}^2.
\end{aligned}\tag{2.12}$$

Therefore, inserting (2.11) and (2.12) in (2.8) leads to

$$\begin{aligned}
&\Delta P+(p-2)P_{11}\\
&\ge\frac{2}{u^{2\beta}}\Big\{(4-p)\sum_{j=2}^N u_{1j}^2+(p-2)u_{11}fu^{2-p}-f'u_1^{4-p}\\
&\quad+\frac{(p-1)(N-1)+1}{N-1}u_{11}^2-\frac{2}{N-1}\Delta u u_{11}+\frac{1}{N-1}(\Delta u)^2\\
&\quad-\beta[\Delta u+(5p-6)u_{11}]\frac{u_1^2}{u}+\beta(2\beta+1)(p-1)\frac{u_1^4}{u^2}\Big\}.
\end{aligned}\tag{2.13}$$

From (2.3) and (2.5) we have

$$P_i=\frac{2}{u^{2\beta}}u_{i1}u_1\quad\text{for } i=2,\dots,N.\tag{2.14}$$

Therefore

$$\frac{4}{u^{4\beta}}u_1^2\sum_{i=2}^N u_{i1}u_{i1}=\sum_{i=2}^N P_i^2\le|\nabla P|^2,\tag{2.15}$$

so that

$$\frac{2}{u^{2\beta}}(4-p)\sum_{j=2}^{N} u_{1j}^2 \geq -|4-p|\frac{|\nabla P|^2}{2P}. \tag{2.16}$$

Finally, since

$$\begin{gathered}\frac{(p-1)(N-1)+1+2(p-2)+(p-2)^2}{N-1} = \frac{(p-1)(p+N-2)}{N-1},\\ \frac{2}{N-1}+\frac{2(p-2)}{N-1} = \frac{2(p-1)}{N-1},\end{gathered} \tag{2.17}$$

using (2.10) and (2.16) in (2.13) we are lead to

$$\begin{aligned}&\Delta P + (p-2)P_{11}\\ &\geq -|4-p|\frac{|\nabla P|^2}{2P} + \frac{2}{u^{2\beta}}\Big\{-f'u_1^{4-p} + \frac{(p-1)(p+N-2)}{N-1}u_{11}^2\\ &\quad + \big(p-2+\frac{2(p-1)}{N-1}\big)fu_1^{2-p}u_{11} + \frac{1}{N-1}(fu_1^{2-p})^2\\ &\quad - \beta[4(p-1)u_{11} - fu_1^{2-p}]\frac{u_1^2}{u} + \beta(2\beta+1)(p-1)\frac{u_1^4}{u^2}\Big\}.\end{aligned} \tag{2.18}$$

Now, evaluating (2.3) at $\mathbf{x}^*$, by using (2.5), we have

$$P_1 = \frac{2}{u^{2\beta}}u_{11}u_1 - \frac{2\beta}{u^{2\beta+1}}u_1^3, \tag{2.19}$$

so that

$$u_{11} = \frac{P_1u^{2\beta}}{2u_1} + \beta\frac{u_1^2}{u}. \tag{2.20}$$

Inserting (2.20) into (2.18) we obtain

$$\begin{aligned}&\Delta P + (p-2)P_{11}\\ &\geq -|4-p|\frac{|\nabla P|^2}{2P} + \frac{2}{u^{2\beta}}\Big\{-f'u_1^{4-p}\\ &\quad + \frac{(p-1)(p+N-2)}{N-1}\Big(\frac{P_1u^{2\beta}}{2u_1} + \beta\frac{u_1^2}{u}\Big)^2\\ &\quad + \Big(p-2+\frac{2(p-1)}{N-1}\Big)fu_1^{2-p}\Big(\frac{P_1u^{2\beta}}{2u_1} + \beta\frac{u_1^2}{u}\Big)\\ &\quad + \frac{1}{N-1}(fu_1^{2-p})^2 - \beta\Big[4(p-1)\Big(\frac{P_1u^{2\beta}}{2u_1} + \beta\frac{u_1^2}{u}\Big) - fu_1^{2-p}\Big]\frac{u_1^2}{u}\\ &\quad + \beta(2\beta+1)(p-1)\frac{u_1^4}{u^2}\Big\}.\end{aligned} \tag{2.21}$$

Next, using the restriction (1.4) on f we note that

$$\begin{aligned}&-f'u^{4-p} + \Big(p-2+\frac{2(p-1)}{N-1}\Big)\beta\frac{f}{u}u_1^{4-p} + \beta\frac{f}{u}u_1^{4-p}\\ &= u^{4-p}\big[-f' + \beta(p-1)\frac{N+1}{N-1}\frac{f}{u}\big] \geq 0.\end{aligned} \tag{2.22}$$

We also note that

$$\frac{(p-1)(p+N-2)}{N-1}\Big(\frac{P_1u^{2\beta}}{2u_1}\Big)^2 \geq 0. \tag{2.23}$$

