Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 127, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SUPER-QUADRATIC CONDITIONS FOR PERIODIC ELLIPTIC SYSTEM ON $\mathbb{R}^N$

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Abstract. This article concerns the elliptic system

$$\begin{gathered} -\Delta u + V(x)u = W_v(x,u,v), \quad x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = W_u(x,u,v), \quad x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{gathered}$$

where V and W are periodic in x, and $W(x,z)$ is super-linear in $z=(u,v)$. We use a new technique to show that the above system has a nontrivial solution under concise super-quadratic conditions. These conditions show that the existence of a nontrivial solution depends mainly on the behavior of $W(x,u,v)$ as $|u+v| \to 0$ and $|au+bv| \to \infty$ for some positive constants a, b.

1. Introduction

In this article, we study the elliptic system

$$\begin{gathered} -\Delta u + V(x)u = W_v(x,u,v), \quad x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = W_u(x,u,v), \quad x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $z := (u,v) \in \mathbb{R}^2$, $V : \mathbb{R}^N \to \mathbb{R}$ and $W : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$.

Systems similar to (1.1) have been considered recently; see for instance [1,2,3,4, 6,7,11,12,13,16,17,19,20,21,22,23,25,26,24,27,28,29,30,31] and references therein. For the superquadratic case, it always assumed that W satisfies the Ambrosetti-Rabinowitz condition

(AR) there is a $\mu > 2$ such that

$$0 < \mu W(x,z) \le W_z(x,z)\cdot z, \quad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}^2,\ z \ne 0\,. \tag{1.2}$$

We use the assumption that there exist $c > 0$ and $\nu \in (2N/(N+2), 2)$ such that

$$|W_z(x,z)|^\nu \le c[1 + W_z(x,z)\cdot z], \quad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}^2\,, \tag{1.3}$$

2010 *Mathematics Subject Classification.* 35J10, 35J20.
Key words and phrases. Elliptic system; super-quadratic; nontrivial solution; strongly indefinite functionals.

Submitted January 31, 2015. Published May 6, 2015.

or the super-quadratic condition

$$\lim_{|z|\to\infty} \frac{|W(x,z)|}{|z|^2} = \infty, \quad \text{uniformly in } x \in \mathbb{R}^N, \tag{1.4}$$

and a condition of the Ding-Lee type,

(DL) $\widetilde{W}(x,z) := \frac{1}{2}W_z(x,z)\cdot z - W(x,z) > 0$ for $z \neq 0$ and there exist $\hat{c} > 0$ and $\kappa > \max\{1, N/2\}$ such that

$$|W_z(x,z)|^\kappa \le \hat{c}|z|^\kappa \widetilde{W}(x,z), \quad \text{for large } |z|. \tag{1.5}$$

Observe that conditions (1.4) and $W(x,z) > 0, \forall z \neq 0$ in (AR) or $\widetilde{W}(x,z) > 0, \forall\ z \neq 0$ in (DL) play an important role for showing that any Palais-Smale sequence or Cerami sequence is bounded in the aforementioned works. However, there are many functions that do not satisfy these conditions, for example,

$$W(x,u,v) = (u^2 + uv + v^2)\ln(1+u^2),$$

or

$$W(x,u,v) = (u+2v)^2\sqrt{u^2+v^2}.$$

In a recent paper Liao, Tang and Zhang [11] studied the existence of solutions for system (1.1) under the following assumptions on V and W:

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V(x)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, and $\min_{\mathbb{R}^N} V \ge \beta_0 > 0$;

(W1) $W \in C(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^+)$, $W(x,z)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, continuously differentiable on $z \in \mathbb{R}^2$ for every $x \in \mathbb{R}^N$, and there exist constants $p \in (2, 2^*)$ and $C_0 > 0$ such that

$$|W_z(x,z)| \le C_0\left(1 + |z|^{p-1}\right), \quad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}^2;$$

(W2) $|W_z(x,z)| = o(|z|)$, as $|z| \to 0$, uniformly in $x \in \mathbb{R}^N$;

(W3) $\lim_{|u+v|\to\infty} \frac{|W(x,u,v)|}{|u+v|^2} = \infty$, a.e. $x \in \mathbb{R}^N$;

(W4) $\widetilde{W}(x,z) \ge 0$ for all $(x,z) \in \mathbb{R}^N \times \mathbb{R}^2$, and there exist $c_0, R_0 > 0$ and $\kappa > \max\{1, N/2\}$ such that

$$|W_u(x,u,v) + W_v(x,u,v)| \le \frac{2\beta_0}{3}\sqrt{u^2+v^2}, \quad u^2+v^2 \le R_0^2$$

and

$$|W_u(x,u,v) + W_v(x,u,v)|^\kappa \le c_0\left(u^2+v^2\right)^{\kappa/2}\widetilde{W}(x,u,v), \quad u^2+v^2 \ge R_0^2.$$

Specifically, Liao, Tang and Zhang [11] established the following theorem.

