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AN APPLICATION OF VARIATIONAL METHODS TO SECOND-ORDER IMPULSIVE DIFFERENTIAL EQUATION WITH DERIVATIVE DEPENDENCE

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Abstract. In this article, we study the existence of solutions for nonlinear impulsive problems. We show the existence of classical solutions by using variational methods.

1. Introduction

Variational methods are used in the modeling of certain nonlinear problems from biological neural networks, elastic mechanics, anisotropic problems, and so forth. During the previous decade, variational methods have been applied to boundary value problems for differential equations. Recently, impulsive differential equation has been studied in many classical works, for example, [1, 2, 6, 8, 15, 17]. The study of impulsive differential equation via variational methods was initiated by Nieto and O'Regan [5], Tian and Ge [11]. The study of second order impulsive differential equation with derivative dependence ordinary differential equations via variational methods was initiated by Nieto [4]. Since then there is a trend to study differential equation via variational methods which leads to many meaningful results, see [9, 10, 13, 12, 14, 18, 19, 20, 21] and the references therein.

Recently, Nieto [4] studied the following damped linear Dirichlet boundary value problem with impulses:

$$\begin{gathered} -u''(t)+g(t)u'(t)+\lambda u(t)=\sigma(t), \quad \text{a.e. } t\in[0,T], \\ \Delta u'(t_j)=d_j, \quad j=1,2,\dots,p, \\ u(0)=u(T)=0, \end{gathered} \tag{1.1}$$

where $\lambda, d_j \in \mathbb{R}$, $\sigma \in C[0,1]$, the author introduced a variational formulation for the damped linear Dirichlet problem with impulses and the concept of a weak solution for such a problem.

We would also like to mention that Xiao and Nieto [13], considered the following nonlinear boundary value problems for second order impulsive differential

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equations:

$$\begin{gathered} -u''(t)+g(t)u'(t)+\lambda u(t)=f(t,u(t)), \quad \text{a.e. } t\in[0,T], \\ -\Delta u'(t_j)=I_j(u(t_j)), \quad j=1,2,\dots,p, \\ u(0)=u(T)=0, \end{gathered} \tag{1.2}$$

where $T>0$, $0=t_0<t_1<\dots<t_p<t_{p+1}=T$, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous, $g\in C[0,T]$, and $I_j:\mathbb{R}\to\mathbb{R}$, $j=1,2,\dots,p$ are continuous, and $\Delta u'(t_j)=u'(t_j^+)-u'(t_j^-)$, for $u'(t_j^{\pm})=\lim_{t\to t_j^{\pm}}u'(t)$. Authors used critical point theory and variational methods to obtain the above second order impulsive differential equations has at least one positive solution.

Motivated by the above mentioned work, in this paper we consider the impulsive boundary value problem

$$\begin{gathered} -u''(t)+\lambda u(t)+g(t)u'(t)=f(t,u), \quad \text{a.e. } t\in[0,T], \\ -\Delta u'(t_i)=I_i(u(t_i)), \quad i=1,2,\dots,p, \\ u(0)=0, \quad \alpha u(T)+\beta u'(T)=0, \end{gathered} \tag{1.3}$$

where λ is a parameter, $T>0$, $g\in C[0,T]$, $f\in C([0,T]\times\mathbb{R},\mathbb{R})$ and $I_{ij}:\mathbb{R}\to\mathbb{R}$, $i=1,2,\dots,p$ are continuous, $0=t_0<t_1<\dots<t_p<t_{p+1}=T$, $\Delta u'(t_i)=u'(t_i^+)-u'(t_i^-)=\lim_{t\to t_i^+}u'(t)-\lim_{t\to t_i^-}u'(t)$, $\alpha\ge 0,\beta>0$ (or $\beta=0$).

We consider the existence of classical solutions for the nonlinear impulsive problems and obtain some new existence theorems of solutions by using variational methods. We obtain the equation (1.3) has at least one classical solution, at least two classical solutions and infinitely many classical solutions under different conditions, and the conditions in our paper is easy to verify compared with the papers in the literature.

The rest of the article is organized as follows: In Section 2, we give some preliminaries, lemmas and variational structure. The main theorems are formulated and proved in Section 3. In Section 4, some examples are presented to illustrate our results.

2. Preliminaries, and variational structure

We first recall some basic results in eigenvalue problems. For linear problem

$$\begin{gathered} -u''(t)=\lambda u(t), \quad \text{a.e. } t\in[0,T], \\ u(0)=0, \quad \alpha u(T)+\beta u'(T)=0, \quad \alpha\ge 0,\ \beta>0, \end{gathered} \tag{2.1}$$

the eigenvalue λ satisfies

$$\alpha\sin\sqrt{\lambda}T+\beta\sqrt{\lambda}\cos\sqrt{\lambda}T=0. \tag{2.2}$$

Solving (2.2), we obtain $\lambda=(\frac{k\pi-\alpha}{T})^2$, $k=0,1,\dots$, where α satisfies $\cos\alpha=\frac{\alpha}{\sqrt{\alpha^2+\lambda\beta^2}}$.

