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BOUNDEDNESS IN A CHEMOTAXIS SYSTEM WITH CONSUMPTION OF CHEMOATTRACTANT AND LOGISTIC SOURCE

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Abstract. In this article, we consider a chemotaxis system with consumption of chemoattractant and logistic source

$$u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + f(u), \quad x\in\Omega,\ t>0,$$
$$v_t = \Delta v - uv, \quad x\in\Omega,\ t>0,$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subset\mathbb{R}^n$, with non-negative initial data u_0 and v_0 satisfying $(u_0,v_0)\in(W^{1,\theta}(\Omega))^2$ (for some $\theta>n$). $\chi>0$ is a parameter referred to as chemosensitivity and $f(s)$ is assumed to generalize the logistic function

$$f(s) = as - bs^2, \quad s\geq 0,\ \text{with } a>0,\ b>0.$$

It is proved that if $\|v_0\|_{L^\infty(\Omega)}>0$ is sufficiently small then the corresponding initial-boundary value problem possesses a unique global classical solution that is uniformly bounded.

1. Introduction

This article considers the following chemotaxis system with consumption of chemoattractant and logistic source

$$\begin{gathered}
u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + f(u), \quad x\in\Omega,\ t>0,\\
v_t = \Delta v - uv, \quad x\in\Omega,\ t>0,\\
\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0,\\
u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad x\in\Omega,
\end{gathered}\tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $\partial/\partial\nu$ denotes the derivative with respect to the outer normal of $\partial\Omega$. The parameter $\chi>0$ is referred as chemosensitivity, and the function $f\in C^1([0,\infty))$ with $f(0)=0$. Moreover, we shall suppose that

$$f(u)\leq au - bu^p \quad \text{for all } u\geq 0 \tag{1.2}$$

with some $a>0$, $b>0$ and $p>1$.

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Equations (1.1) is the well-known Keller-Segel model, and the origin of this fundamental model was introduced by Keller and Segel [13] to describe the motion of cells which are diffusing and moving towards the concentration gradient of a chemical signal substance called chemoattractant, the latter being produced by the cells themselves. We refer the reader to the paper [10] where a comprehensive information of further examples illustrating the outstanding biological relevance of chemotaxis can be found. In this paper, we consider a mathematical model for the motion of cells which towards the higher concentration of oxygen that is consumed by the cells, where $u = u(x,t)$ denotes the density of the cells and $v = v(x,t)$ represents the concentration of the oxygen.

During the past four decades, the Keller-Segel models have become one of the best-study models in mathematical biology the Keller-Segel models have been studied extensively by many authors. For example, Keller and Segel [13] proposed the following classical chemotaxis model

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot (u \nabla v), \quad x \in \Omega,\ t > 0, \\ v_t &= \Delta v - v + u, \quad x \in \Omega,\ t > 0, \end{aligned} \tag{1.3}$$

which has been investigated successfully up to now and the main issue of the investigation was the solutions of the model are bounded or blow-up. If $n = 1$, in [19], it was shown all solutions of (1.3) are global in time and bounded; if $n = 2$, then all solutions of (1.3) are global in time and bounded provided that $\|u_0\|_{L^1(\Omega)} < 4\pi$ in [15], however, for almost every $\|u_0\|_{L^1(\Omega)} > 4\pi$, then the corresponding solutions of (1.3) blow up either in finite or infinite time in [11] and that some radially symmetric solutions blow up in finite time in [9, 24]; if $n \geq 3$, Winkler [25] showed that $\|u_0\|_{L^{n/2+\epsilon}(\Omega)}$ and $\|\nabla v_0\|_{L^{n+\epsilon}(\Omega)}$ are small for all $\epsilon > 0$, then the solution is global in time and bounded, however, for any $\|u_0\|_{L^1(\Omega)} > 0$, then the radially symmetric solution of (1.3) blows up either in finite or infinite time (see also [24, 26, 5]).