Theorem 1.1 ([11, Theorem1.2]). *Assume that* (V1), (W1)–(W4) *are satisfied. Then* (1.1) *has a nontrivial solution.*

As shown in [11], (W3) is different from usual superquadratic conditions (AR) and (1.4), and is weaker than (1.4). Clearly, (W4) is significantly weaker than (DL). By a variable substitution, instead of (W3) and (W4), the following more general conditions were used:

(W3') there exist $a, b > 0$ such that

$$\lim_{|au+bv|\to\infty} \frac{|W(x,u,v)|}{|au+bv|^2} = \infty, \quad \text{a.e. } x \in \mathbb{R}^N;$$

(W4') $\widetilde{W}(x,z) \geq 0$ for all $(x,z) \in \mathbb{R}^N \times \mathbb{R}^2$, and there exist $c_1, R_1 > 0$ and $\kappa > \max\{1, N/2\}$ such that

$$|bW_u(x,u,v) + aW_v(x,u,v)| \leq \frac{2\beta_0}{3}\sqrt{u^2+v^2}, \quad a^2u^2+b^2v^2 \leq R_1^2,$$

$$|bW_u(x,u,v) + aW_v(x,u,v)|^\kappa \leq c_1\left(a^2u^2+b^2v^2\right)^{\kappa/2}\widetilde{W}(x,u,v),$$
$$a^2u^2+b^2v^2 \geq R_1^2.$$

Motivated by [11], we obtain a super-quadratic condition more concise than (W4'):

(W5) $\widetilde{W}(x,z) \geq 0$ for all $(x,z) \in \mathbb{R}^N \times \mathbb{R}^2$, and there exist $\theta \in (0,1)$, $\alpha_0 > 0$, and $\kappa > \max\{1, N/2\}$ such that

$$\frac{|bW_u(x,z) + aW_v(x,z)|}{|z|} \geq \theta\beta_0 \min\{a,b\}$$
$$\Rightarrow \Big(\frac{|bW_u(x,z) + aW_v(x,z)|}{|z|}\Big)^\kappa \leq \alpha_0 \widetilde{W}(x,z).$$

By introducing new techniques, under (W3') and (W5), we obtain the linking structure and the boundedness of a Cerami sequence of the energy functional associated with (1.1). Specifically, we obtain the following theorem.

Theorem 1.2. *Assume that*(V1), (W1), (W2), (W3'), (W5) *hold. Then* (1.1) *has a nontrivial solution.*

Remark 1.3. Note that (W5) is weaker than (DL) and than (AR). Since

$$|bW_u + aW_v| \leq \sqrt{a^2+b^2}|W_z(x,z)|,$$

in view of (W2), it is clear that (DL) implies (W5). If $W(x,z)$ satisfies (1.2), then there exist $c_1, R_1 > 0$ such that

$$W_z(x,z)\cdot z \geq \mu W(x,z) \geq c_1|z|^\mu, \quad |z| \geq R_1, \tag{1.6}$$

$$\widetilde{W}(x,z) \geq \frac{\mu-2}{2}W_z(x,z)\cdot z > 0, \quad \forall z \in \mathbb{R}^2 \setminus \{0\}. \tag{1.7}$$

Let $\kappa = \nu/(2-\nu)$. Then $\kappa > \max\{1, N/2\}$. Hence, it follows from (1.3), (1.6) and (1.7) that

$$\begin{aligned} |W_z(x,z)|^\kappa &\leq c_2|W_z(x,z)|^{\kappa-\nu}W_z(x,z)\cdot z \\ &\leq c_3|z|^{(\kappa-\nu)/(\nu-1)}\widetilde{W}(x,z) \\ &= c_3|z|^\kappa\widetilde{W}(x,z), \quad |z| \geq R_1. \end{aligned} \tag{1.8}$$

This shows that (DL) holds, and so (W5) holds.

Before proceeding with the proof of Theorem 1.2, we give a nonlinear example to illustrate the assumptions.

