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HOMOCLINIC ORBITS OF SECOND-ORDER NONLINEAR DIFFERENCE EQUATIONS

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Abstract. We establish existence criteria for homoclinic orbits of second-order nonlinear difference equations by using the critical point theory in combination with periodic approximations.

1. Introduction

Homoclinic orbits play an important role in analyzing the chaos of dynamical systems, and have been the subject of many investigations. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation, its perturbed system probably produce chaotic phenomenon. So homoclinic orbits have been extensively investigated since the time of Poincaré, see [12, 13, 14, 15, 16, 17, 26, 28] and the references therein.

Difference equations [1, 9] are closely related to differential equations in the sense that a differential equation model is often derived from a difference equation, and numerical solutions of a differential equation are obtained by discretizing the differential equation. Therefore, the study of homoclinic orbits [4, 5, 6, 7, 8, 10, 11, 20, 21, 22, 23, 30] of difference equation is meaningful.

Here $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For any $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a \leq b$. The symbol l^2 denotes the space of real functions whose second powers are summable on $\mathbb{Z}$. Also, * denotes the transpose of a vector.

This article considers the existence for homoclinic orbits of second-order nonlinear difference equation

$$Lu(t) = f(t, u(t+T), u(t), u(t-T)), \quad t \in \mathbb{Z} \tag{1.1}$$

containing both advance and retardation. Here the operator L is the Jacobi operator

$$Lu(t) = a(t)u(t+1) + a(t-1)u(t-1) + b(t)u(t),$$

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where $a(t)$ and $b(t)$ are real valued for each $t \in \mathbb{Z}$, T is a given nonnegative integer, $f \in C(\mathbb{Z} \times \mathbb{R}^3, \mathbb{R})$, $a(t)$, $b(t)$ and $f(t, v_1, v_2, v_3)$ are M-periodic in t for a given positive integer M. Jacobi operators appear in a variety of applications [29].

We may think of (1.1) as being a discrete analogue of the second-order nonlinear differential equation

$$Su(s) = f(s, u(s+T), u(s), u(s-T)), \quad s \in \mathbb{R}, \tag{1.2}$$

where S is the Sturm-Liouville differential expression, $f \in C(\mathbb{R}^4, \mathbb{R})$. Equations similar in structure to (1.2) arise in the study of homoclinic orbits [13, 15, 16, 17] of functional differential equations.

For the case $T = 1$, Chen and Fang [3] obtained the existence of periodic and subharmonic solutions of the second-order p-Laplacian difference equation

$$\Delta(\varphi_p(\Delta u(t-1))) + f(t, u(t+1), u(t), u(t-1)) = 0, \quad t \in \mathbb{Z},$$

and Chen and Tang [4] obtained the existence of infinitely many homoclinic orbits of the fourth-order difference equation

$$\Delta^4 u(t-2) + q(t)u(t) = f(t, u(t+1), u(t), u(t-1)), \quad t \in \mathbb{Z}$$

containing both advance and retardation.

It is well known that critical point theory is a powerful tool that deals with the problems of differential equations [2, 12, 13, 14, 15, 16, 17, 24, 27]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions for second-order difference equations [18, 19]. Along this direction, Ma and Guo [22] (without periodicity assumption) and [23] (with periodicity assumption) applied variational methods to prove the existence of homoclinic orbits for the special form of (1.1) (with $T = 0$). The Ambrosetti-Rabinowitz condition plays a crucial role to ensure the boundedness of Palais-Smale sequences. This is very crucial in applying the critical point theory.

Some special cases of (1.1) have been studied by many researchers via variational methods, see [18, 19, 22, 23]. However, to our best knowledge, the results on homoclinic orbits of (1.1) are scarce in the literature. Since (1.1) contains both advance and retardation, there are very few manuscripts dealing with this subject, the traditional ways of establishing the functional in [10, 18, 19, 20, 22, 23, 25] are inapplicable to our case.

The main purpose of this article is to give some sufficient conditions for the existence of a nontrivial homoclinic orbit for (1.1) without the classical Ambrosetti-Rabinowitz condition. In particular, our results generalize and improve the existing results; see Remarks 1.3 and 1.4. The motivation for the present work stems from the recent papers [3, 7, 17].

Let

$$\underline{\lambda} = \min_{t \in \mathbb{Z}(1,M)} (b(t) - |a(t-1)| - |a(t)|), \quad \bar{\lambda} = \max_{t \in \mathbb{Z}(1,M)} (b(t) + |a(t-1)| + |a(t)|).$$

In this article we use the following hypotheses:

- (H1) $b(t) - |a(t-1)| - |a(t)| > 0$, for all $t \in \mathbb{Z}$;
- (H2) there exists a functional $F(t, v_1, v_2) \in C^1(\mathbb{Z} \times \mathbb{R}^2, \mathbb{R})$ with $F(t+M, v_1, v_2) = F(t, v_1, v_2)$ and it satisfies

$$\frac{\partial F(t-T, v_2, v_3)}{\partial v_2} + \frac{\partial F(t, v_1, v_2)}{\partial v_2} = f(t, v_1, v_2, v_3);$$

(H3) there exist positive constants δ_1 and $a_1 < \underline{\lambda}/4$ such that

$$|F(t, v_1, v_2)| \leq a_1 \left(v_1^2 + v_2^2\right)$$

for all $t \in \mathbb{Z}$ and $\sqrt{v_1^2 + v_2^2} \leq \delta_1$;

(H4) there exist constants $\rho_1, c_1 > \bar{\lambda}/4$ and b_1 such that

$$F(t, v_1, v_2) \geq c_1 \left(v_1^2 + v_2^2\right) + b_1$$

for all $t \in \mathbb{Z}$ and $\sqrt{v_1^2 + v_2^2} \geq \rho_1$;

(H5)

$$\frac{\partial F(t, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(t, v_1, v_2)}{\partial v_2} v_2 - 2F(t, v_1, v_2) > 0,$$

for all $(t, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2 \setminus \{(0,0)\}$;

(H6)

$$\frac{\partial F(t, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(t, v_1, v_2)}{\partial v_2} v_2 - 2F(t, v_1, v_2) \to +\infty$$

as $\sqrt{v_1^2 + v_2^2} \to +\infty$.