Involving a source term of logistic type in chemotaxis system have been studied [10, 28, 18, 23, 16, 14]. The following initial-boundary value chemotaxis model with logistic source

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot (u \chi(v) \nabla v) + f(u), \quad x \in \Omega,\ t > 0, \\ v_t &= \Delta v - v + u, \quad x \in \Omega,\ t > 0. \end{aligned} \tag{1.4}$$

If $\chi(v)$ is a constant, Winkler [28] studied proved that the solutions of problem (1.4) are global and bounded provided that $f(0) \geq 0$ as well as $f(u) \leq a - bu^2$ with some $a \geq 0$ and b is sufficiently large. If $\chi(v) \leq \frac{\chi_0}{(1+\beta v)^\delta}$ for all $v \geq 0$ and some $\delta > 1$, $\chi_0 > 0$ and $\beta > 0$, the authors [14] shown that the model (1.4) with logistic source $f(u)$ satisfies (1.2) with $p = 2$ then solutions are global and bounded provided that χ_0 and a are sufficiently small, the authors [3] recent obtain the same result for all positive values of χ_0 and a, which improved the previous result.

The model (1.1) deals with the chemotaxis process where the signal is consumed by the cells, rather than produced by the cells. In the absence of the logistic source (i.e. $f(u) \equiv 0$) for problem (1.1), Tao [20] proved that the classical solution of model (1.1) is uniformly bounded provided that $\|v_0\|_{L^\infty(\Omega)}$ is sufficiently small. In particular, if $\Omega \subset \mathbb{R}^3$ is a bounded convex domains, Tao and Winkler [22] showed that there exists $T > 0$ such that the problem has global weak solution which is bounded and smooth in $\Omega \times (T, +\infty)$. It is the goal of this paper to prove that model

(1.1) has global and bounded solutions provided that $\|v_0\|_{L^\infty(\Omega)} > 0$ is sufficiently small (Theorem 3.3).

2. Preliminaries

We first state one result concerning local-in-time existence of a classical solution to problem (1.1).

Theorem 2.1. *Let the non-negative functions u_0 and v_0 satisfy (u_0, v_0) belong to $(W^{1,\theta}(\Omega))^2$, for some $\theta > n$. Moreover, $f(s)$ with $s \geq 0$ is smooth and $f(0) = 0$. Then problem (1.1) has a unique local-in-time non-negative classical solution*

$$(u, v) \in (C([0, T_{\max}); W^{1,\theta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})))^2, \tag{2.1}$$

where $T_{\max}$ denotes the maximal existence time. If for every $T < +\infty$ both $u(\cdot, t)$ and $v(\cdot, t)$ are a priori bounded for all $0 < t < \min\{T, T_{\max}\}$; i.e.,

$$\|(u(\cdot, t), v(\cdot, t))\|_{L^\infty(\Omega)} \leq C(T) \quad \text{for all } 0 < t < \min\{T, T_{\max}\} \tag{2.2}$$

with some constant $C(T)$ depending on T and $\|(u_0, v_0)\|_{W^{1,\theta}(\Omega)}$ only. Then the solution of (1.1) is globally defined and thus $T_{\max} = +\infty$. Moreover, assume the condition (1.2) is satisfied, then there exists a constant $C_0 > 0$ such that $u(x, t)$ and $v(x, t)$ satisfy

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq C_0, \tag{2.3}$$

$$0 \leq v \leq \|v_0\|_{L^\infty(\Omega)} \tag{2.4}$$

for all $t \in (0, T_{\max})$.

Proof. As in [4, 20, 7], let $V = (u, v) \in \mathbb{R}^2$. Then the initial-boundary value problem (1.1) can be reformulated as

$$\begin{gathered} V_t = \nabla \cdot (F(V)\nabla V) + H(V) \\ \frac{\partial V}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ V(x, 0) = (u_0(x), v_0(x)), \quad x \in \Omega, \end{gathered}$$

where

$$F(V) = \begin{pmatrix} 1 & -\chi u \\ 0 & 1 \end{pmatrix}, \quad H(V) = \begin{pmatrix} f(u) \\ -uv \end{pmatrix}.$$

Then applying [2, Theorems 14.4, 14.6, 15.5], statements (2.1) and (2.2) can be proved. Since the initial data $u_0 \geq 0$, $v_0 \geq 0$ and $f(0) = 0$, the maximum principle ensures that both u and v are non-negative.