Let λ_1 be the first eigenvalue of the above linear problem (2.1). For linear problem

$$\begin{gathered} -u''(t)=\lambda u(t), \quad \text{a.e. } t\in[0,T], \\ u(0)=0, \quad \alpha u(T)+\beta u'(T)=0, \quad \alpha\ge 0,\ \beta=0, \end{gathered} \tag{2.3}$$

the eigenvalue λ satisfies

$$\alpha\sin(\sqrt{\lambda}T)=0.$$

Solving this equation, we have $\lambda = (\frac{k\pi}{T})^2$, $k = 1, 2, \dots$, thus the first eigenvalue of the linear problem is $\frac{\pi^2}{T^2}$; i.e. $\lambda_1 = \frac{\pi^2}{T^2}$. In the remaining part of this paper, we assume that $\lambda > -m\lambda_1/M$, where $m = \min_{t\in[0,T]} e^{G(t)}$, $M = \max_{t\in[0,T]} e^{G(t)}$, and $G(t) = -\int_0^t g(s)ds$.

We denote the Sobolev space $H := H_0^1(0,T) = \big\{u : [0,T] \to \mathbb{R} | u$ is absolutely continuous, $u' \in L^2(0,T)$ and $u(0) = 0\big\}$ with the inner product and the corresponding norm

$$
\begin{gathered}
(u,v) = \int_0^T e^{G(t)} u'(t)v'(t)dt, \\
\|u\| = \Big(\int_0^T e^{G(t)} (u'(t))^2 dt\Big)^{1/2}.
\end{gathered}
$$

Let $H^2(0,T) = \{u : [0,T] \to \mathbb{R} | u, u'$ are absolutely continuous, $u'' \in L^2(0,T)\}$. For $u \in H^2(0,T)$, we have that u, u' are both absolutely continuous, and $u'' \in L^2(0,T)$. Hence $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ for any $t \in (0,T)$. If $u \in H^1(0,T)$, we have that u is absolutely continuous, and $u' \in L^2(0,T)$, thus the one side derivatives $u'(t^+)$, $u'(t^-)$ may not exist, which leads to the impulsive effects.

So by a classical solution to (1.3) we mean a function $u \in C[0,T]$ satisfying the differential equation in (1.3) such that $u_i = u_{|(t_i,t_{i+1})} \in H^2(t_i, t_{i+1})$ and $u'(t_i^-), u'(t_i^+)$ exist for every $i = 1, 2, \dots, p$ and verify the impulsive and the boundary conditions. The weak solution to (1.3) is given below and it is inspired by the weak solution defined in [4].

Multiply the first equation of (1.3) by $e^{G(t)}$, we obtain

$$
-(e^{G(t)}u'(t))' + \lambda e^{G(t)}u(t) = e^{G(t)} f(t, u(t)).
$$

Now multiply by $v \in H$ at both sides,

$$
-(e^{G(t)}u'(t))' v(t) + \lambda e^{G(t)}u(t)v(t) = e^{G(t)} f(t, u(t))v(t). \tag{2.4}
$$

Integrate (2.4) on the interval $[0,T]$ and use the boundary condition $u(0) = 0$, $\alpha u(T) + \beta u'(T) = 0$ to obtain

$$
\begin{aligned}
&\int_0^T e^{G(t)}u'(t)v'(t)dt + \lambda \int_0^T e^{G(t)}u(t)v(t)dt \\
&-\sum_{i=1}^p e^{G(t_i)} I_i(u(t_i))v(t_i) + \frac{\alpha}{\beta} e^{G(T)}u(T)v(T) \\
&= \int_0^T e^{G(t)} f(t,u(t))v(t)dt.
\end{aligned} \tag{2.5}
$$

Thus, a weak solution of the impulsive boundary value problem (1.3) is a function $u \in H$ such that (2.5) holds for any $v \in H$.

Define

$$
\begin{gathered}
A(u,v) = \int_0^T e^{G(t)}u'(t)v'(t)dt + \lambda \int_0^T e^{G(t)}u(t)v(t)dt + \frac{\alpha}{\beta} e^{G(T)}u(T)v(T), \\
F(t,u) = \int_0^u f(t,s)ds.
\end{gathered}
$$

Consider $\varphi : H \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{2}A(u,u) - \sum_{i=1}^{p} e^{G(t_i)} \int_0^{u(t_i)} I_i(t)dt - \int_0^T e^{G(t)} F(t,u(t))dt. \tag{2.6}$$

Using the continuity of f and $I_i, i = 1, 2, \dots p$, we obtain the continuity and differentiability of φ and $\varphi \in C^1(H, \mathbb{R})$. For any $v \in H$, one has

$$\begin{aligned} \varphi'(u)v = \int_0^T e^{G(t)} u'(t)v'(t)dt + \lambda \int_0^T e^{G(t)} u(t)v(t)dt + \frac{\alpha}{\beta} e^{G(T)} u(T)v(T) \\ - \sum_{i=1}^{p} e^{G(t_i)} I_i(u(t_i))v(t_i) - \int_0^T e^{G(t)} f(t,u(t))v(t)dt. \end{aligned} \tag{2.7}$$

Hence, a critical point of φ defined by (2.6), gives us a weak solution of (1.3).