Our main results are the following theorem.

Theorem 1.1. *Suppose that* (H1)–(H6) *are satisfied. Then* (1.1) *has a nontrivial homoclinic orbit.*

Remark 1.2. By (H4), it is easy to see that there exists a constant $\zeta_1 > 0$ such that

(H4') $F(t, v_1, v_2) \geq c_1 \left(v_1^2 + v_2^2\right) + b_1 - \zeta_1$, for all $(t, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2$.

As a matter of fact, letting

$$\zeta_1 = \max \left\{ |F(n, v_1, v_2) - c_1 \left(v_1^2 + v_2^2\right) - b_1| : n \in \mathbb{Z}, \sqrt{v_1^2 + v_2^2} \leq \rho_1 \right\},$$

we can easily get the desired result.

Remark 1.3. As a special case of Theorem 1.1 with $T = 0$ and $a(t) < 0$, we obtain [23, Theorem 1.1].

Remark 1.4. In many studies (see e.g. [18, 19, 22, 23]) of second-order difference equations, the following classical Ambrosetti-Rabinowitz condition is assumed.

(AR) There exists a constant $\beta > 2$ such that $0 < \beta F(t, u) \leq u f(t, u)$ for all $t \in \mathbb{Z}$ and $u \in \mathbb{R} \setminus \{0\}$.

Note that (H4)–(H6) are much weaker than (AR). Thus our result improves that the existing results.

For the next theorem, we use the hypotheses:

(H7) there exist positive constants δ_2 and $a_2 > \frac{\bar{\lambda}}{4}$ such that

$$|F(t, v_1, v_2)| \geq a_2 \left(v_1^2 + v_2^2\right)$$

for all $t \in \mathbb{Z}$ and $\sqrt{v_1^2 + v_2^2} \leq \delta_2$;

(H8) there exists a constant $1 < \mu < 2$ such that

$$0 < \frac{\partial F(t, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(t, v_1, v_2)}{\partial v_2} v_2 \leq \mu F(t, v_1, v_2),$$

for all $(t, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2 \setminus \{(0,0)\}$.

Theorem 1.5. *Suppose that* (H1), (H2), (H7), (H8) *are satisfied. Then* (1.1) *has a nontrivial homoclinic orbit.*

Remark 1.6. By (H8), there exist constants $a_3 > 0$ and b_2 such that

$$F(t, v_1, v_2) \le a_3 \left(v_1^2 + v_2^2\right)^{\mu/2} + b_2 \quad \text{for all } t \in \mathbb{Z},$$

which implies that there exist constants $\rho_2 > 0$ and $c_2 < \frac{\lambda}{4}$ such that

(H9) $F(t, v_1, v_2) \le c_2 \left(v_1^2 + v_2^2\right) + b_2$ for all $t \in \mathbb{Z}$ and $\sqrt{v_1^2 + v_2^2} \ge \rho_2$.

By (H9), it is easy to see that there exists a constant $\zeta_2 > 0$ such that

(H9') $F(t, v_1, v_2) \le c_2 \left(v_1^2 + v_2^2\right) + b_2 + \zeta_2$, for all $(t, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2$.

The remainder of this paper is organized as follows. In Section 2, we shall establish the variational framework associated with (1.1) and transfer the problem of the existence of homoclinic orbits of (1.1) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. In Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give two examples to illustrate the results.

2. Preliminaries

To apply the critical point theory, we shall establish the corresponding variational framework for (1.1) and give some lemmas which will be of fundamental importance in proving our results. We start by giving the basic notation.

Let S be the set of sequences

$$u = \{u(t)\}_{t\in\mathbb{Z}} = (\dots, u(-t), \dots, u(-1), u(0), u(1), \dots, u(t), \dots);$$

that is,

$$S = \{\{u(t)\} : u(t) \in \mathbb{R},\ t \in \mathbb{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbb{R}$, $au + bv$ is defined by

$$au + bv = \{au(t) + bv(t)\}_{t=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers M and m, we define

$$E_m = \{u \in S | u(t + 2mM) = u(t),\ \forall t \in \mathbb{Z}\}.$$

Clearly, E_m is isomorphic to $\mathbb{R}^{2mM}$. E_m can be equipped with the inner product

$$(u, v) = \sum_{t=-mM}^{mM-1} u(t) \cdot v(t), \quad \forall u, v \in E_m, \tag{2.1}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \Big(\sum_{t=-mM}^{mM-1} u^2(t)\Big)^{1/2}, \quad \forall u \in E_m. \tag{2.2}$$

It is obvious that E_m with the inner product (2.1) is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbb{R}^{2mM}$.