Example 1.4. $W(x,u,v) = h(x)(u+2v)^2\sqrt{u^2+v^2}$, where $h \in C(\mathbb{R}^N, (0,\infty))$ is 1-periodic in each of the variables $x_1, x_2, \dots, x_N$. Then

$$\widetilde{W}(x,u,v) = \frac{1}{2}h(x)(u+2v)^2\sqrt{u^2+v^2}, \quad u,\, v \in \mathbb{R}.$$

Therefore all conditions (W1), (W2), (W3'), (W5) are satisfied with $a = 1, b = 2$ and $\kappa \leq 3$. Note that $W(x,u,v) = \widetilde{W}(x,u,v) = 0$ for $u = -2v$, $v \in \mathbb{R}$, thus W does not satisfy (AR) and (DL).

The rest of this article is organized as below. In Section 2, we provide a variational setting. The proofs of our main results are given in the last section.

2. Variational setting

Under assumption (V1), we can define the Hilbert space

$$E_V = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x < +\infty\}$$

equipped with the inner product

$$(u,v)_{E_V} = \int_{\mathbb{R}^N} [\nabla u \cdot \nabla v + V(x)uv]\,\mathrm{d}x, \quad \forall u, v \in E_V,$$

and the corresponding norm

$$\|u\|_{E_V} = \Big(\int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)u^2]\,\mathrm{d}x \Big)^{1/2}, \quad \forall u \in E_V. \tag{2.1}$$

By the Sobolev embedding theorem, there exists constant $\gamma_s > 0$ such that

$$\|u\|_s \le \gamma_s \|u\|_{E_V}, \quad \forall u \in H^1(\mathbb{R}^N),\ 2 \le s \le 2^*, \tag{2.2}$$

here and in the sequel, by $\|\cdot\|_s$ we denote the usual norm in space $L^s(\mathbb{R}^N)$.

Let $E = E_V \times E_V$ with the inner product

$$(z_1, z_2) = ((u_1, v_1), (u_2, v_2)) = (u_1, u_2)_{E_V} + (v_1, v_2)_{E_V},$$

for $z_i = (u_i, v_i) \in E$, $i = 1, 2$, and the corresponding norm $\|\cdot\|$. Then there hold

$$\|z\|^2 = \|u\|_{E_V}^2 + \|v\|_{E_V}^2, \quad \forall z = (u,v) \in E \tag{2.3}$$

and

$$\begin{aligned} \|z\|_s^s &= \int_{\mathbb{R}^N} (u^2 + v^2)^{s/2}\,\mathrm{d}x \le 2^{(s-2)/2}(\|u\|_s^s + \|v\|_s^s) \\ &\le 2^{(s-2)/2}\gamma_s^s \left(\|u\|_{E_V}^s + \|v\|_{E_V}^s\right) \\ &\le 2^{(s-2)/2}\gamma_s^s \left(\|u\|_{E_V}^2 + \|v\|_{E_V}^2\right)^{s/2} \\ &= 2^{(s-2)/2}\gamma_s^s \|z\|^s, \quad \forall s \in [2, 2^*],\ z = (u,v) \in E. \end{aligned} \tag{2.4}$$

Now we define a functional Φ on E by

$$\Phi(z) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)\ \mathrm{d}x - \int_{\mathbb{R}^N} W(x,u,v)\,\mathrm{d}x, \quad \forall z = (u,v) \in E. \tag{2.5}$$

Consequently, under assumptions (V1), (V2), (W1), (W2), (W3'), it is well known that Φ is $C^1(E, \mathbb{R})$, and

$$\begin{aligned} \langle \Phi'(z), \zeta \rangle &= \int_{\mathbb{R}^N} [\nabla u \cdot \nabla \psi + \nabla v \cdot \nabla \varphi + V(x)(u\psi + v\varphi)]\ \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} [W_u(x,u,v)\varphi + W_v(x,u,v)\psi]\,\mathrm{d}x, \end{aligned} \tag{2.6}$$

for all $z = (u,v)$, $\zeta = (\varphi, \psi) \in E$. Let

$$E^- = \{(u, -u) : u \in H^1(\mathbb{R}^N)\}, \quad E^+ = \{(u,u) : u \in H^1(\mathbb{R}^N)\}.$$

For any $z = (u,v) \in E$, set

$$z^- = \Big(\frac{u-v}{2}, \frac{v-u}{2}\Big), \quad z^+ = \Big(\frac{u+v}{2}, \frac{u+v}{2}\Big). \tag{2.7}$$

It is obvious that $z = z^- + z^+$, z^- and z^+ are orthogonal with respect to the inner products $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$. Thus we have $E = E^- \oplus E^+$. By a simple calculation, one can get that

$$\frac{1}{2}\left(\|z^+\|^2 - \|z^-\|^2\right) = \int_{\mathbb{R}^N} [\nabla u \cdot \nabla v + V(x)uv]\,\mathrm{d}x.$$