By the maximum principle we have (2.4). Now, we prove (2.3). Integrating the first equation in (1.1) and using (1.2), we obtain

$$\frac{d}{dt}\int_\Omega u dx = \int_\Omega f(u) dx \leq \int_\Omega au - bu^p dx. \tag{2.5}$$

By Young's inequality, since $b > 0$ and $p > 1$, we obtain

$$(a+1)\int_\Omega u\,dx = \int_\Omega b^{-1/p}(a+1)b^{1/p}u\,dx \leq \int_\Omega bu^p dx + (a+1)^{\frac{p}{p-1}} b^{\frac{1}{1-p}} |\Omega|. \tag{2.6}$$

Combining (2.5) and (2.6), we conclude

$$\frac{d}{dt}\int_\Omega u dx + \int_\Omega u\,dx \leq (a+1)^{\frac{p}{p-1}} b^{\frac{1}{1-p}} |\Omega|. \tag{2.7}$$

Integrating, we have

$$\int_\Omega u dx \le c_0, \quad c_0 = \max\{\|u_0\|_{L^1(\Omega)}, (a+1)^{\frac{p}{p-1}} b^{\frac{1}{1-p}} |\Omega|\} > 0.$$

□

Let us collect some basic statements about the Gagliardo-Nirenberg inequality which will be used in forthcoming proofs. For details, we refer the reader to [29, 8, 17] (see also [26, 5]).

Lemma 2.2. *Let*

$$\alpha^* = \begin{cases} \frac{2n}{n-2}, & \text{if } n > 2, \\ \infty, & \text{if } n = 1, 2. \end{cases}$$

Then for all $l^* \in (2, \infty)$ satisfying $l^* \le \alpha^*$ and $h \in (0, 2)$, $\alpha \in [h, l^*]$, there exists a constant $c_{GN} > 0$ such that

$$\|\psi\|_{L^\alpha(\Omega)} \le c_{GN}(\|\nabla\psi\|_{L^2(\Omega)}^{\lambda^*} \|\psi\|_{L^h(\Omega)}^{1-\lambda^*} + \|\psi\|_{L^h(\Omega)})$$

holds for any $\psi \in W^{1,2}(\Omega)$, where $\lambda^* = \frac{\frac{n}{h} - \frac{n}{\alpha}}{1 - \frac{n}{2} + \frac{n}{h}}$.

3. Global bounded solutions

The main step towards the existence and boundedness of a global solution is to establish uniform bound of the cells population density $u(x,t)$ in the space $L^{n+1}(\Omega)$. This is accomplished by providing some associated weighted bounds involving weight functions $\phi(v)$ which are uniformly bounded both from above and below by positive constants. This approach was developed by Winkler in [27] (see also [14, 20]).

Lemma 3.1. *Let $f(u)$ satisfy (1.2), $\|v_0\|_{L^\infty(\Omega)} > 0$ and $\chi > 0$. Then there exists a constant $C > 0$ such that the first component of the solution of (1.1) satisfies*

$$\|u(\cdot,t)\|_{L^{n+1}(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max}). \tag{3.1}$$

Proof. Set $k := n+1$ and fix $\|v_0\|_{L^\infty(\Omega)} > 0$ small such that

$$\|v_0\|_{L^\infty(\Omega)} \le \frac{1}{6(n+1)\chi}. \tag{3.2}$$

Define

$$\phi(s) := e^{(\alpha s)^2} \quad \text{for all } 0 \le s \le \|v_0\|_{L^\infty(\Omega)},$$