In what follows, we define a norm in E_m by

$$\|u\|_\infty = \max_{t\in\mathbb{Z}(-mM, mM-1)} |u(t)|, \quad \forall u \in E_m.$$

For $u \in E_m$, we define the functional J_m by

$$J_m(u) = \frac{1}{2}\sum_{t=-mM}^{mM-1} Lu(t)\cdot u(t) - \sum_{t=-mM}^{mM-1} F(t, u(t+T), u(t)). \tag{2.3}$$

Clearly, $J_m \in C^1(E_m, \mathbb{R})$ and for any $u = \{u(t)\}_{t\in\mathbb{Z}} \in E_m$, by the periodicity of $\{u(t)\}_{t\in\mathbb{Z}}$, we can compute the partial derivative as

$$\frac{\partial J_m(u)}{\partial u(t)} = Lu(t) - f(t, u(t+T), u(t), u(t-T)),\ \forall t \in \mathbb{Z}(-mM, mM-1). \tag{2.4}$$

Thus, u is a critical point of J_m on E_m if and only if

$$Lu(t) = f(t, u(t+T), u(t), u(t-T)),\ \forall t \in \mathbb{Z}(-mM, mM-1).$$

Due to the periodicity of $u = \{u(t)\}_{t\in\mathbb{Z}} \in E_m$ and $f(t, v_1, v_2, v_3)$ in the first variable t, we reduce the existence of periodic solutions of (1.1) to the existence of critical points of J_m on E_m. That is, the functional J_m is just the variational framework of (1.1).

Let E be a real Banach space, $J \in C^1(E, \mathbb{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (PS condition for short) if any sequence $\{u(t)\} \subset E$ for which $\{J(u(t))\}$ is bounded and $J'(u(t)) \to 0$ $(t \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Mountain Pass Lemma [27]). *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfy the PS condition. If $J(0) = 0$ and*

- (J1) *there exist constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$, and*
- (J2) *there exists $e \in E \setminus B_\rho$ such that $J(e) \leq 0$.*

Then J possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} J(g(s)), \tag{2.5}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0,\ g(1) = e\}. \tag{2.6}$$

Lemma 2.2. *Assume that* (H1) *holds. Then there exist constants $\underline{\lambda}$ and $\bar{\lambda}$ independent of m, such that*

$$\underline{\lambda}\|u\|^2 \leq \sum_{t=-mM}^{mM-1} Lu(t)\cdot u(t) \leq \bar{\lambda}\|u\|^2. \tag{2.7}$$

Proof. Let

$$\sum_{t=-mM}^{mM-1} Lu(t)\cdot u(t) = (P_m u, u),$$

where $u = (u(-mM), \dots, u(-1), u(0), u(1), \dots, u(mM-1))^*$ and

$$P_m = \begin{pmatrix} b(-mM) & a(-mM) & 0 & \dots & 0 & a(-mM-1) \\ a(-mM) & b(-mM+1) & a(-mM+1) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b(mM-2) & a(mM-2) \\ a(mM-1) & 0 & 0 & \dots & a(mM-2) & b(mM-1) \end{pmatrix}$$

which is a $2mM \times 2mM$ matrix. By (H1), P_m is positive definite.

Let $\lambda_{-mM}, \lambda_{-mM+1}, \dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots, \lambda_{mM-2}, \lambda_{mM-1}$ be the eigenvalues of P_m. Applying matrix theory, we see that $\underline{\lambda} \le \lambda_i \le \overline{\lambda}$, $i \in \mathbb{Z}(-mM, mM-1)$. From the definition of the norm $\|\cdot\|$, (2.7) is obviously true. □

3. Proof of main results

In this section, we shall prove the results stated in Section 1 by using the critical point theory.

3.1. **Proof of Theorem 1.1.**

Lemma 3.1. *Suppose that* (H1), (H2)–(H6) *are satisfied. Then* J_m *satisfies the* PS *condition.*

Proof. Assume that $\{u_j\}_{j\in\mathbb{N}}$ in E_m is a sequence such that $\{J_m(u_j)\}_{j\in\mathbb{N}}$ is bounded. Then there exists a constant $K_1 > 0$ such that $-K_1 \le J_m(u_j)$. By (2.7) and (H4'), it is easy to see that

$$\begin{aligned} -K_1 \le J_m(u_j) &\le \frac{\bar{\lambda}}{2}\|u_j\|^2 - \sum_{t=-mM}^{mM-1} \{c_1[u_j^2(t+T) + u_j^2(t)] + b_1 - \zeta_1\} \\ &= \frac{\bar{\lambda}}{2}\|u_j\|^2 - 2c_1\|u_j\|^2 + 2mM\left(\zeta_1 - b_1\right), \ \forall j \in \mathbb{N}. \end{aligned}$$

Therefore,

$$\big(2c_1 - \frac{\bar{\lambda}}{2}\big)\|u_j\|^2 \le 2mM(\zeta_1 - b_1) + K_1. \tag{3.1}$$

Since $c_1 > \bar{\lambda}/4$, (3.1) implies that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E_m. Thus, $\{u_j\}_{j\in\mathbb{N}}$ possesses a convergence subsequence in E_m. The desired result follows. □

Lemma 3.2. *Suppose that* (H1)–(H6) *are satisfied. Then for any given positive integer* m*,* (1.1) *possesses a* $2mM$*-periodic solution* $u_m \in E_m$*.*

Proof. In our case, it is clear that $J_m(0) = 0$. By Lemma 3.1, J_m satisfies the PS condition. By (H3), we have

$$\begin{aligned} J_m(u) &\ge \frac{\underline{\lambda}}{2}\|u\|^2 - a_1 \sum_{t=-mM}^{mM-1} [u^2(t) + u^2(t+T)] \\ &\ge \frac{\underline{\lambda}}{2}\|u\|^2 - 2a_1\|u\|^2 \\ &= \big(\frac{\underline{\lambda}}{2} - 2a_1\big)\|u\|^2. \end{aligned}$$

Taking $\alpha_1 = (\frac{\underline{\lambda}}{2} - 2a_1)\delta_1^2 > 0$, we obtain

$$J_m(u)|_{\partial B_{\delta_1}} \ge \alpha_1 > 0,$$

which implies that J_m satisfies the condition (J1) of the Mountain Pass Lemma.