Therefore, the functional Φ defined in (2.5) can be rewritten in a standard way

$$\Phi(z) = \frac{1}{2}\left(\|z^+\|^2 - \|z^-\|^2\right) - \int_{\mathbb{R}^N} W(x,z)\,\mathrm{d}x, \quad \forall z = (u,v) \in E. \tag{2.8}$$

Moreover

$$\langle \Phi'(z), z\rangle = \|z^+\|^2 - \|z^-\|^2 - \int_{\mathbb{R}^N} [W_u(x,u,v)u + W_v(x,u,v)v]\,\mathrm{d}x, \tag{2.9}$$

for all $z = (u,v) \in E$.

3. Proofs of main resutls

To give the proofs of our results, we set

$$\Psi(z) = \int_{\mathbb{R}^N} W(x,z)\,\mathrm{d}x, \quad \forall z \in E. \tag{3.1}$$

Lemma 3.1. *Suppose that* (W1), (W2) *are satisfied. Then Ψ is nonnegative, weakly sequentially lower semi-continuous, and Ψ' is weakly sequentially continuous.*

Using the Sobolev's imbedding theorem, one can easily check the above lemma, so we omit the proof.

Lemma 3.2. *Suppose that* (V1), (W1), (W2), (W3') *are satisfied. Then there is a $\rho > 0$ such that $\kappa_1 := \inf \Phi(S_\rho^+) > 0$, where $S_\rho^+ = \partial B_\rho \cap E^+$.*

The above lemma can be proved in standard way; we omit its proof.

Lemma 3.3. *Suppose that* (V1), (W1), (W2), (W3') *are satisfied. Let $e = (e_0, e_0)$ belong to E^+ with $\|e\| = 1$. Then there is a constant $r > 0$ such that $\sup \Phi(\partial Q) \le 0$, where*

$$Q = \left\{\zeta + se : \zeta = (w, -w) \in E^-,\ s \ge 0,\ \|\zeta + se\| \le r\right\}. \tag{3.2}$$

Proof. By (W1) and (2.8), $\Phi(z) \le 0$ for $z \in E^-$. Next, it is sufficient to show that $\Phi(z) \to -\infty$ as $z \in E^- \oplus \mathbb{R}e$ for $\|z\| \to \infty$. Arguing indirectly, assume that for some sequence $\{\zeta_n + s_n e\} \subset E^- \oplus \mathbb{R}e$ with $\|\zeta_n + s_n e\| \to \infty$, there is $M > 0$ such that $\Phi(\zeta_n + s_n e) \ge -M$ for all $n \in \mathbb{N}$. Set $\zeta_n = (w_n, -w_n)$, $\xi_n = (\zeta_n + s_n e)/\|\zeta_n + s_n e\| = \xi_n^- + t_n e$, then $\|\xi_n^- + t_n e\| = 1$. Passing to a subsequence, we may assume that $t_n \to \bar{t}$ and $\xi_n \rightharpoonup \xi$ in E, then $\xi_n \to \xi$ a.e. on $\mathbb{R}^N$, $\xi_n^- \rightharpoonup \xi^-$ in E, $\xi_n^- := (\tilde{w_n}, -\tilde{w_n}) \rightharpoonup \xi^- := (\tilde{w}, -\tilde{w})$, and

$$\begin{aligned} -\frac{M}{\|\zeta_n + s_n e\|^2} &\le \frac{\Phi(\zeta_n + s_n e)}{\|\zeta_n + s_n e\|^2} \\ &= \frac{t_n^2}{2} - \frac{1}{2}\|\xi_n^-\|^2 - \int_{\mathbb{R}^N} \frac{W(x, w_n + s_n e_0, -w_n + s_n e_0)}{\|\zeta_n + s_n e\|^2}\,\mathrm{d}x. \end{aligned} \tag{3.3}$$

If $\bar{t} = 0$, then it follows from (3.3) that

$$0 \le \frac{1}{2}\|\xi_n^-\|^2 + \int_{\mathbb{R}^N} \frac{W(x, w_n + s_n e_0, -w_n + s_n e_0)}{\|\zeta_n + s_n e\|^2}\,\mathrm{d}x \le \frac{t_n^2}{2} + \frac{M}{\|\zeta_n + s_n e\|^2} \to 0,$$

which yields $\|\xi_n^-\| \to 0$, and so $1 = \|\xi_n\| \to 0$, a contradiction.