where

$$\alpha = \sqrt{\frac{n}{24(n+1)}} \frac{1}{\|v_0\|_{L^\infty(\Omega)}}.$$

By direct calculation, from (1.1), we obtain

$$\begin{aligned}
&\frac{1}{k}\frac{d}{dt}\int_\Omega u^k \phi(v) dx \\
&= \int_\Omega u^{k-1}\phi(v) u_t dx + \frac{1}{k}\int_\Omega u^k \phi'(v) v_t \, dx \\
&= \int_\Omega u^{k-1}\phi(v)\Delta u dx - \int_\Omega u^{k-1}\phi(v)\chi\nabla\cdot(u\nabla v) dx + \int_\Omega u^{k-1}\phi(v) f(u) dx \\
&\quad + \frac{1}{k}\int_\Omega u^k \phi'(v)\Delta v \, dx - \frac{1}{k}\int_\Omega u^{k+1} v\phi'(v) dx
\end{aligned}$$

$$
\begin{aligned}
&= -(k-1)\int_\Omega u^{k-2}\phi(v)|\nabla u|^2dx - \int_\Omega u^{k-1}\phi'(v)\nabla u\cdot\nabla v\,dx \\
&\quad + \chi(k-1)\int_\Omega u^{k-1}\phi(v)\nabla u\cdot\nabla v\,dx + \chi\int_\Omega u^k\phi'(v)|\nabla v|^2dx \\
&\quad + \int_\Omega u^{k-1}\phi(v)f(u)dx - \int_\Omega u^{k-1}\phi'(v)\nabla u\cdot\nabla v\,dx \\
&\quad - \frac{1}{k}\int_\Omega u^k\phi''(v)|\nabla v|^2dx - \frac{1}{k}\int_\Omega u^{k+1}v\phi'(v)dx.
\end{aligned} \tag{3.3}
$$

Since $f(s)\le as-bs^p$ and $\phi'(s)\ge 0$ for all $s\ge 0$, we have

$$
\begin{aligned}
&\frac{1}{k}\frac{d}{dt}\int_\Omega u^k\phi(v)dx + (k-1)\int_\Omega u^{k-2}\phi(v)|\nabla u|^2dx + \frac{1}{k}\int_\Omega u^k\phi''(v)|\nabla v|^2dx \\
&\le -2\int_\Omega u^{k-1}\phi'(v)\nabla u\cdot\nabla v\,dx + \chi(k-1)\int_\Omega u^{k-1}\phi(v)\nabla u\cdot\nabla v\,dx \\
&\quad + \chi\int_\Omega u^k\phi'(v)|\nabla v|^2dx + a\int_\Omega u^k\phi(v)dx - b\int_\Omega u^{k+p-1}\phi(v)dx.
\end{aligned} \tag{3.4}
$$

By Young's inequality, we obtain

$$
\begin{aligned}
-2\int_\Omega u^{k-1}\phi'(v)\nabla u\cdot\nabla v\,dx &\le \frac{k-1}{4}\int_\Omega u^{k-2}\phi(v)|\nabla u|^2dx \\
&\quad + \frac{4}{k-1}\int_\Omega u^k\frac{\phi'^2(v)}{\phi(v)}|\nabla v|^2dx
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
\chi(k-1)\int_\Omega u^{k-1}\phi(v)\nabla u\cdot\nabla v\,dx &\le \frac{k-1}{4}\int_\Omega u^{k-2}\phi(v)|\nabla u|^2dx \\
&\quad + \chi^2(k-1)\int_\Omega u^k\phi(v)|\nabla v|^2dx.
\end{aligned} \tag{3.6}
$$

Thus, from (3.4)–(3.6) we obtain

$$
\begin{aligned}
&\frac{1}{k}\frac{d}{dt}\int_\Omega u^k\phi(v)dx + \frac{k-1}{2}\int_\Omega u^{k-2}\phi(v)|\nabla u|^2dx + \frac{1}{k}\int_\Omega u^k\phi''(v)|\nabla v|^2dx \\
&\le \frac{4}{k-1}\int_\Omega u^k\frac{\phi'^2(v)}{\phi(v)}|\nabla v|^2dx + \chi^2(k-1)\int_\Omega u^k\phi(v)|\nabla v|^2dx \\
&\quad + \chi\int_\Omega u^k\phi'(v)|\nabla v|^2dx + a\int_\Omega u^k\phi(v)dx - b\int_\Omega u^{k+p-1}\phi(v)dx.
\end{aligned} \tag{3.7}
$$