Definition 2.1. Let E be a Banach space and $\varphi : E \to \mathbb{R}$, is said to be sequentially weakly lower semi-continuous if $\lim_{k\to+\infty} \inf \varphi(x_k) \geq \varphi(x)$ as $x_k \rightharpoonup x$ in E.

Definition 2.2 ([16, p. 81])**.** Let E be a real reflexive Banach space. For any sequence $\{u_k\} \subset E$, if $\varphi(u_k)$ is bounded and $\varphi'(u_k) \to 0$, as $k \to +\infty$ possesses a convergent subsequence, then we say φ satisfies the Palais-Smale condition.

Lemma 2.3. *If $u \in H$ is a weak solution of* (1.3)*, then u is a classical solution of* (1.3)*.*

Proof. By the definition of weak solution, one has

$$\begin{aligned} &\int_0^T e^{G(t)} u'(t)v'(t)dt + \lambda \int_0^T e^{G(t)} u(t)v(t)dt + \frac{\alpha}{\beta} e^{G(T)} u(T)v(T) \\ &- \sum_{i=1}^{p} e^{G(t_i)} I_i(u(t_i))v(t_i) - \int_0^T e^{G(t)} f(t,u(t))v(t)dt = 0. \end{aligned} \tag{2.8}$$

For $i \in \{0, 1, 2, \dots p\}$, we choose $v \in H$ with $v(t) = 0$ for every $t \in [0, t_i] \cup [t_{i+1}, T]$. Then we have

$$\int_{t_i}^{t_{i+1}} e^{G(t)} u'(t)v'(t)dt + \lambda \int_{t_i}^{t_{i+1}} e^{G(t)} u(t)v(t)dt = \int_{t_i}^{t_{i+1}} e^{G(t)} f(t,u(t))v(t)dt.$$

By the definition of weak derivative, the above equality implies

$$-\left(e^{G(t)}u'(t)\right)' + \lambda e^{G(t)} u(t) = e^{G(t)} f(t,u(t)), \quad \text{a.e. } t \in (t_i, t_{i+1}). \tag{2.9}$$

i.e.

$$-u''(t) + \lambda u(t) + g(t)u'(t) = f(t,u), \quad \text{a.e. } t \in (t_i, t_{i+1}).$$

Hence, $u_i = u_{|(t_i,t_{i+1})} \in H^2(t_i, t_{i+1})$ and u satisfies the first equation in (1.3) a.e. on $[0, T]$. Now, multiplying by $v \in H,\ v(T) = 0$ and integrating between 0 and T, we obtain

$$-\sum_{i=1}^{p} \Delta(e^{G(t_i)} u'(t_i))v'(t_i) = \sum_{i=1}^{p} (e^{G(t_i)} I(t_i))v'(t_i).$$

Hence

$$-\Delta u'(t_i) = I_i(u(t_i)), \quad i = 1, 2, \dots, p,$$

thus, u satisfies the impulsive conditions. It is easy to verify u satisfies the boundary conditions $u(0) = 0,\ \alpha u(T) + \beta u'(T) = 0$. Therefore, u is a classical solution to (1.3). $\square$

Lemma 2.4 ([16, 7, Theorem 38]). *For the functional $F : M \subseteq X \to [-\infty, +\infty]$ with $M \neq \emptyset$, $\min_{u\in M} F(u) = \alpha$ has a solution when the following conditions hold:*

- (i) *X is a real reflexive Banach space;*
- (ii) *M is bounded and weak sequentially closed; i.e., by definition, for each sequence u_n in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we always have $u \in M$;*
- (iii) *F is weak sequentially lower semi-continuous on M.*

Next we sate the mountain pass theorem [3, Theorem 4.10].

Lemma 2.5. *Let E be a Banach space and $\varphi \in C^1(E, \mathbb{R})$ satisfy Palais-Smale condition. Assume there exist $x_0, x_1 \in E$, and a bounded open neighborhood Ω of x_0 such that $x_1 \notin \overline{\Omega}$ and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x\in\partial\Omega} \varphi(x).$$

Then there exists a critical value of φ; that is, there exists $u \in E$ such that $\varphi'(u) = 0$ and $\varphi(u) > \max\{\varphi(x_0), \varphi(x_1)\}$.

Now we have the symmetric mountain pass theorem [7, Theorem 9.12].

Lemma 2.6. *Let E be an infinite dimensional real Banach space. Let $\varphi \in C^1(E, \mathbb{R})$ be an even functional which satisfies the Palais-Smale condition, and $\varphi(0) = 0$. Suppose that $E = V \oplus X$, where V is infinite dimensional, and φ satisfies that*

- (i) *there exist $\gamma > 0$ and $\rho > 0$ such that $\varphi(u) \geq \gamma$ for all $u \in X$ with $\|u\| = \rho$,*
- (ii) *for any finite dimensional subspace $W \subset E$ there is $R = R(W)$ such that $\varphi(u) \leq 0$ on $W \setminus B_R(W)$.*

Then φ possesses an unbounded sequence of critical values.