Next, we shall verify the condition (J2) of the Mountain Pass Lemma.. There exists a sufficiently large number $\rho > \max\{\rho_1, \delta_1\}$ such that

$$\big(2c_1 - \frac{\bar{\lambda}}{2}\big)\rho^2 \ge |b_1|. \tag{3.2}$$

Let $e_m^{(1)} \in E_m$ and

$$e_m^{(1)}(t) = \begin{cases} \rho, & \text{if } t = 0, \\ 0, & \text{if } t \in \{j \in \mathbb{Z} : -mM \le j \le mM-1 \text{ and } j \ne 0\}, \end{cases}$$

$$e_m^{(1)}(t+T) = \begin{cases} \rho, & \text{if } t = 0, \\ 0, & \text{if } t \in \{j \in \mathbb{Z} : -mM \le j \le mM-1 \text{ and } j \ne 0\}. \end{cases}$$

Then

$$\begin{aligned} &F\big(t, e_m^{(1)}(t+T), e_m^{(1)}(t)\big) \\ &= \begin{cases} F(0,\rho,\rho), & \text{if } t = 0, \\ 0, & \text{if } t \in \{j \in \mathbb{Z} : -mM \le j \le mM-1 \text{ and } j \ne 0\}. \end{cases} \end{aligned}$$

With (3.2) and (H4), we have

$$\begin{aligned} J_m\big(e_m^{(1)}\big) &= \frac{1}{2}\sum_{t=-mM}^{mM-1} L\big(e_m^{(1)}(t)\big)\cdot\big(e_m^{(1)}(t)\big) \\ &\quad - \sum_{t=-mM}^{mM-1} F\Big(t, \big(e_m^{(1)}(t+T)\big), \big(e_m^{(1)}(t)\big)\Big) \\ &\le \frac{\bar\lambda}{2}\|e_m^{(1)}\|^2 - 2c_1\rho^2 - b_1 \\ &= -\big(2c_1 - \frac{\bar\lambda}{2}\big)\rho^2 - b_1 \le 0. \end{aligned} \tag{3.3}$$

All the assumptions of the Mountain Pass Lemma have been verified. Consequently, J_m possesses a critical value c_m given by (2.5) and (2.6) with $E = E_m$ and $\Gamma = \Gamma_m$, where

$$\Gamma_m = \big\{g_m \in C([0,1], E_m) | g_m(0) = 0,\ g_m(1) = e_m^{(1)},\ e_m^{(1)} \in E_m \backslash B_\rho\big\}.$$

Let u_m denote the corresponding critical point of J_m on E_m. Note that $\|u_m\| \ne 0$ since $c_m > 0$. □

Lemma 3.3. *Suppose that* (H1)–(H6) *are satisfied. Then there exist positive constants δ_1 and η_1 independent of m such that*

$$\delta_1 \le \|u_m\|_\infty \le \eta_1. \tag{3.4}$$

Proof. The continuity of $F(0, v_1, v_2)$ with respect to the second and third variables implies that there exists a constant $\tau_1 > 0$ such that $|F(0, v_1, v_2)| \le \tau_1$ for $\sqrt{v_1^2 + v_2^2} \le \delta_1$. It is clear that

$$\begin{aligned} J_m(u_m) &\le \max_{0\le s\le 1}\Big\{\big|\frac{1}{2}\sum_{t=-mM}^{mM-1} L\big(se_m^{(1)}(t)\big)\cdot\big(se_m^{(1)}(t)\big)\big| \\ &\quad - \sum_{t=-mM}^{mM-1} F\Big(t, se_m^{(1)}(t+T), se_m^{(1)}(t)\Big)\Big\} \\ &\le \frac{\bar\lambda}{2}\|e_m^{(1)}\|^2 + \tau_1 \\ &= \frac{\bar\lambda}{2}\rho^2 + \tau_1. \end{aligned}$$

Let $\xi_1 = \frac{\bar{\lambda}}{2}\rho^2 + \tau_1$. Then $J_m(u_m) \le \xi_1$, which is a bound independent of m. From (2.3) and (2.4), we have

$$\begin{aligned}
J_m(u_m) &= \frac{1}{2}\sum_{t=-mM}^{mM-1}\Big[\frac{\partial F(t-T, u_m(t), u_m(t-T))}{\partial v_2}u_m(t) \\
&\quad + \frac{\partial F(t, u_m(t+T), u_m(t))}{\partial v_2}u_m(t)\Big] - \sum_{t=-mM}^{mM-1} F(t, u_m(t+T), u_m(t)) \\
&= \frac{1}{2}\sum_{t=-mM}^{mM-1}\Big[\frac{\partial F(t, u_m(t+T), u_m(t))}{\partial v_1}u_m(t+T) \\
&\quad + \frac{\partial F(t, u_m(t+T), u_m(t))}{\partial v_2}u_m(t)\Big] - \sum_{t=-mM}^{mM-1} F(t, u_m(t+T), u_m(t)) \\
&\le \xi_1.
\end{aligned}$$

By (H5) and (H6), there exists a constant $\eta_1 > 0$ such that

$$\frac{1}{2}\Big(\frac{\partial F(t, v_1, v_2)}{\partial v_1}v_1 + \frac{\partial F(t, v_1, v_2)}{\partial v_2}v_2\Big) - F(t, v_1, v_2) > \xi_1,$$

for all $t \in \mathbb{Z}$ and $\sqrt{v_1^2 + v_2^2} \ge \eta_1$, which implies that $|u_m(t)| \le \eta_1$ for all $t \in \mathbb{Z}$; that is, $\|u_m\|_\infty \le \eta_1$.