Next we show that the three terms on the right-hand side of (3.7) are dominated by $\frac{1}{k}\int_\Omega u^k\phi''(v)|\nabla v|^2dx$. To this end, for $s\ge 0$, we compute

$$
\begin{gathered}
y_1(s) := \frac{\phi''(s)}{k} = \frac{2}{k}\alpha^2e^{(\alpha s)^2} + \frac{4}{k}\alpha^4s^2e^{(\alpha s)^2}, \\
y_2(s) := \frac{4}{k-1}\frac{\phi'^2(s)}{\phi(s)} = \frac{16}{k-1}\alpha^4s^2e^{(\alpha s)^2}, \\
y_3(s) := \chi^2(k-1)\phi(s) = \chi^2(k-1)e^{(\alpha s)^2}, \\
y_4(s) := \chi\phi'(s) = 2\chi\alpha^2se^{(\alpha s)^2}.
\end{gathered}
$$

By a direct calculation, we obtain

$$\frac{y_2(s)}{\frac{1}{3}y_1(s)} \le \frac{\frac{16}{k-1}\alpha^4 s^2 e^{(\alpha s)^2}}{\frac{2}{3k}\alpha^2 e^{(\alpha s)^2}} = \frac{24k}{k-1}(\alpha s)^2 \le \frac{24(n+1)}{n}(\alpha\|v_0\|_{L^\infty(\Omega)})^2 = 1, \tag{3.8}$$

where we have used that $\alpha = \sqrt{\frac{n}{24(n+1)}}\frac{1}{\|v_0\|_{L^\infty(\Omega)}}$. Using (3.2),

$$\frac{y_3(s)}{\frac{1}{3}y_1(s)} \le \frac{\chi^2(k-1)e^{(\alpha s)^2}}{\frac{2}{3k}\alpha^2 e^{(\alpha s)^2}} \le \frac{3k(k-1)\chi^2}{2\alpha^2} = 36(n+1)^2\|v_0\|^2_{L^\infty(\Omega)}\chi^2 \le 1 \tag{3.9}$$

and

$$\frac{y_4(s)}{\frac{1}{3}y_1(s)} \le \frac{2\chi\alpha^2 s e^{(\alpha s)^2}}{\frac{2}{3k}\alpha^2 e^{(\alpha s)^2}} \le 3k\chi s \le 3(n+1)\chi\|v_0\|_{L^\infty(\Omega)} \le \frac{1}{2}. \tag{3.10}$$

Therefore, from (3.7)-(3.10), it follows easily that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^k\phi(v)dx + kb\int_\Omega u^{k+p-1}\phi(v)dx + \frac{2(k-1)}{k}\int_\Omega |\nabla u^{k/2}|^2\phi(v)dx \\ &\le ka\int_\Omega u^k\phi(v)dx. \end{aligned} \tag{3.11}$$

Since $0 \le s \le \|v_0\|_{L^\infty(\Omega)}$, we have $1 \le \phi(s) \le e^{(\alpha\|v_0\|_{L^\infty(\Omega)})^2} := d$, it is not difficult to obtain

$$kb\int_\Omega u^k\phi(v)dx \le kb\int_\Omega u^{k+p-1}\phi(v)dx + kbd|\Omega|. \tag{3.12}$$

Combining (3.11) with (3.12) yields

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^k\phi(v)dx + kb\int_\Omega u^k\phi(v)dx + \frac{2(k-1)}{k}\int_\Omega |\nabla u^{k/2}|^2\phi(v)dx \\ &\le ka\int_\Omega u^k\phi(v)dx + kbd|\Omega|. \end{aligned} \tag{3.13}$$

