Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 218, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

WEAK ROLEWICZ'S THEOREM IN HILBERT SPACES

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Abstract. Let $\phi : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}_+$ be a nondecreasing function which is positive on $(0, \infty)$ and let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be a positive strongly continuous periodic evolution family of bounded linear operators acting on a complex Hilbert space H. We prove that $\mathcal{U}$ is uniformly exponentially stable if for each unit vector $x \in H$, one has

$$\int_0^\infty \phi(|\langle U(t, 0)x, x\rangle|)dt < \infty.$$

The result seems to be new and it generalizes others of the same topic. Moreover, the proof is surprisingly simple.

1. Introduction

The classical theorem of Datko [9] states that a strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t \geq 0}$, acting on a real or complex Banach space X, is uniformly exponentially stable; i.e., there are two positive constants N and ν such that

$$\|T(t)\| \leq Ne^{-\nu t}, \quad \forall t \geq 0,$$

if and only if

$$\int_0^\infty \|T(t)x\|^2 dt < \infty, \quad \forall x \in X. \tag{1.1}$$

Obviously, for strongly continuous selfadjoint semigroups acting on a Hilbert space H, the integral condition (1.1) is equivalent to

$$\int_0^\infty |\langle T(t)x, x\rangle| dt < \infty, \quad \forall x \in H. \tag{1.2}$$

In this article we prove a result of this type for positive periodic evolution families, in a more general form, that is related to a theorem of Rolewicz.

The history of the Rolewicz theorem is well known among experts in the field. However, we recall, in the following, a few facts to help readers compare results.

The theorem of Datko has been extended by Pazy, who states that all trajectories of a strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t \geq 0}$, of bounded linear operators acting on a Banach space X, belongs to the space $L^p(\mathbb{R}_+, X)$ (for some, and then

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2000 *Mathematics Subject Classification.* 47A30, 46A30.
Key words and phrases. Uniform exponential stability; Rolewicz's type theorems; weak integral stability boundedness.

Submitted October 3, 2012. Published November 29, 2012.

for all $p \geq 1$) if and only if the semigroup $\mathbf{T}$ is uniformly exponentially stable, or equivalently, its growth bound

$$\omega_0(\mathbf{T}) = \lim_{t\to\infty} \frac{\ln \|T(t)\|}{t},$$

is negative. See [19] and [20] for further details.

A real valued nondecreasing function ϕ, defined on $\mathbb{R}_+$, which is positive on $(0, \infty)$, will be called (ad hoc) $\mathcal{R}$-function.

An important generalization of Datko-Pazy's theorem was given by Rolewicz, [24]. He showed that a strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ of bounded linear operators acting on a Banach space X is uniformly exponentially stable, provided that for a given continuous, $\mathcal{R}$-function ϕ, one has

$$\int_0^\infty \phi(\|T(t)x\|)dt < \infty, \quad \forall x \in X.$$

Earlier, special cases of this result were obtained by Zabczyk, [29] and Przyluski, [23]. Zheng and Littman subsequently provided new proofs for theorem of Rolewicz and, moreover, they removed the assumption of continuity of ϕ. See [13] and [30].

Jan van Neerven emphasized a new method of demonstration for Rolewicz's theorem using the theory of Orlicz spaces. See [16].

Recently, Storozhuk [25] has given a very short proof for a Rolewicz's type theorem. See also [4, 5, 6] for different approaches of the Rolewicz theorem.

Next, we introduce the "weak" version (in the sense of the functional analysis) of the Rolewicz theorem.

Let X be a Banach space, X^* be its dual, and let $p \geq 1$ be a given real number. The semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is called *weak-L^p-stable* if

$$\int_0^\infty |\langle T(t)x, x^*\rangle|^p dt < \infty, \quad \text{for all } x \in X \text{ and all } x^* \in X^*.$$

The weak-L^p-stability of a semigroup $\mathbf{T}$ does not imply the uniform exponential stability of $\mathbf{T}$. See [11, 18] for counterexamples.

In 1983, Pritchard and Zabczyk [21] raised the following problem:

> Is the uniform exponential stability of a semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$, acting on a Hilbert space, a consequence of its weak-L^p-stability?

The positive answer to the Pritchard and Zabczyk question was given by Falun Huang. Further details could be found in [12]. The general case has been treated independently by Weiss [27].

Also, it is known [16] that a bounded strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$, of bounded linear operators acting on a complex Hilbert space H, is uniformly exponentially stable, if, for a given function φ as above, one has

$$\int_0^\infty \varphi(|\langle T(t)x, y\rangle|)dt < \infty, \quad \text{for all } x, y \in H. \tag{1.3}$$

It is natural to ask whether the condition of boundedness of $\mathbf{T}$ can be removed? A positive answer to this question was given recently by Storozhuk. In order to describe the Storozhuk theorem we briefly resume some results targeting the same problem, but on Banach spaces.