Lemma 2.7. *There exists $\delta > 0$ such that if $u \in H$, then $\|u\|_\infty \leq \delta\|u\|$, where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$.*

Proof. It follows from Hölder's inequality that

$$\begin{aligned}
|u(t)| &= \Big|\int_0^t u'(s)ds\Big| \\
&\leq \int_0^t |u'(s)|ds \leq \int_0^T |u'(t)|dt \\
&\leq \Big(\int_0^T \frac{1}{e^{G(t)}}ds\Big)^{1/2}\Big(\int_0^T e^{G(t)}|u'(t)|^2 ds\Big)^{1/2} \\
&\leq \sqrt{\frac{T}{m}}\|u\|,
\end{aligned}$$

thus, we can choose $\delta = \sqrt{\frac{T}{m}}$ such that Lemma 2.7 holds. □

Lemma 2.8. *There exist two constants $\theta_2 > \theta_1 > 0$ such that if $u \in H$, then*

$$\theta_1\|u\|^2 \leq A(u, u) \leq \theta_2\|u\|^2.$$

Proof. Firstly, when $\lambda \geq 0$, we obtain the following results by Poincaré's inequality,

$$\begin{aligned}
A(u, u) &= \int_0^T e^{G(t)}(u'(t))^2 dt + \lambda \int_0^T e^{G(t)}(u(t))^2 dt + \frac{\alpha}{\beta} e^{G(T)} u^2(T) \\
&\leq \int_0^T e^{G(t)}(u'(t))^2 dt + \lambda M \int_0^T (u(t))^2 dt + \frac{\alpha}{\beta} e^{G(T)} u^2(T)
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^T e^{G(t)}(u'(t))^2dt + \frac{\lambda M}{\lambda_1}\int_0^T \frac{e^{G(t)}}{m}(u'(t))^2dt + \frac{\alpha}{\beta}e^{G(T)}u^2(T)\\
&\le (1+\frac{\lambda M}{\lambda_1 m})\int_0^T e^{G(t)}(u'(t))^2dt + \frac{M\alpha}{\beta}\|u(t)\|_\infty^2\\
&\le (1+\frac{\lambda M}{\lambda_1 m}+\frac{M\alpha\delta^2}{\beta})\|u\|^2.
\end{aligned}
$$

and

$$
\begin{aligned}
A(u,u) &= \int_0^T e^{G(t)}(u'(t))^2dt + \lambda\int_0^T e^{G(t)}(u(t))^2dt + \frac{\alpha}{\beta}e^{G(T)}u^2(T)\\
&\ge \int_0^T e^{G(t)}(u'(t))^2dt\\
&= \|u\|^2.
\end{aligned}
$$

thus, $\theta_1 = 1, \theta_2 = 1 + \frac{\lambda M}{\lambda_1 m} + \frac{M\alpha\delta^2}{\beta}$.

Secondly, when $0 > \lambda > -m\lambda_1/M$, by using Poincaré's inequality, one has

$$
\begin{aligned}
A(u,u) &= \int_0^T e^{G(t)}(u'(t))^2dt + \lambda\int_0^T e^{G(t)}(u(t))^2dt + \frac{\alpha}{\beta}e^{G(T)}u^2(T)\\
&\ge \int_0^T e^{G(t)}(u'(t))^2dt + \lambda\int_0^T e^{G(t)}(u(t))^2dt\\
&\ge \int_0^T e^{G(t)}(u'(t))^2dt + \lambda M\int_0^T (u(t))^2dt\\
&\ge \int_0^T e^{G(t)}(u'(t))^2dt + \frac{\lambda M}{\lambda_1}\int_0^T (u'(t))^2dt\\
&\ge \int_0^T e^{G(t)}(u'(t))^2dt + \frac{\lambda M}{\lambda_1 m}\int_0^T e^{G(t)}(u'(t))^2dt\\
&= (1+\frac{\lambda M}{\lambda_1 m})\|u\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
A(u,u) &= \int_0^T e^{G(t)}(u'(t))^2dt + \lambda\int_0^T e^{G(t)}(u(t))^2dt + \frac{\alpha}{\beta}e^{G(T)}u^2(T)\\
&\le \int_0^T e^{G(t)}(u'(t))^2dt + \frac{\alpha}{\beta}e^{G(T)}u^2(T)\\
&\le (1+\frac{M\alpha\delta^2}{\beta})\|u\|^2,
\end{aligned}
$$

thus, $\theta_1 = 1 + \frac{\lambda M}{\lambda_1 m}$, $\theta_2 = 1 + \frac{M\alpha\delta^2}{\beta}$. □

Lemma 2.9. *The functional φ is continuous, continuously differentiable and weakly lower semi-continuous.*

Proof. By the continuity of f and I_i $(i = 1, 2, \dots, p)$, it is easy to check that functional φ is continuous, continuously differentiable. To show that φ is weakly lower semi-continuous, let $\{u_n\}$ be a weakly convergent sequence to u in H, then