From the definition of J_m, we have

$$\begin{aligned}
0 &= (J_m'(u_m), u_m) \\
&\ge \underline{\lambda}\|u_m\|^2 - \sum_{t=-mM}^{mM-1}\Big[\frac{\partial F(t-T, u_m(t), u_m(t-T))}{\partial v_2}u_m(t) \\
&\quad + \frac{\partial F(t, u_m(t+T), u_m(t))}{\partial v_2}u_m(t)\Big].
\end{aligned}$$

This inequality and (H3) yield

$$\begin{aligned}
\underline{\lambda}\|u_m\|^2 &\le \sum_{t=-mM}^{mM-1}\Big[\frac{\partial F(t, u_m(t+T), u_m(t))}{\partial v_1}u_m(t+T) \\
&\quad + \frac{\partial F(t, u_m(t+T), u_m(t))}{\partial v_2}u_m(t)\Big] \\
&\le \Big\{\sum_{t=-mM}^{mM-1}\big[\frac{\partial F(t, u_m(t+T), u_m(t))}{\partial v_1}\big]^2\Big\}^{1/2}\|u_m\| \\
&\quad + \Big\{\sum_{t=-mM}^{mM-1}\big[\frac{\partial F(t, u_m(t+T), u_m(t))}{\partial v_2}\big]^2\Big\}^{1/2}\|u_m\|.
\end{aligned}$$

That is,

$$\underline{\lambda}\|u_m\| \le \Big\{\sum_{t=-mM}^{mM-1}\big[\frac{\partial F(t, u_m(t+T), u_m(t))}{\partial v_1}\big]^2\Big\}^{1/2}$$

$$+\Big\{\sum_{t=-mM}^{mM-1}\big[\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_2}\big]^2\Big\}^{1/2}.$$

Thus,

$$\begin{aligned}\underline{\lambda}^2\|u_m\|^2 &\le 2\sum_{t=-mM}^{mM-1}\big[\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_1}\big]^2\\ &\quad+2\sum_{t=-mM}^{mM-1}\big[\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_2}\big]^2.\end{aligned}\tag{3.5}$$

From this inequality and (H3), we obtain

$$\underline{\lambda}^2\|u_m\|^2\le 2\sum_{t=-mM}^{mM-1}[2a_1u_m(t+T)]^2+2\sum_{t=-mM}^{mM-1}[2a_1u_m(t)]^2=16a_1^2\|u_m\|^2.$$

Thus, we have $u_m=0$. This contradicts $\|u_m\|\neq 0$, which shows that

$$\|u_m\|_\infty\ge\delta_1,$$

and the proof is complete. □

Proof of Theorem 1.1. Now we shall give the existence of a nontrivial homoclinic orbit. Consider the sequence $\{u_m(t)\}_{t\in\mathbb{Z}}$ of $2mM$-periodic solutions found in Section 3.1. First, by (3.4), for any $m\in\mathbb{N}$, there exists a constant $t_m\in\mathbb{Z}$ independent of m such that

$$|u_m(t_m)|\ge\delta_1.\tag{3.6}$$

Since $a(t)$, $b(t)$ and $f(t,v_1,v_2,v_3)$ are M-periodic in t, $\{u_m(t+jM)\}$ is also $2mM$-periodic solution of (1.1) (for all $j\in\mathbb{N}$). Hence, making such shifts, we can assume that $t_m\in\mathbb{Z}(0,M-1)$ in (3.6). Moreover, passing to a subsequence of ms, we can even assume that $t_m=t_0$ is independent of m.

Next, we extract a subsequence, still denote by u_m, such that

$$u_m(t)\to u(t),\text{as } m\to\infty,\ \forall t\in\mathbb{Z}.$$

Inequality (3.6) implies that $|u(t_0)|\ge\xi$ and, hence, $u=\{u(t)\}$ is a nonzero sequence. Moreover,

$$\begin{aligned}&Lu(t)-f(t,u(t+T),u(t),u(t-T))\\ &=\lim_{m\to\infty}[Lu_m(t)-f(t,u_m(t+T),u_m(t),u_m(t-T))]=0.\end{aligned}$$

So $u=\{u(t)\}$ is a solution of (1.1).