Therefore

$$\begin{aligned}\Delta P+(p-2)P_{11} \geq & -|4-p|\frac{|\nabla P|^2}{2P} \\ & +\frac{(p-1)(p+N-2)}{N-1}\Big(2\beta\frac{P_1u_1}{u}+2\beta^2\frac{u_1^4}{u^{2\beta+2}}\Big) \\ & +\Big(p-2+\frac{2(p-1)}{N-1}\Big)P_1fu_1^{1-p}+\frac{2}{N-1}\frac{(fu_1^{2-p})^2}{u^{2\beta}} \\ & -4\beta(p-1)\frac{P_1u_1}{u}+2(p-1)(\beta-2\beta^2)\frac{u_1^4}{u^{2\beta+2}}\,.\end{aligned} \tag{2.24}$$

Moreover, using the following two identities

$$\frac{2(p-1)(p+N-2)}{N-1}-4(p-1)=\frac{2(p-1)(p-N)}{N-1}, \tag{2.25}$$

$$\frac{2(p-1)(p+N-2)}{N-1}\beta^2+2(p-1)(\beta-2\beta^2)=\frac{p-1}{N-1}[2\beta^2(p-N)+2\beta(N-1)], \tag{2.26}$$

one may easily see that (2.24) becomes

$$\begin{aligned}\frac{\Delta P+(p-2)P_{11}}{P} \geq & -|4-p|\frac{|\nabla P|^2}{2P^2}+2\beta\frac{(p-1)(p-N)}{N-1}\frac{P_1u_1}{Pu} \\ & +\Big(p-2+\frac{2(p-1)}{N-1}\Big)\frac{P_1fu_1^{1-p}}{P}+\frac{2}{N-1}(fu_1^{1-p})^2 \\ & +2\beta[\beta(p-N)+N-1]\frac{p-1}{N-1}\frac{u_1^2}{u^2}.\end{aligned} \tag{2.27}$$

Next, let us consider a point $\mathbf{x_0}\in\mathbb{R}^N$ and define

$$J(\mathbf{x})=(a^2-r^2)^2P, \tag{2.28}$$

where $a>0$ is a constant and $r:=|\mathbf{x}-\mathbf{x_0}|$. Let us denote by B the ball centered at $\mathbf{x_0}$ and of radius a. Then we immediately notice that

$$J(\mathbf{x})\geq 0 \text{ in } B,\quad J(\mathbf{x})=0 \text{ on } \partial B. \tag{2.29}$$

Consequently, $J(\mathbf{x})$ must attain its maximum at some (interior) point $\mathbf{x}^*$.

Now, if $|\nabla u|(\mathbf{x}^*)=0$, then $P\equiv 0$ in B. Since the ball was chosen arbitrarily, $P\equiv 0$ in every ball, so that $\nabla u\equiv 0$ in $\mathbb{R}^N$ and our theorem follows. It thus remain to analyze the case $|\nabla u|(\mathbf{x}^*)>0$. In such a case, we have the following complementary inequality at x^* (see Cuccu-Mohammed-Porru [1, p. 227] for the proof; they used a different auxiliary function P, but the proof is identical, since the form of P does not really play a role in the proof):

$$\frac{\Delta P+(p-2)P_{11}}{P}\leq\frac{Ca^2}{(a^2-r^2)^2},\quad C:=24+4N+28|p-2|. \tag{2.30}$$

Combining (2.27) and (2.30) we obtain

$$\begin{aligned}\frac{Ca^2}{(a^2-r^2)^2} \geq & -|4-p|\frac{|\nabla P|^2}{2P^2}+2\beta\frac{(p-1)(p-N)}{N-1}\frac{P_1u_1}{Pu} \\ & +\Big(p-2+\frac{2(p-1)}{N-1}\Big)\frac{P_1fu_1^{1-p}}{P}+\frac{2}{N-1}(fu_1^{1-p})^2 \\ & +2\beta[\beta(p-N)+N-1]\frac{p-1}{N-1}\frac{u_1^2}{u^2}.\end{aligned} \tag{2.31}$$

On the other hand, differentiating (2.28) we obtain that at $\mathbf{x}^*$ (the point of maximum for J in B) we have

$$J_i = -2(a^2 - r^2)(r^2)_i P + (a^2 - r^2)^2 P_i = 0, \tag{2.32}$$

so that

$$P_1 = 2\frac{(r^2)_1 P}{a^2 - r^2}, \quad \nabla P = 2\frac{\nabla r^2 P}{a^2 - r^2}. \tag{2.33}$$

From (2.5) and (2.33) we then conclude that

$$\frac{P_1 u_1}{P} = \frac{\nabla P \nabla u}{P} = 2\frac{\nabla r^2 \nabla u}{a^2 - r^2}, \tag{2.34}$$

$$\frac{|\nabla P|}{P} = \frac{2|\nabla(r^2)|}{a^2 - r^2} = \frac{4r}{a^2 - r^2}. \tag{2.35}$$