$\|u\| \leq \liminf_{n\to\infty} \|u_n\|$, and $\{u_n\}$ converges uniformly to u in $[0,T]$, so when $n \to \infty$, we have

$$\begin{aligned}
\liminf_{n\to\infty} \varphi(u_n) &= \liminf_{n\to\infty}\Big(\frac{1}{2}\|u_n\|^2 + \frac{\lambda}{2}\int_0^T e^{G(t)}(u_n(t))^2dt + \frac{\alpha}{2\beta}e^{G(T)}u_n^2(T) \\
&\quad - \int_0^T e^{G(t)}F(t,u_n)dt - \sum_{i=1}^p e^{G(t_i)}\int_0^{u_n(t_i)} I_i(t)dt\Big) \\
&\geq \frac{1}{2}\|u\|^2 + \frac{\lambda}{2}\int_0^T e^{G(t)}(u(t))^2dt + \frac{\alpha}{2\beta}e^{G(T)}u^2(T) \\
&\quad - \int_0^T e^{G(t)}F(t,u)dt - \sum_{i=1}^p e^{G(t_i)}\int_0^{u(t_i)} I_i(t)dt \\
&= \varphi(u).
\end{aligned}$$

Thus, by Definition 2.1, φ is weakly lower semi-continuous. □

Now, we introduce the well-known Ambrosetti-Rabinowitz condition: There exist $\mu > 2$ and $r > 0$ such that

$$0 < \mu F(t,u) \leq f(t,u)u, \quad \forall u \in \mathbb{R}\setminus\{0\},\ t \in [0,T].$$

It is well known that the Ambrosetti-Rabinowitz condition is quite natural and convenient not only to ensure the Palais-Smale sequence of the functional φ is bounded but also to guarantee the functional φ has a mountain pass geometry.

Lemma 2.10. *Suppose that Ambrosetti-Rabinowitz condition holds. Furthermore, we assume*

$$I_i(u)u \geq \mu\int_0^u I_i(t)dt, \quad u \in \mathbb{R}\setminus\{0\}.$$

then the functional φ satisfies Palais-Smale condition.

Proof. Let $\{u_k\}$ be a sequence in H such that $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k) \to 0$, as $k \to +\infty$, then we will prove $\{u_k\}$ possesses a convergent subsequence.

First we prove that $\{u_k\}$ is bounded. By the Ambrosetti-Rabinowitz condition and $I_i(u)u \geq \mu\int_0^u I_i(t)dt$, we have

$$\begin{aligned}
&\mu\varphi(u_k) - \varphi'(u_k)u_k \\
&= \Big(\frac{\mu}{2} - 1\Big)A(u_k,u_k) - \mu\sum_{i=1}^p e^{G(t_i)}\int_0^{u_k(t_i)} I_i(t)dt + \sum_{i=1}^p e^{G(t_i)}I_i(u_k(t_i))u_k(t_i) \\
&\quad - \mu\int_0^T e^{G(t)}F(t,u_k)dt + \int_0^T e^{G(t)}f(t,u_k)u_kdt \\
&\geq \Big(\frac{\mu}{2} - 1\Big)\theta_1\|u_k\|^2,
\end{aligned}$$

which implies that $\{u_k\}$ is bounded. Hence there exists a subsequence of $\{u_k\}$ (for simplicity denoted again by $\{u_k\}$) such that $\{u_k\}$ weakly converges to some u in H, then the sequence $\{u_k\}$ converges uniformly to u in $[0,T]$. Hence

$$(\varphi'(u_k) - \varphi'(u))(u_k - u) \to 0,$$

$$\int_0^T e^{G(t)}(F(t,u) - f(t,u))(u_k - u)dt \to 0,$$

$$[I_i(u_k(t_i)) - I_i(u(t_i))](u_k(t_i) - u(t_i)) \to 0,$$

as $k \to +\infty$. Thus, we have

$$\begin{aligned}
&(\varphi'(u_k) - \varphi'(u))(u_k - u) \\
&= \varphi'(u_k)(u_k - u) - \varphi'(u)(u_k - u) \\
&= \int_0^T e^{G(t)}(u_k'(t) - u'(t))^2 dt + \lambda \int_0^T e^{G(t)}(u_k(t) - u(t))^2 dt + \frac{\alpha}{\beta}(u_k(T) - u(T))^2 \\
&\quad - \sum_{i=1}^p e^{G(t_i)}[I_i(u_k(t_i)) - I_i(u(t_i))](u_k(t_i) - u(t_i)) \\
&\quad - \int_0^T e^{G(t)}(f(t,u_k) - f(t,u))(u_k(t) - u(t))dt \\
&= A(u_k(t) - u(t), u_k(t) - u(t)) - \sum_{i=1}^p e^{G(t_i)}[I_i(u_k(t_i)) \\
&\quad - I_i(u(t_i))](u_k(t_i) - u(t_i)) - \int_0^T e^{G(t)}(f(t,u_k) - f(t,u))(u_k(t) - u(t))dt \\
&\geq \theta_1\|u_k - u\|^2 - \sum_{i=1}^p e^{G(t_i)}[I_i(u_k)(t_i) \\
&\quad - I_i(u(t_i))](u_k(t_i) - u(t_i)) - \int_0^T e^{G(t)}(f(t,u_k) - f(t,u))(u_k(t) - u(t))dt,
\end{aligned}$$

which means $\|u_k - u\| \to 0$, as $k \to +\infty$. That is, $\{u_k\}$ converges strongly to u in H. $\square$

The following Lemma was proved in [20].