Finally, we show that $u\in l^2$. For $u_m\in E_m$, let

$$\begin{aligned}P_m&=\big\{t\in\mathbb{Z}:|u_m(t)|<\frac{\sqrt{2}}{2}\delta_1,-mM\le t\le mM-1\big\},\\ Q_m&=\big\{t\in\mathbb{Z}:|u_m(t)|\ge\frac{\sqrt{2}}{2}\delta_1,-mM\le t\le mM-1\big\}.\end{aligned}$$

Since $F(t,v_1,v_2)\in C^1(\mathbb{Z}\times\mathbb{R}^2,\mathbb{R})$, there exist constants $\bar{\xi}>0$, $\underline{\xi}>0$ such that

$$\max\Big\{\Big[\frac{\partial F(t,v_1,v_2)}{\partial v_1}\Big]^2+\Big[\frac{\partial F(t,v_1,v_2)}{\partial v_2}\Big]^2:\delta_1\le\sqrt{v_1^2+v_2^2}\le\eta_1,\ t\in\mathbb{Z}\Big\}\le\bar{\xi},$$

$$
\begin{aligned}
\min\Big\{\frac{1}{2}\Big[\frac{\partial F(t,v_1,v_2)}{\partial v_1}v_1+\frac{\partial F(t,v_1,v_2)}{\partial v_2}v_2\Big]-F(t,v_1,v_2):\\
\delta_1\le\sqrt{v_1^2+v_2^2}\le\eta_1,\ t\in\mathbb{Z}\Big\}\ge\underline{\xi}.
\end{aligned}
$$

For $t\in Q_m$,

$$
\begin{aligned}
&\Big[\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_1}\Big]^2+\Big[\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_2}\Big]^2\\
&\le\frac{\bar{\xi}}{\underline{\xi}}\Big\{\frac{1}{2}\Big[\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_1}u_m(t+T)+\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_2}u_m(t)\Big]\\
&\quad-F(t,u_m(t+T),u_m(t))\Big\}.
\end{aligned}
$$

By (3.5), we have

$$
\begin{aligned}
&\underline{\lambda}^2\|u_m\|^2\\
&\le 2\sum_{t\in P_m}\Big[\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_1}\Big]^2+2\sum_{t\in P_m}\Big[\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_2}\Big]^2\\
&\quad+2\sum_{t\in Q_m}\Big[\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_1}\Big]^2+2\sum_{t\in Q_m}\Big[\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_2}\Big]^2\\
&\le 2\sum_{t\in P_m}[2a_1u_m(t+T)]^2+2\sum_{t\in P_m}[2a_1u_m(t)]^2\\
&\quad+\frac{\bar{\xi}}{\underline{\xi}}\sum_{t\in Q_m}\Big\{\frac{1}{2}\Big[\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_1}u_m(t+T)\\
&\quad+\frac{\partial F(t,u_m(t+T),u_m(t))}{\partial v_2}u_m(t)\Big]-F(t,u_m(t+T),u_m(t))\Big\}\\
&\le 16a_1^2\|u_m\|^2+\frac{\bar{\xi}\xi_1}{\underline{\xi}}.
\end{aligned}
$$

Thus,

$$
\|u_m\|^2\le\frac{\bar{\xi}\xi_1}{\underline{\xi}\left(\underline{\lambda}^2-16a_1^2\right)}.
$$

For any fixed $D\in\mathbb{Z}$ and m large enough, we have

$$
\sum_{t=-D}^{D}u_m^2(t)\le\|u_m\|^2\le\frac{\bar{\xi}\xi_1}{\underline{\xi}\left(\underline{\lambda}^2-16a_1^2\right)}.
$$

Since $\bar{\xi},\underline{\xi},\xi_1,\underline{\lambda}$ and a_1 are constants independent of m, passing to the limit, we have

$$
\sum_{t=-D}^{D}u^2(t)\le\frac{\bar{\xi}\xi_1}{\underline{\xi}\left(\underline{\lambda}^2-16a_1^2\right)}.
$$

By the arbitrariness of D, $u\in l^2$. Therefore, u satisfies $u(t)\to 0$ as $|t|\to\infty$. The existence of a nontrivial homoclinic orbit is obtained. $\square$

3.2. **Proof Theorem 1.5.** Let

$$J_m^*(u) = -\frac{1}{2}\sum_{t=-mM}^{mM-1} Lu(t)\cdot u(t) + \sum_{t=-mM}^{mM-1} F(t,u(t+T),u(t)). \tag{3.7}$$

Then

$$\frac{\partial J_m^*(u)}{\partial u(t)} = -Lu(t) + f(t,u(t+T),u(t),u(t-T)), \tag{3.8}$$

for all $t \in \mathbb{Z}(-mM, mM-1)$.

Lemma 3.4. *Suppose that* (H1), (H2), (H7), (H8) *are satisfied. Then* J_m^* *satisfies the PS condition.*

.

Proof. Assume that $\{u_j\}_{j\in\mathbb{N}}$ in E_m is a sequence such that $\{J_m^*(u_j)\}_{j\in\mathbb{N}}$ is bounded. Then there exists a constant $K_2 > 0$ such that $-K_2 \le J_m^*(u_j)$. By (2.7) and (H9'), it is easy to see that

$$-K_2 \le J_m^*(u_j) \le -\frac{\lambda}{2}\|u_j\|^2 + 2c_2\|u_j\|^2 + 2mM\left(\zeta_2 + b_2\right), \quad \forall j\in\mathbb{N}.$$

Therefore,

$$-\big(2c_2 - \frac{\lambda}{2}\big)\|u_j\|^2 \le 2mM\left(\zeta_2 + b_2\right) + K_2. \tag{3.9}$$

Since $c_2 < \underline{\lambda}/4$, (3.9) implies that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E_m. Thus, $\{u_j\}_{j\in\mathbb{N}}$ possesses a convergence subsequence in E_m. The desired result follows. □

Lemma 3.5. *Suppose that* (H1), (H2), (H7), (H8) *are satisfied. Then for any given positive integer* m, (1.1) *possesses a* $2mM$*-periodic solution* $u_m^* \in E_m$.