If $\bar{t} \neq 0$, then

$$(a-b)\tilde{w} + (a+b)\bar{t}e_0 \neq 0. \tag{3.4}$$

Arguing indirectly, assume that $(a-b)\tilde{w} + (a+b)\bar{t}e_0 = 0$, then $a \neq b$ and

$$\begin{aligned}
\bar{t}^2 &= \lim_{n\to\infty} t_n^2 \\
&\geq \liminf_{n\to\infty} \Big(-\frac{2M}{\|\zeta_n + s_n e\|^2} + \|\xi_n^-\|^2 \Big) \\
&\geq \|\xi^-\|^2 \\
&= \int_{\mathbb{R}^N} [|\nabla \xi^-|^2 + V(x)|\xi^-|^2]\,\mathrm{d}x \\
&= \frac{(a+b)^2}{(b-a)^2} \bar{t}^2 \int_{\mathbb{R}^N} \big[|\nabla e|^2 + V(x)|e|^2\big]\,\mathrm{d}x \\
&> \bar{t}^2 \int_{\mathbb{R}^N} [|\nabla e|^2 + V(x)|e|^2]\,\mathrm{d}x = \bar{t}^2,
\end{aligned}$$

which is a contradiction.

Let $\Omega := \{x \in \mathbb{R}^N : (a-b)\tilde{w}(x) + (a+b)\bar{t}e_0(x) \neq 0\}$. Then (3.4) shows that $|\Omega| > 0$. Since $\|\zeta_n + s_n e\| \to \infty$, for any $x \in \Omega$, one has

$$\begin{aligned}
&|a(w_n(x) + s_n e_0(x)) + b(-w_n(x) + s_n e_0(x))| \\
&= \|\zeta_n + s_n e\| |(a-b)\tilde{w}_n(x) + (a+b)t_n e_0(x)| \to \infty.
\end{aligned}$$

Let $\eta_n := a(\tilde{w}_n + t_n e_0) + b(-\tilde{w}_n + t_n e_0)$. It follows from (3.3), (3.4), (W3') and Fatou's lemma that

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \Big[\frac{t_n^2}{2} - \frac{1}{2}\|\xi_n^-\|^2 - \int_{\mathbb{R}^N} \frac{W(x, w_n + s_n e_0, -w_n + s_n e_0)}{\|\zeta_n + s_n e\|^2}\,\mathrm{d}x \Big] \\
&= \limsup_{n\to\infty} \Big[\frac{t_n^2}{2} - \frac{1}{2}\|\xi_n^-\|^2 - \int_{\mathbb{R}^N} \frac{W(x, w_n + s_n e_0, -w_n + s_n e_0)}{|a(w_n + s_n e_0) + b(-w_n + s_n e_0)|^2} |\eta_n|^2\,\mathrm{d}x \Big] \\
&\leq \frac{1}{2} \lim_{n\to\infty} t_n^2 - \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{W(x, w_n + s_n e_0, -w_n + s_n e_0)}{|a(w_n + s_n e_0) + b(-w_n + s_n e_0)|^2} |\eta_n|^2\,\mathrm{d}x \\
&\leq \frac{\bar{t}^2}{2} - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{W(x, w_n + s_n e_0, -w_n + s_n e_0)}{|a(w_n + s_n e_0) + b(-w_n + s_n e_0)|^2} |\eta_n|^2\,\mathrm{d}x \\
&= -\infty,
\end{aligned}$$

a contradiction. □

Applying the generalized linking theorem [8,10] and standard arguments, we can prove the following lemma.

Lemma 3.4. *Suppose that* (V1), (W1), (W2), (W3') *are satisfied. Then there exist a constant* $c_* \in [\kappa_0, \sup \Phi(Q)]$ *and a sequence* $\{z_n\} = \{(u_n, v_n)\} \subset E$ *satisfying*

$$\Phi(z_n) \to c_*, \quad \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0. \tag{3.5}$$

where Q *is defined by* (3.2).

Lemma 3.5. *Suppose that* (V1), (W1), (W2), (W3'), (W5) *are satisfied. Then any sequence* $\{z_n\} = \{(u_n, v_n)\} \subset E$ *satisfying* (3.5) *is bounded in* E.