Lemma 2.11. *Denote $\bar{M} = \max_{t\in[0,T],|u|=1} F(t,u)$, $\bar{m} = \min_{t\in[0,T],|u|=1} F(t,u)$. Suppose that Ambrosetti-Rabinowitz condition holds. Then, for every $t \in [0,T]$, the following inequalities hold.*

- (i) *$F(t,u) \leq \bar{M}|u|^\mu$, if $|u| < 1$,*
- (ii) *For any finite dimensional subspace $W \in H$ and any $u \in W$, there exist constants $A, B > 0$, such that $\int_0^T F(t,u)dt \geq \bar{m}B^\mu\|u\|^\mu - AT$.*

3. Main results

Our main results are the following theorems.

Theorem 3.1. *Suppose that $\lambda > -m\lambda_1/M$, f and I_i $(i = 1,2,\dots,p)$ are bounded, and furthermore $f(t,0) \not\equiv 0$, then (1.3) has at least one classical solution.*

Proof. Take $C > 0$, $C_i > 0$, $i = 1,2,\dots,p$, such that

$$\begin{gathered}
|f(t,u)| \leq C, \quad \forall (t,u) \in [0,T] \times \mathbb{R}, \\
|I_i(u)| \leq C_i, \quad \forall (t,u) \in [0,T] \times \mathbb{R},\ i = 1,2,\dots,p.
\end{gathered}$$

For any $u \in H$, one has

$$\varphi(u) = \frac{1}{2}A(u,u) - \sum_{i=1}^p e^{G(t_i)} \int_0^{u(t_i)} I_i(t)dt - \int_0^T e^{G(t)}F(t,u(t))dt$$

$$
\begin{aligned}
&\geq \frac{1}{2}\theta_1\|u\|^2-\sum_{i=1}^{p}e^{G(t_i)}\int_0^{u(t_i)}I_i(t)dt-\int_0^T e^{G(t)}F(t,u(t))dt\\
&\geq \frac{1}{2}\theta_1\|u\|^2-\sum_{i=1}^{p}e^{G(t_i)}C_i|u(t_i)|-\int_0^T e^{G(t)}F(t,u(t))dt\\
&\geq \frac{1}{2}\theta_1\|u\|^2-\sum_{i=1}^{p}e^{G(t_i)}C_i|u(t_i)|-CM\int_0^T|u(t)|dt\\
&\geq \frac{1}{2}\theta_1\|u\|^2-\sum_{i=1}^{p}e^{G(t_i)}C_i\|u\|_\infty-CMT\|u\|_\infty\\
&\geq \frac{1}{2}\theta_1\|u\|^2-M\sum_{i=1}^{p}C_i\|u\|_\infty-CMT\|u\|_\infty\\
&\geq \frac{1}{2}\theta_1\|u\|^2-M\sum_{i=1}^{p}C_i\delta\|u\|-CMT\delta\|u\|,
\end{aligned}
$$

which implies that $\liminf_{\|u\|\to\infty}\varphi(u)=+\infty$, thus, φ is coercive. Hence, by [3, Lemma 2.7 and Theorem 1.1], φ has a minimum, which is a critical point of φ, then (1.3) has at least one solution. $\square$

Analogously we have the following result.

Theorem 3.2. *Suppose that $\lambda>-m\lambda_1/M$, f and I_i $(i=1,2,\dots,p)$ have sublinear growth, and furthermore $f(t,0)\not\equiv 0$, then* (1.3) *has at least one classical solution.*

Proof. Let $a,b,a_i,b_i>0$, and $\gamma,\gamma_i\in[0,1)$, $i=1,2,\dots,p$, such that

$$
\begin{gathered}
|f(t,u)|\leq a+b|u|^\gamma,\quad \forall (t,u)\in[0,T]\times\mathbb{R},\\
|I_i(u)|\leq a_i+b_i|u|^{\gamma_i},\quad \forall u\in\mathbb{R},\ i=1,2,\dots,p.
\end{gathered}
$$

By using the same methods as in the above proof, there exists $\eta>0$, such that

$$
\varphi(u)\geq\frac{1}{2}\theta_1\|u\|^2-\eta\|u\|^{\gamma+1},
$$

which implies that $\liminf_{\|u\|\to\infty}\varphi(u)=+\infty$, thus, φ is coercive. Hence, by [3, Lemma 2.7 and Theorem 1.1], φ has a minimum, which is a critical point of φ, then (1.3) has at least one solution. $\square$

Theorem 3.3. *Suppose the Ambrosetti-Rabinowitz condition holds, $\lambda>-m\lambda_1/M$, and there exist $\delta_i>0$, $\mu>2$, $i=1,2,\dots p$ such that $\int_0^u I_i(t)dt\leq\delta_i|u|^\mu$, $I_i(u)u\geq\mu\int_0^u I_i(t)dt>0$, $u\in\mathbb{R}\setminus\{0\}$. Then the impulsive problem* (1.3) *has at least two classical solutions.*

Proof. Firstly, We will show that there exists $\rho>0$ such that the functional φ has a local minimum $u_0\in B_\rho=\{u\in H:\|u\|<\rho\}$. By the same methods used in [20] show that $\overline{B}_\rho$ is a bounded and weak sequentially closed. Noting that φ is weak sequentially lower semi-continuous on $\overline{B}_\rho$ and H is a reflexive Banach space. Then by Lemma 2.4 we can know that φ has a local minimum $u_0\in B_\rho$; that is, $\varphi(u_0)=\min_{u\in\overline{B}_\rho}\varphi(u)$.