We refer to [16, Theorems 4.6.3(i), 4.6.4], for results concerning the exponential stability (rather than the uniform exponential stability) of bounded linear semigroups acting on a Banach space. Recently Storozhuk [26] settled a more general problem. He stated that, if $s_0(\mathbf{T}) \geq 0$, then there exist an $\mathcal{R}$-function ϕ and two vectors $x_0, y_0 \in H$, such that

$$\int_0^\infty \phi(|\langle T(t)x_0, y_0\rangle|)dt = \infty.$$

More details on strongly continuous semigroups of operators, including the precise definitions and characterizations of the semigroups growth bounds $\omega_0(\mathbf{T})$ and $s_0(\mathbf{T})$, can be found, for example, in the monographs [20, 15, 1]. In [10] and [22] it is shown that $s_0(\mathbf{T}) = \omega_0(\mathbf{T})$ for all strongly continuous semigroups $\mathbf{T}$ of bounded linear operators acting on complex Hilbert spaces.

2. Notation and preliminary results

We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{C}$ the set of complex numbers. Also we denote by $\mathbb{Z}_+$ the set of nonnegative integer numbers. As usual, $\sigma(L)$ denotes the spectrum of the bounded linear operator acting on a Banach space X. The spectral radius of L, denoted by $r(L)$, is given by

$$r(L) := \sup\{|z| : z \in \sigma(L)\} = \lim_{n\to\infty} \|L^n\|^{1/n}.$$

By $\mathcal{L}(X)$ we denote the Banach algebra of all bounded linear operators acting on X. As usual, $\langle \cdot, \cdot\rangle$ denotes the scalar product on a Hilbert space H. The norms in $X, H, \mathcal{L}(X), \mathcal{L}(H)$ will be denoted by the same symbol, namely by $\|\cdot\|$.

We recall that a family $\mathcal{U} := \{U(t,s)\}_{t\geq s\geq 0} \subset \mathcal{L}(H)$ is called strongly continuous q-periodic evolution family (for some $q \geq 1$) if it satisfy the following conditions:

- (i) $U(t,t) = I$ for all $t \geq 0$.
- (ii) $U(t,r)U(r,s) = U(t,s)$ for all $t \geq r \geq s \geq 0$.
- (iii) $U(t+q, s+q) = U(t,s)$ for all $t \geq s \geq 0$.
- (iv) The map $(t,s) \to U(t,s)x : \{(t,s) : t \geq s\} \to H$ is continuous for all $t \geq s \geq 0$ and every $x \in H$.

It is well known that any such evolution family $\mathcal{U}$ is exponentially bounded, that is, there exist $\omega \in \mathbb{R}$ and $M_\omega \geq 0$ such that

$$\|U(t,s)\| \leq M_\omega e^{\omega(t-s)} \quad \text{for } t \geq s \geq 0. \tag{2.1}$$

See [8]. Whenever the evolution family $\mathcal{U}$ is exponentially bounded its growth bound is defined by

$$\omega_0(\mathcal{U}) := \inf\{\omega \in \mathbb{R} : \text{ there is } M_\omega \geq 0 \text{ such that (2.1) holds}\}.$$

A bounded linear operator L, acting on a Hilbert space H, is positive if $\langle Lx, x\rangle \geq 0$ for every $x \in H$.

An evolution family $\{U(t,s) : t \geq s \geq 0\}$ is called selfadjoint, (positive), if each operator $U(t,s)$, with $t \geq s \geq 0$, is selfadjoint and (respectively positive).

The family $\mathcal{U}$ is uniformly exponentially stable if its growth bound is negative.

In the following we use a deep result from the theory of operators, originally given by Müller and Jan van Neerven. See [14] and [17], and also [2] for related results in the framework of Hilbert spaces. For the reader convenience, we state this result here.

Lemma 2.1. *Let X be a complex Banach space and let $V \in \mathcal{L}(X)$. If the spectral radius of V is greater or equal to 1, then for all $0 < \varepsilon < 1$ and any sequence (a_n)*

with $a_n \to 0$ (as $n \to \infty$) and $\|(a_n)\|_\infty \leq 1$, there exists a unit vector $u_0 \in X$, such that

$$\|V^n u_0\| \geq (1-\varepsilon)\cdot |a_n|, \quad \textit{for all } n \in \mathbb{Z}_+.$$

Throughout this article, (t_n) will be a sequence of nonnegative real numbers, such that $1 \leq q \leq t_{n+1} - t_n \leq \alpha$ for every $n \in \mathbb{Z}_+$ and some positive real number α.