Proof. To prove the boundedness of $\{z_n\}$, arguing by contradiction, suppose that $\|z_n\| \to \infty$. Let

$$\xi_n = \frac{z_n}{\|z_n\|} = (\varphi_n, \psi_n), \quad \hat{z}_n = (\hat{u}_n, \hat{v}_n) := \Big(\frac{au_n + bv_n}{2a}, \frac{au_n + bv_n}{2b}\Big),$$
$$\hat{\xi}_n = (\hat{\varphi}_n, \hat{\psi}_n) := \frac{\hat{z}_n}{\|z_n\|} = \Big(\frac{a\varphi_n + b\psi_n}{2a}, \frac{a\varphi_n + b\psi_n}{2b}\Big).$$

By (W1), (2.3), (2.5), (2.8), (2.9) and (3.5), one obtains

$$2c_* + o(1) = \|z_n^+\|^2 - \|z_n^-\|^2 - 2\int_{\mathbb{R}^N} W(x, z_n)\,\mathrm{d}x \le \|z_n^+\|^2 - \|z_n^-\|^2, \tag{3.6}$$
$$c_* + o(1) = \int_{\mathbb{R}^N} \widetilde{W}(x, z_n)\,\mathrm{d}x, \tag{3.7}$$

and

$$\begin{aligned}
\|\hat{z}_n\|^2 &= \frac{a^2+b^2}{4a^2b^2}\|au_n + bv_n\|_{E_V}^2 \\
&= \frac{a^2+b^2}{4a^2b^2}\Big[a^2\|u_n\|_{E_V}^2 + b^2\|v_n\|_{E_V}^2 + 2ab\int_{\mathbb{R}^N}(\nabla u_n \nabla v_n + V(x)u_n v_n)\Big] \\
&= \frac{a^2+b^2}{4a^2b^2}\Big[a^2\|u_n\|_{E_V}^2 + b^2\|v_n\|_{E_V}^2 + 2ab\Big(\Phi(z_n) + \int_{\mathbb{R}^N} W(x, u_n, v_n)\,\mathrm{d}x\Big)\Big] \\
&\ge \frac{a^2+b^2}{4a^2b^2}[\min\{a^2, b^2\}\|z_n\|^2 + 2ab(c_* + o(1))],
\end{aligned}$$

which implies

$$\|z_n\| \le \frac{2ab}{\sqrt{a^2+b^2}\min\{a,b\}}\|\hat{z}_n\|, \quad a, b > 0. \tag{3.8}$$

Note that

$$\begin{aligned}
\|\hat{\xi}_n\|^2 &= \frac{a^2+b^2}{4a^2b^2}\|a\varphi_n + b\psi_n\|_{E_V}^2 \\
&\le \frac{a^2+b^2}{4a^2b^2}\left(a\|\varphi_n\|_{E_V} + b\|\psi_n\|_{E_V}\right)^2 \\
&\le \frac{a^2+b^2}{2a^2b^2}\left(a^2\|\varphi_n\|_{E_V}^2 + b^2\|\psi_n\|_{E_V}^2\right) \\
&\le \frac{(a^2+b^2)^2}{2a^2b^2}\left(\|\varphi_n\|_{E_V}^2 + \|\psi_n\|_{E_V}^2\right) \\
&= \frac{(a^2+b^2)^2}{2a^2b^2}\|\xi_n\|^2 = \frac{(a^2+b^2)^2}{2a^2b^2},
\end{aligned} \tag{3.9}$$

which implies that $\{\hat{\xi}_n\}$ is bounded. If $\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}|\hat{\xi_n}|^2\,\mathrm{d}x = 0$, then by Lions's concentration compactness principle [18, Lemma 1.21], $a\varphi_n + b\psi_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Set $\kappa' = \kappa/(\kappa-1)$ and

$$\Omega_n := \Big\{x \in \mathbb{R}^N : \frac{|bW_u(x, u_n, v_n) + aW_v(x, u_n, v_n)|}{|z_n|} \le \theta\beta_0\min\{a,b\}\Big\},$$

then $2 < 2\kappa' < 2^*$. Hence, by (W1), (W2), (W3'), it follows from (2.1), (2.3), (3.8) and Hölder inequality that