In the following, we will show that $\varphi(u_0) < \inf_{u\in\partial B_\rho}\varphi(u)$. Choose ρ small enough such that

$$\frac{\theta_1}{2}\rho^2 - M\sum_{i=1}^{p}\delta_i\rho^\mu - M\bar{M}\rho^\mu\delta^\mu T > 0.$$

For all $u = \rho\omega, \omega \in H$ with $\|\omega\| = 1$, we have $\|u\| = \|\rho\omega\| = \rho\|\omega\| = \rho$, thus $u \in \partial B_\rho$. By Lemma 2.8 and (i) of Lemma 2.11, one has

$$\begin{aligned}
\varphi(u) &= \varphi(\rho\omega)\\
&= \frac{1}{2}A(\rho\omega,\rho\omega) - \sum_{i=1}^{p} e^{G(t_i)}\int_0^{\rho\omega(t_i)} I_i(t)dt - \int_0^T e^{G(t)}F(t,\rho\omega(t))dt\\
&\geq \frac{\theta_1}{2}\rho^2 - \sum_{i=1}^{p} e^{G(t_i)}\int_0^{\rho\omega(t_i)} I_i(t)dt - M\int_0^T \bar{M}|\rho\omega|^\mu dt\\
&\geq \frac{\theta_1}{2}\rho^2 - M\sum_{i=1}^{p}\delta_i|\rho\omega|^\mu - M\bar{M}\rho^\mu\int_0^T|\omega|^\mu dt\\
&\geq \frac{\theta_1}{2}\rho^2 - M\sum_{i=1}^{p}\delta_i\rho^\mu - M\bar{M}\rho^\mu\delta^\mu T,
\end{aligned}$$

thus we obtain $\varphi(u) > 0 = \varphi(0) \geq \varphi(u_0)$ for $u \in \partial B_\rho$, which implies $\varphi(u_0) < \inf_{u\in\partial B_\rho}\varphi(u)$.

Secondly, we will show that there exists u_1 with $\|u\| > \rho$, such that $\varphi(u_1) < \inf_{u\in\partial B_\rho}\varphi(u)$. By Lemma 2.8, the sublinear growth of I_i, $i = 1, 2\dots, p$ and (ii) of Lemma 2.11, one has

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}A(u,u) - \sum_{i=1}^{p} e^{G(t_i)}\int_0^{u(t_i)} I_i(t)dt - \int_0^T e^{G(t)}F(t,u(t))dt\\
&\leq \frac{1}{2}\theta_2\|u\|^2 - m(\bar{m}B^\mu\|u\|^\mu - AT).
\end{aligned}$$

Therefore, we can choose u_1 with $\|u_1\|$ sufficiently large such that $\varphi(u_1) < 0$. Then we have

$$\max\{\varphi(u_0),\varphi(u_1)\} < \inf_{u\in\partial B_\rho}\varphi(u).$$

Lemma 2.10 shows that u satisfies Palais-smale condition. Hence, by Lemma 2.5 there exists a critical point $\hat{u}$. Therefore, u_0 and $\hat{u}$ are two critical points of φ, and they are also classical solutions of (1.3). □

Theorem 3.4. *Suppose the Ambrosetti-Rabinowitz condition holds, $\lambda > -m\lambda_1/M$, and there exist $\delta_i > 0$, $\mu > 2$, $i = 1, 2, \dots p$ such that $\int_0^u I_i(t)dt \leq \delta_i|u|^\mu$, $I_i(u)u \geq \mu\int_0^u I_i(t)dt > 0$, $u \in \mathbb{R}\setminus\{0\}$. Moreover, $f(t,u)$ and I_i are odd about u, then the impulsive problem (1.3) has infinitely many classical solutions.*

Proof. For any $u \in H$, we know that $\|u\| \leq \frac{1}{\delta}$ implies $\|u\|_\infty \leq 1$ by Lemma 2.7, thus when $\|u\| \leq \frac{1}{\delta}$, one has the following inequality by (i) of Lemma 2.11,

$$\varphi(u) = \frac{1}{2}A(u,u) - \sum_{i=1}^{p} e^{G(t_i)}\int_0^{u(t_i)} I_i(t)dt - \int_0^T e^{G(t)}F(t,u(t))dt$$

$$
\begin{aligned}
&\geq \frac{1}{2}\theta_1\|u\|^2 - M\sum_{i=1}^{p}\delta_i|u|^{\mu} - \int_0^T e^{G(t)}F(t,u(t))dt \\
&\geq \frac{1}{2}\theta_1\|u\|^2 - M\sum_{i=1}^{p}\delta_i\|u\|_{\infty}^{\mu} - \int_0^T e^{G(t)}F(t,u(t))dt \\
&\geq \frac{1}{2}\theta_1\|u\|^2 - M\sum_{i=1}^{p}\delta_i\delta\|u\|^{\mu} - M\bar{M}T\delta^{\mu}\|u\|^{\mu}.
\end{aligned}
$$

Thus we can choose u with $\|u\|$ sufficiently small such that $\varphi(u) \geq \gamma > 0$. Thus φ satisfies condition (i) of Lemma 2.6.