Using Lemma 2.2 and $(x+y)^\gamma \le 2^\gamma(x^\gamma + y^\gamma)$ for all $x, y \ge 0$ and $\gamma > 0$, we obtain

$$\begin{aligned} ka\int_\Omega u^k\phi(v)dx &\le kad\int_\Omega u^k dx \\ &= kad\|u^{k/2}\|^2_{L^2(\Omega)} \\ &\le kad(c_{GN}\|\nabla u^{k/2}\|^\lambda_{L^2(\Omega)}\|u^{k/2}\|^{1-\lambda}_{L^{2/k}(\Omega)} + c_{GN}\|u^{k/2}\|_{L^{2/k}(\Omega)})^2 \\ &\le 4kad(c^2_{GN}c_0^{k(1-\lambda)}\|\nabla u^{k/2}\|^{2\lambda}_{L^2(\Omega)} + c^2_{GN}c_0^k) \end{aligned} \tag{3.14}$$

holds with some constant $c_{GN} > 0$ and

$$\lambda = \frac{\frac{kn}{2} - \frac{n}{2}}{1 - \frac{n}{2} + \frac{kn}{2}} \in (0,1).$$

By Young's inequality, we derive

$$\begin{aligned} ka\int_\Omega u^k\phi(v)dx &\le 4kadc^2_{GN}c_0^{k(1-\lambda)}\|\nabla u^{k/2}\|^{2\lambda}_{L^2(\Omega)} + 4kadc^2_{GN}c_0^k \\ &\le \frac{2(k-1)}{k}\int_\Omega |\nabla u^{k/2}|^2dx + c_1 \\ &\le \frac{2(k-1)}{k}\int_\Omega |\nabla u^{k/2}|^2\phi(v)dx + c_1, \end{aligned} \tag{3.15}$$

where

$$c_1 = c_0^k (4kadc_{GN}^2 (\frac{2k-2}{k})^{-\lambda})^{\frac{1}{1-\lambda}} + 4kadc_{GN}^2 c_0^k > 0.$$

Hence, substituting (3.15) into (3.13) yields

$$\frac{d}{dt}\int_\Omega u^k \phi(v)dx + kb\int_\Omega u^k\phi(v)dx \le c_1 + kbd|\Omega|. \tag{3.16}$$

Integrating (3.16), we have

$$\int_\Omega u^k dx \le \int_\Omega u^k\phi(v)dx \le \max\big\{d\int_\Omega u_0^k, \frac{c_1 + kbd|\Omega|}{kb}\big\},$$

we arrive at the desired result. □

Remark 3.2. To prove that the three terms on the right-hand side of (3.7) are dominated by $\frac{1}{k}\int_\Omega u^k\phi''(v)|\nabla v|^2dx$, we need $\frac{y_i(s)}{\frac{1}{3}y_1(s)} \le 1$ $(i = 2,3,4)$, so we have $0 < \|v_0\|_{L^\infty(\Omega)} \le \frac{1}{6k\chi}$, in such a way that $\frac{1}{6k\chi} \to 0$ as $k \to \infty$. In fact, in the proof of Theorem 3.3 is only applied to one fixed $k > n$. So to avoid this situation, we choose $k := n+1$ in Lemma 3.1.

We are now in a position to prove our main results, which are as follows.

Theorem 3.3. *Assume that $u_0(x)$ and $v_0(x)$ are non-negative functions and that (u_0, v_0) belongs to $(W^{1,\theta}(\Omega))^2$ for some $\theta > n$, $\chi > 0$, $f(u)$ satisfies (1.2). Then problem (1.1) possesses a unique global classical solution (u, v) for which both u and v are non-negative and uniformly bounded in $\Omega \times (0,\infty)$ provided that*

$$0 < \|v_0\|_{L^\infty(\Omega)} \le \frac{1}{6(n+1)\chi}.$$

Proof. With the aid of Lemma 3.1 and its proof, based on a Moser-Alikakos-type iterative procedure [1, 6] (for detailed calculations we refer to [14, 20, 21]), we can establish a uniform bound on the solution u in time $(0, T_{\max})$. Combining (2.4) and (2.2) we obtain the desired result of Theorem 3.3. □

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