$$
\begin{aligned}
&\int_{\Omega_n} |bW_u(x,u_n,v_n) + aW_v(x,u_n,v_n)||au_n + bv_n|\,\mathrm{d}x \\
&\le \int_{\Omega_n} \frac{|bW_u(x,u_n,v_n) + aW_v(x,u_n,v_n)|}{|z_n|}|z_n||au_n + bv_n|\,\mathrm{d}x \\
&\le \theta\beta_0 \min\{a,b\} \int_{\Omega_n} |z_n||au_n + bv_n|\,\mathrm{d}x \\
&\le \theta\beta_0 \min\{a,b\}\Big(\int_{\mathbb{R}^N} |z_n|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_{\mathbb{R}^N} |au_n + bv_n|^2\,\mathrm{d}x\Big)^{1/2} \\
&\le \theta \min\{a,b\}\|z_n\|\|au_n + bv_n\|_{E_V} \\
&= \theta \min\{a,b\}\|z_n\| \frac{2ab}{\sqrt{a^2+b^2}}\|\hat{z}_n\| \\
&\le \theta \frac{2ab\min\{a,b\}}{\sqrt{a^2+b^2}} \times \frac{2ab}{\sqrt{a^2+b^2}\min\{a,b\}}\|\hat{z}_n\|^2 \\
&= \theta \frac{4a^2b^2}{a^2+b^2}\|\hat{z}_n\|^2.
\end{aligned} \tag{3.10}
$$

On the other hand, by (W5), (2.4), (3.7), (3.8) and Hölder inequality, one obtains that

$$
\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega_n} \frac{|bW_u(x,u_n,v_n) + aW_v(x,u_n,v_n)||au_n + bv_n|}{\|z_n\|^2}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N\setminus\Omega_n} \frac{|bW_u(x,u_n,v_n) + aW_v(x,u_n,v_n)||\xi_n||a\varphi_n + b\psi_n|}{|z_n|}\,\mathrm{d}x \\
&\le \Big[\int_{\mathbb{R}^N\setminus\Omega_n} \Big(\frac{|aW_u(x,u_n,v_n) + bW_v(x,u_n,v_n)|}{|z_n|}\Big)^{\kappa}\,\mathrm{d}x\Big]^{1/\kappa} \\
&\quad \times \Big(\int_{\mathbb{R}^N\setminus\Omega_n} |\xi_n|^{2\kappa'}\,\mathrm{d}x\Big)^{1/2\kappa'}\Big(\int_{\mathbb{R}^N\setminus\Omega_n} |a\varphi_n + b\psi_n|^{2\kappa'}\,\mathrm{d}x\Big)^{1/2\kappa'} \\
&\le \Big(\int_{\mathbb{R}^N\setminus\Omega_n} \alpha_0 \widetilde{W}(x,z_n)\,\mathrm{d}x\Big)^{1/\kappa}\|\xi_n\|_{2\kappa'}\|a\varphi_n + b\psi_n\|_{2\kappa'} \\
&\le (c_*\alpha_0 + o(1))^{1/\kappa}\,\|\xi_n\|_{2\kappa'}\,\|a\varphi_n + b\psi_n\|_{2\kappa'} \\
&\le (c_*\alpha_0 + o(1))^{1/\kappa} 2^{(\kappa'-1)/2\kappa'}\gamma_{2\kappa'}\,\|\xi_n\|\,\|a\varphi_n + b\psi_n\|_{2\kappa'} \\
&= o(1).
\end{aligned} \tag{3.11}
$$

Combining (3.10) with (3.11) and using (2.6), and (3.8), we have

$$
\begin{aligned}
&\frac{4a^2b^2}{a^2+b^2} + o(1) \\
&= \frac{4a^2b^2}{a^2+b^2} - 2ab\frac{\langle \Phi'(z_n), \hat{z}_n\rangle}{\|\hat{z}_n\|^2} \\
&= \frac{1}{\|\hat{z}_n\|^2}\int_{\mathbb{R}^N} [bW_u(x,u_n,v_n) + aW_v(x,u_n,v_n)]\,(au_n + bv_n)\,\mathrm{d}x \\
&= \frac{1}{\|\hat{z}_n\|^2}\int_{\Omega_n} [bW_u(x,u_n,v_n) + aW_v(x,u_n,v_n)]\,(au_n + bv_n)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\|\hat{z}_n\|^2}\int_{\mathbb{R}^N\setminus\Omega_n}\left[bW_u(x,u_n,v_n)+aW_v(x,u_n,v_n)\right](au_n+bv_n)\,\mathrm{d}x\\
&\le \theta\frac{4a^2b^2}{a^2+b^2}+o(1). \qquad (3.12)
\end{aligned}
$$

This contradiction shows that $\delta\neq 0$.