Proof. In our case, it is clear that $J_m^*(0) = 0$. By Lemma 3.4, J_m^* satisfies the PS condition. By (H7), we have

$$\begin{aligned}
J_m^*(u) &\ge -\frac{\bar\lambda}{2}\|u\|^2 + a_2\sum_{t=-mM}^{mM-1}[u^2(t) + u^2(t+T)] \\
&\ge -\frac{\bar\lambda}{2}\|u\|^2 + 2a_2\|u\|^2 \\
&= -\big(\frac{\bar\lambda}{2} - 2a_2\big)\|u\|^2.
\end{aligned}$$

Taking $\alpha_2 = -(\frac{\bar\lambda}{2} - 2a_2)\delta_2^2 > 0$, we obtain

$$J_m^*(u)|_{\partial B_{\delta_2}} \ge \alpha_2 > 0,$$

which implies that J_m^* satisfies the condition (J1) of the Mountain Pass Lemma.

Next, we shall verify the condition (J2) of the Mountain Pass Lemma. There exists a sufficiently large number $\eta > \max\{\rho_2, \delta_2\}$ such that

$$\big(2c_2 - \frac{\bar\lambda}{2}\big)\eta^2 \ge |b_2|. \tag{3.10}$$

Let $e_m^{(2)} \in E_m$ and

$$e_m^{(2)}(t) = \begin{cases} \eta, & \text{if } t = 0,\\ 0, & \text{if } t \in \{j\in\mathbb{Z} : -mM \le j \le mM-1 \text{ and } j \ne 0\},\end{cases}$$

$$e_m^{(2)}(t+T)=\begin{cases}\eta, & \text{if } t=0,\\ 0, & \text{if } t\in\{j\in\mathbb{Z}: -mM\le j\le mM-1 \text{ and } j\neq 0\}.\end{cases}$$

Then

$$\begin{aligned} &F\big(t,e_m^{(2)}(t+T),e_m^{(2)}(t)\big)\\ &=\begin{cases}F(0,\eta,\eta), & \text{if } t=0,\\ 0, & \text{if } t\in\{j\in\mathbb{Z}: -mM\le j\le mM-1 \text{ and } j\neq 0\},\end{cases}\end{aligned}$$

With (3.10) and (H9), we have

$$\begin{aligned} J_m^*\big(e_m^{(2)}\big) &= -\frac{1}{2}\sum_{t=-mM}^{mM-1} L\big(e_m^{(2)}(t)\big)\cdot\big(e_m^{(2)}(t)\big)\\ &\quad+\sum_{t=-mM}^{mM-1} F\Big(t,e_m^{(2)}(t+T),e_m^{(2)}(t)\Big)\\ &\le -\frac{\lambda}{2}\|e_m^{(2)}\|^2+2c_2\eta^2+b_2\\ &= -\big(\frac{\lambda}{2}-2c_2\big)\eta^2+b_2\le 0. \end{aligned} \tag{3.11}$$

All the assumptions of the Mountain Pass Lemma have been verified. Consequently, J_m^* possesses a critical value c_m^* given by (2.5) and (2.6) with $E=E_m$ and $\Gamma=\Gamma_m$, where

$$\Gamma_m=\big\{g_m\in C([0,1],E_m)|g_m(0)=0,\ g_m(1)=e_m^{(2)},\ e_m^{(2)}\in E_m\backslash B_\eta\big\}.$$

Let u_m^* denote the corresponding critical point of J_m^* on E_m. Note that $\|u_m^*\|\neq 0$ since $c_m^*>0$. □

Lemma 3.6. *Suppose that* (H1), (H2), (H7), (H8) *are satisfied. Then there exist positive constants* δ_2 *and* η_2 *independent of* m *such that*

$$\delta_2\le\|u_m^*\|_\infty\le\eta_2. \tag{3.12}$$

Proof. The continuity of $F(0,v_1,v_2)$ with respect to the second and third variables implies that there exists a constant $\tau_2>0$ such that $|F(0,v_1,v_2)|\le\tau_2$ for $\sqrt{v_1^2+v_2^2}\le\delta_2$. It is clear that

$$\begin{aligned} |J_m^*(u_m^*)| &\le \max_{0\le s\le 1}\Big\{\big|-\frac{1}{2}\sum_{t=-mM}^{mM-1} L\big(se_m^{(2)}(t)\big)\cdot\big(se_m^{(2)}(t)\big)\big|\\ &\quad+\sum_{t=-mM}^{mM-1} F\Big(t,se_m^{(2)}(t+T),se_m^{(2)}(t)\Big)\Big\}\\ &\le \frac{\bar\lambda}{2}\|e_m^{(2)}\|^2+\tau_2\\ &= \frac{\bar\lambda}{2}\eta^2+\tau_2. \end{aligned} \tag{3.13}$$

Let $\xi_2=\frac{\bar\lambda}{2}\eta^2+\tau_2$, we have that $|J_m^*(u_m^*)|\le\xi_2$, which is a bound independent of m. Then by (3.7) and (3.8), we have

$$\xi_2\ge J_m^*(u_m)$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{t=-mM}^{mM-1}\Big[\frac{\partial F(t-T,u_m^*(t),u_m^*(t-T))}{\partial v_2}u_m^*(t)\\
&\quad + \frac{\partial F(t,u_m^*(t+T),u_m^*(t))}{\partial v_2}u_m^*(t)\Big] + \sum_{t=-mM}^{mM-1}F(t,u_m^*(t+T),u_m^*(t))\\
&= -\frac{1}{2}\sum_{t=-mM}^{mM-1}\Big[\frac{\partial F(t,u_m^*(t+T),u_m^*(t))}{\partial v_1}u_m^*(t+T)\\
&\quad + \frac{\partial F(t,u_m^*(t+T),u_m^*(t))}{\partial v_2}u_m^*(t)\Big] + \sum_{t=-mM}^{mM-1}F(t,u_m^*(t+T),u_m^*(t))\\
&\geq \big(\frac{2-\mu}{2}\big)\sum_{t=-mM}^{mM-1}F(t,u_m^*(t+T),u_m^*(t))\,.
\end{aligned}
$$