In the following, it is turn to verify condition (ii) of Lemma 2.6. In fact, we can get the following inequality by (ii) of Lemma 2.11,

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}A(u,u) - \sum_{i=1}^{p}e^{G(t_i)}\int_0^{u(t_i)} I_i(t)dt - \int_0^T e^{G(t)}F(t,u(t))dt \\
&\leq \frac{1}{2}\theta_2\|u\|^2 - m\int_0^T F(t,u(t))dt \\
&\leq \frac{1}{2}\theta_2\|u\|^2 - m(\bar{m}B^{\mu}\|u\|^{\mu} - AT).
\end{aligned}
$$

Noting that $\mu > 2$, the above inequality implies that $\varphi(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in W$. Therefore, there exists $R = R(W)$ such that $\varphi(u) \leq 0$ on $W \setminus B_R$. According to Lemma 2.6, the functional $\varphi(u)$ possesses infinitely many critical points; i.e., the impulsive problem (1.3) has infinitely many classical solutions. $\square$

Remark 3.5. Equation (1.3) when $\beta = 0$, i.e. (1.2), which has been studied in [13]. By defining a new functional

$$
\begin{aligned}
\tilde{\varphi}(u) = \frac{1}{2}\int_0^T e^{G(t)}u'(t)v'(t)dt + \frac{\lambda}{2}\int_0^T e^{G(t)}u(t)v(t)dt \\
- \sum_{i=1}^{p}e^{G(t_i)}\int_0^{u(t_i)} I_i(t)dt - \int_0^T e^{G(t)}F(t,u(t))dt,
\end{aligned}
$$

and using the same methods, we can obtain the same results as the above-proved four theorems when $\lambda > -\frac{m\pi^2}{MT^2}$.

4. Examples

Example 4.1. Take $T > 0$, $t_1 \in (0,T)$, $g(t) = t$, $a(t), b(t) \in C([0,T],\mathbb{R})$, $c \in \mathbb{R}$, $\alpha \geq 0$, $\beta > 0$. Consider the equation

$$
\begin{gathered}
-u''(t) + \lambda u(t) + g(t)u'(t) = a(t)\sin u(t) + b(t), \quad \text{a.e. } t \in [0,T], \\
-\Delta u'(t_1) = c\cos u(t_1), \\
u(0) = 0, \quad \alpha u(T) + \beta u'(T) = 0,
\end{gathered} \tag{4.1}
$$

when $\lambda > -e^{T^2/2}\lambda_1$, equation (4.1) is solvable according to Theorem 3.1.

Example 4.2. Take $T > 0$, $t_1 \in (0, T)$, $g(t) = t$, $a(t), b(t) \in C([0, T], \mathbb{R})$, $c, d \in \mathbb{R}$, $\alpha \geq 0$, $\beta > 0$. Consider the equation

$$\begin{gathered} -u''(t) + \lambda u(t) + g(t)u'(t) = a(t)\sqrt[3]{u(t)} + b(t)\sin u(t), \quad \text{a.e. } t \in [0, T], \\ -\Delta u'(t_1) = c\sqrt[5]{u(t_1)} + d\cos u(t_1), \\ u(0) = 0, \quad \alpha u(T) + \beta u'(T) = 0, \end{gathered} \tag{4.2}$$

when $\lambda > -e^{T^2/2}\lambda_1$, equation (4.2) is solvable according to Theorem 3.2.

Example 4.3. Take $T > 0$, $t_1 \in (0, T)$, $g(t) = t$, $a(t) \in C([0, T], (0, +\infty))$, $c > 0$, $\mu = 3$, $\delta_1 = 1/6$, $\alpha \geq 0$, $\beta > 0$. Consider the equation

$$\begin{gathered} -u''(t) + \lambda u(t) + g(t)u'(t) = 2a(t)(e^{u^2} - e^{-u^2})u^5 + 4a(t)(e^{u^2} - e^{-u^2})u^3, \\ \text{a.e. } t \in [0, T], \\ -\Delta u'(t_1) = cu^5(t_1), \\ u(0) = 0, \quad \alpha u(T) + \beta u'(T) = 0, \end{gathered} \tag{4.3}$$

when $\lambda > -e^{T^2/2}\lambda_1$, equation (4.3) has at least two classical solutions according to Theorem 3.3.

Example 4.4. Take $T > 0$, $t_1 \in (0, T)$, $g(t) = t$, $a(t), b(t) \in C([0, T], (0, +\infty))$, $c > 0$, $\mu = 3$, $\delta_1 = 1/4$, $\alpha \geq 0$, $\beta > 0$. Consider the equation

$$\begin{gathered} -u''(t) + \lambda u(t) + g(t)u'(t) = a(t)u^5(t) + b(t)u^7(t), \quad \text{a.e. } t \in [0, T], \\ -\Delta u'(t_1) = cu^3(t_1), \\ u(0) = 0, \quad \alpha u(T) + \beta u'(T) = 0, \end{gathered} \tag{4.4}$$

when $\lambda > -e^{T^2/2}\lambda_1$, equation (4.4) has infinitely many classical solutions according to Theorem 3.4.

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