Now using (2.34) and (2.35) in (2.31) we obtain

$$\begin{aligned} \frac{Ca^2}{(a^2 - r^2)^2} &\geq -|4 - p|\frac{8r^2}{(a^2 - r^2)^2} + 4\beta\frac{(p-1)(p-N)}{N-1}\frac{\nabla r^2 \nabla u}{(a^2 - r^2)u} \\ &\quad + 2\Big(p - 2 + \frac{2(p-1)}{N-1}\Big) f u_1^{1-p}\frac{\nabla r^2 \nabla u}{a^2 - r^2} + \frac{2}{N-1}(f u_1^{1-p})^2 \\ &\quad + 2\beta[\beta(p - N) + N - 1]\frac{p-1}{N-1}\frac{|\nabla u|^2}{u^2}. \end{aligned} \tag{2.36}$$

Moreover, by classical inequalities we have

$$\begin{aligned} &4\beta(p-1)(p-N)\frac{\nabla r^2 \nabla u}{(a^2 - r^2)u} \\ &\geq -\beta^2\gamma(p-1)^2\frac{|\nabla u|^2}{u^2} - 4(p-N)^2\frac{4r^2}{\gamma(a^2 - r^2)^2}, \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} &2(p-2) + \frac{2(p-1)}{N-1} f u_1^{-p}\frac{\nabla r^2 \nabla u}{a^2 - r^2} \\ &\geq -4(|p-2| + \frac{2(p-1)}{N-1})|f|u_1^{p-1}\frac{r}{a^2 - r^2} \\ &\geq -\frac{2}{N-1}(f u_1^{1-p})^2 - \widetilde{C}\frac{r^2}{(a^2 - r^2)^2}, \end{aligned} \tag{2.38}$$

with $\gamma > 0$ to be chosen and $\widetilde{C} := 2[(N-1)(p-2) + 2(p-1)]^2/(N-1)$. Inserting now estimates (2.37) and (2.38) into (2.36) we find

$$\begin{aligned} \frac{Ca^2}{(a^2 - r^2)^2} &\geq -|4 - p|\frac{8r^2}{(a^2 - r^2)^2} - 4(p-N)^2\frac{4r^2}{\gamma(a^2 - r^2)^2} - \widetilde{C}\frac{r^2}{(a^2 - r^2)^2} \\ &\quad + [2\beta^2(p-N) + 2\beta(N-1) - \beta^2\gamma(p-1)]\frac{p-1}{N-1}\frac{|\nabla u|^2}{u^2}. \end{aligned} \tag{2.39}$$

Now let us analyze separately the following two cases:

Case 1. when $1 < p < N$ and $\beta \in [1, \frac{N-1}{N-p})$, we have

$$2\beta^2(p-N) + 2\beta(N-1) > 0. \tag{2.40}$$

Therefore, we can choose γ small enough so that we have

$$[2\beta^2(p-N) + 2\beta(N-1) - \beta^2\gamma(p-1)] > 0. \tag{2.41}$$

smallskip

Case 2. when $p \geq N$ and $\beta \in [1, +\infty)$, as above, we can again choose a small enough γ so that (2.41) holds.

In conclusion, for a well chosen γ, there exists a constant $K = K(N, p, \beta, \gamma)$ such that

$$\frac{|\nabla u|^2}{u^2} \leq \frac{Ka^2}{(a^2 - r^2)^2}. \tag{2.42}$$

Moreover, since $u(\mathbf{x})$ is positive, there exists a constant $L > 0$ such that $u^{2-2\beta} \leq L$. Therefore, at some point $\mathbf{x}^*$ we have

$$J(\mathbf{x}^*) = \frac{|\nabla u|^2}{u^2} \frac{1}{u^{2\beta-2}} (a^2 - r^2)^2 \leq KLa^2. \tag{2.43}$$

But $\mathbf{x}^*$ is a point of maximum for $J(\mathbf{x})$ in B, so that we have

$$J(\mathbf{x}_0) = \frac{|\nabla u|^2}{u^{2\beta}} a^4 \leq KLa^2. \tag{2.44}$$

It follows that at $\mathbf{x} = \mathbf{x}_0$ we have

$$\frac{|\nabla u|^2}{u^{2\beta}} \leq \frac{KL}{a^2}. \tag{2.45}$$

Letting $a \to \infty$ we find that $\nabla u = 0$ at x_0. Since x_0 is arbitrary, we must have $\nabla u = 0$ in $\mathbb{R}^N$. The proof is thus achieved.

Acknowledgments. This work was supported by a CNCSIS (Romania) research grant (PN-II-ID-PCE-2012-4-0021 : Variable Exponent Analysis: Partial Differential Equations and Calculus of Variations).

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