Then

$$
\sum_{t=-mM}^{mM-1}F(t,u_m^*(t+T),u_m^*(t)) \leq \frac{2\xi_2}{2-\mu}. \tag{3.14}
$$

Since

$$
\begin{aligned}
J_m^*(u_m^*) &= -\frac{1}{2}\sum_{t=-mM}^{mM-1}\Big[\frac{\partial F(t-T,u_m^*(t),u_m^*(t-T))}{\partial v_2}u_m^*(t)\\
&\quad + \frac{\partial F(t,u_m^*(t+T),u_m^*(t))}{\partial v_2}u_m^*(t)\Big] + \sum_{t=-mM}^{mM-1}F(t,u_m^*(t+T),u_m^*(t))\\
&\geq -\xi_2.
\end{aligned}
$$

This inequality combined with (3.14) gives us

$$
\begin{aligned}
\frac{1}{2}\underline{\lambda}\|u_m^*\| &\leq \frac{1}{2}\sum_{t=-mM}^{mM-1}\Big[\frac{\partial F(t-T,u_m^*(t),u_m^*(t-T))}{\partial v_2}u_m^*(t)\\
&\quad + \frac{\partial F(t,u_m^*(t+T),u_m^*(t))}{\partial v_2}u_m^*(t)\Big]\\
&\leq \sum_{t=-mM}^{mM-1}F(t,u_m^*(t+T),u_m^*(t)) + \xi_2\\
&\leq \frac{(4-\mu)\xi_2}{2-\mu},
\end{aligned} \tag{3.15}
$$

$$
\|u_m^*\| \leq \frac{2(4-\mu)\xi_2}{(2-\mu)\underline{\lambda}}, \tag{3.16}
$$

whose right-hand side is independent of m. Then $\|u_m^*\| \leq \eta_2$, which implies

$$
\|u_m^*\|_\infty \leq \eta_2.
$$

From the definition of J_m^*, we have

$$
0 = \big(J_m^{*'}(u_m^*),u_m^*\big) \geq -\bar{\lambda}\|u_m^*\|^2 + \sum_{t=-mM}^{mM-1}f(t,u_m^*(t+T),u_m^*(t),u_m^*(t-T))u_m^*(t).
$$

This inequality combined with (H7) yields

$$\begin{aligned}
\bar{\lambda}\|u_m^*\|^2 &\geq \sum_{t=-mM}^{mM-1}\Big[\frac{\partial F(t-T,u_m^*(t),u_m^*(t-T))}{\partial v_2}u_m^*(t) \\
&\quad + \frac{\partial F(t,u_m^*(t+T),u_m^*(t))}{\partial v_2}u_m^*(t)\Big] \\
&= \sum_{t=-mM}^{mM-1}\Big[\frac{\partial F(t,u_m^*(t+T),u_m^*(t))}{\partial v_1}u_m^*(t+T) \\
&\quad + \frac{\partial F(t,u_m^*(t+T),u_m^*(t))}{\partial v_2}u_m^*(t)\Big] \\
&\geq 2a_2\sum_{t=-mM}^{mM-1}\big[\left(u_m^*(t+T)\right)^2 + \left(u_m^*(t)\right)^2\big] \\
&= 4a_2\|u_m^*\|^2.
\end{aligned}$$

Thus, we have $u_m^* = 0$. This contradicts $\|u_m^*\| \neq 0$, which shows that $\|u_m^*\|_\infty \geq \delta_2$, and the proof is complete. □

The proof of Theorem 1.5 is done similarly to the proof of Theorem 1.1. We omit it here for simplicity.

4. Examples

As an application of Theorems 1.1 and 1.5, we give two examples that illustrate our main results.

Example 4.1. Let

$$\begin{gathered}
f(t,v_1,v_2,v_3) = \gamma v_2\Big(\frac{v_1^2+v_2^2}{v_1^2+v_2^2+1} + \frac{v_2^2+v_3^2}{v_2^2+v_3^2+1}\Big), \\
F(t,v_1,v_2) = \frac{\gamma}{2}[v_1^2+v_2^2-\ln(v_1^2+v_2^2+1)],
\end{gathered}$$

where $\gamma > \bar{\lambda}$. If (H1) is satisfied, then it is easy to verify that all the assumptions of Theorem 1.1 are satisfied. Consequently, there exists a nontrivial homoclinic orbit.

Example 4.2. Let

$$f(t,v_1,v_2,v_3) = \begin{cases} v_2[(v_1^2+v_2^2)^{\frac{\mu}{2}-1} + (v_2^2+v_3^2)^{\frac{\mu}{2}-1}], \\ \quad \text{if } (v_1,v_2)\neq(0,0) \text{ and } (v_2,v_3)\neq(0,0), \\ 0, \quad \text{if } (v_1,v_2)=(0,0) \text{ or } (v_2,v_3)=(0,0), \end{cases}$$

and

$$F(t,v_1,v_2) = \frac{1}{\mu}(v_1^2+v_2^2)^{\mu/2},$$

where $1<\mu<2$. If (H1) is satisfied, then it is easy to verify all the assumptions of Theorem 1.5 are satisfied. Consequently, there exists a nontrivial homoclinic orbit.

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