If necessary going to a subsequence, we may assume the existence of $k_n\in\mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)}|\hat{\xi}_n|^2dx>\frac{\delta}{2}$. Since $|\hat{\xi}_n|^2=\frac{a^2+b^2}{4a^2b^2}|a\varphi_n+b\psi_n|^2$, one can get that

$$
\int_{B_{1+\sqrt{N}}(k_n)}|a\varphi_n+b\psi_n|^2dx>\frac{2a^2b^2}{a^2+b^2}\delta.
$$

Let us define $\tilde{\varphi}_n(x)=\varphi_n(x+k_n)$, $\tilde{\psi}_n(x)=\psi_n(x+k_n)$ so that

$$
\int_{B_{1+\sqrt{N}}(0)}|a\tilde{\varphi}_n+b\tilde{\psi}_n|^2dx>\frac{2a^2b^2}{a^2+b^2}\delta. \qquad (3.13)
$$

Now we define $\tilde{u}_n(x)=u_n(x+k_n)$, $\tilde{v}_n(x)=v_n(x+k_n)$, then $\tilde{\varphi}_n=\tilde{u}_n/\|z_n\|$, $\tilde{\psi}_n=\tilde{v}_n/\|z_n\|$. Passing to a subsequence, we have $a\tilde{\varphi}_n(x)+b\tilde{\psi}_n(x)\rightharpoonup a\tilde{\varphi}(x)+b\tilde{\psi}(x)$ in E_V, $a\tilde{\varphi}_n(x)+b\tilde{\psi}_n(x)\to a\tilde{\varphi}(x)+b\tilde{\psi}(x)$ in $L^s_{\rm loc}(\mathbb{R}^N)$, $2\le s<2^*$ and $a\tilde{\varphi}_n(x)+b\tilde{\psi}_n(x)\to a\tilde{\varphi}(x)+b\tilde{\psi}(x)$ a.e. on $\mathbb{R}^N$. Obviously, (3.13) implies that $a\tilde{\varphi}(x)+b\tilde{\psi}(x)\neq 0$. Since $\|z_n\|\to\infty$, for a.e. $x\in\{y\in\mathbb{R}^N: a\tilde{\varphi}(y)+b\tilde{\psi}(y)\neq 0\}:=\Omega$, we have

$$
\lim_{n\to\infty}|a\tilde{u}_n(x)+b\tilde{v}_n(x)|=\lim_{n\to\infty}\|z_n\||a\tilde{\varphi}_n(x)+b\tilde{\psi}_n(x)|=+\infty.
$$

By (W3'), (3.6) and Fatou's lemma, we have

$$
\begin{aligned}
0&=\lim_{n\to\infty}\frac{c_*+o(1)}{\|z_n\|^2}=\lim_{n\to\infty}\frac{\Phi(z_n)}{\|z_n\|^2}\\
&=\lim_{n\to\infty}\Big[\frac12\|\xi_n^+\|^2-\frac12\|\xi_n^-\|^2-\int_{\mathbb{R}^N}\frac{W(x,z_n)}{\|z_n\|^2}\,\mathrm{d}x\Big]\\
&\le\lim_{n\to\infty}\Big[\frac12\|\xi_n^+\|^2-\frac12\|\xi_n^-\|^2-\int_{\mathbb{R}^N}\frac{W(x,z_n)}{|au_n+bv_n|^2}|a\varphi_n+b\psi_n|^2\,\mathrm{d}x\Big]\\
&=\lim_{n\to\infty}\Big[\frac12\|\xi_n\|^2-\int_{\mathbb{R}^N}\frac{W(x+k_n,\tilde{z}_n)}{|a\tilde{u}_n+b\tilde{v}_n|^2}|a\tilde{\varphi}_n+b\tilde{\psi}_n|^2\,\mathrm{d}x\Big]\\
&\le\frac12-\int_\Omega\liminf_{n\to\infty}\Big[\frac{W(x,\tilde{u}_n,\tilde{v}_n)}{|a\tilde{u}_n+b\tilde{v}_n|^2}|a\tilde{\varphi}_n+b\tilde{\psi}_n|^2\Big]\,\mathrm{d}x=-\infty,
\end{aligned}
$$

which is a contradiction. Thus $\{z_n\}$ is bounded in E. □

Proof of Theorem 1.2. Applying Lemmas 3.4 and 3.5, we deduce that there exists a bounded sequence $\{z_n\}=\{(u_n,v_n)\}\subset E$ satisfying (3.5). Thus there exists a constant $C_2>0$ such that $\|z_n\|_2\le C_2$. By the Lion's concentration compactness principle ([9] or [18, Lemma 1.21]), one can rule out the case of vanishing. So nonvanishing occurs. Using a standard translation argument, we can obtain a nontrivial solution of (1.1). □

Acknowledgments. This work is partially supported by the NNSF of China (No. 11171351), by the Construct Program of the Key Discipline in Hunan Province, and by the Hunan Provincial Innovation Foundation for Postgraduates.

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