Lemma 2.2. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be a strongly continuous q-periodic ($q \geq 1$) evolution family of bounded linear operators acting on a Banach space X and let (t_n) be a sequence as given before. If the evolution family is not uniformly exponentially stable, then there exists a positive constant C, having the properties: for every $\mathbb{C}$-valued sequence (b_n) with $b_n \to 0$ (as $n \to \infty$) and $\|(b_n)\|_\infty \leq 1$, there exists a unit vector $u_0 \in X$, such that*

$$\|U(t_n,0)u_0\| \geq C\cdot |b_{n+1}|, \quad \textit{for all } n \in \mathbb{Z}_+. \tag{2.2}$$

Proof. Let $V := U(q,0)$. Since $\mathcal{U}$ is not uniformly exponentially stable, $r(U(q,0)) \geq 1$. From Lemma 2.1 follows that for every $\varepsilon \in (0,1)$ and every sequence (a_n) with $a_n \to 0$ (as $n \to \infty$) and $\|(a_n)\|_\infty = 1$ there exists a unit vector $y_0 \in X$ such that $\|U(nq,0)y_0\| \geq (1-\varepsilon)\cdot|a_n|$, for any natural number n.

Let $k : \mathbb{Z}_+ \to \mathbb{Z}_+$ be the function defined by, $k(n) = [\frac{t_n}{q}]$. Here $[\frac{t_n}{q}]$ denotes the integer part of the real number $\frac{t_n}{q}$. In view of the properties of the sequence (t_n), the function $k(\cdot)$ is increasing. Let (b_n) be a sequence having the assumed properties, and let us consider a sequence (a_n) defined by

$$a_n = \begin{cases} b_{k^{-1}(n)}, & \text{for } n \in k(\mathbb{Z}_+) \\ 0, & \text{otherwise} \end{cases}$$

Then, we have

$$\|U(qk_n,0)y_0\| \geq (1-\varepsilon)\cdot |a_{k(n)}| \geq (1-\varepsilon)\cdot|b_n|, \quad n \in \mathbb{Z}_+. \tag{2.3}$$

Choose $u_0 := y_0$. Using (2.1) and (2.3), we obtain

$$\begin{aligned}(1-\varepsilon)\cdot|b_{n+1}| &\leq \|U(qk_{n+1},0)u_0\| \\ &= \|U(qk_{n+1},t_n)U(t_n,0)u_0\| \\ &\leq Me^{\omega\cdot\alpha}\cdot\|U(t_n,0)u_0\|.\end{aligned}$$

Hence, (2.2) holds with $C := \frac{(1-\varepsilon)}{Me^{\omega\cdot\alpha}}$. □

A periodic evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ satisfies the *strong discrete Rolewicz condition* related to the $\mathcal{R}$-function φ and the sequence (t_n), if

$$\sum_{n=0}^{\infty} \phi(\|U(t_n,0)x\|) < \infty, \quad \forall x \in X,\ \|x\| = 1. \tag{2.4}$$

Whenever the evolution family $\mathcal{U}$ satisfies the condition (2.4) we can highlight new qualities of the function φ. Such qualities are listed in the following.

- The inequality (2.4) holds for all $x \in X$ with $\|x\| \leq 1$.
- $\varphi(0) = 0$. To justify, putting $x = 0$ in (2.4).
- We may suppose that $\phi(1) = 1$. To justify, putting a suitable multiple ($\alpha\varphi$, with $\alpha > 0$), instead of φ.

- Also, we can put a suitable multiple of the function $\overline{\varphi}$, instead of φ. Here, by $\overline{\varphi}$ we understand the function defined by:
$$\overline{\varphi}(t) = \begin{cases} \int_0^t \varphi(s)ds, & \text{for } t \in [0,1) \\ \frac{at}{at+1-a}, & \text{for } t \geq 1, \end{cases}$$
where $a = \int_0^1 \varphi(s)ds$.

We remark that $\overline{\varphi}$ is continuous and increasing function on $\mathbb{R}_+$, and, moreover, it is convex on the interval $[0,1]$. In addition, the evolution family $\mathcal{U}$ satisfies the condition (2.4) in respect to $\overline{\varphi}$, because $\overline{\varphi}(t) \leq \varphi(t)$, for all $t \in \mathbb{R}_+$.

Given the above, the following lemma proof becomes clear. We insert it for the sake of completeness.

Lemma 2.3. *Let φ be an $\mathcal{R}$-function and let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be a strongly continuous q-periodic ($q \geq 1$) evolution family acting on a Banach space X. If the family $\mathcal{U}$ satisfies* (2.4) *then it is uniformly exponentially stable.*

Proof. Assume that $r(U(q,0)) \geq 1$. We replace φ by $\overline{\varphi}$. Let C be as in Lemma 2.2 and let $|b_n| = \frac{1}{C} \cdot \overline{\varphi}^{-1}(\frac{1}{n}), n \geq 1$. Obviously the sequence (b_n) satisfies the requirements in Lemma 2.2. As a consequence, there exists a unit vector $u_0 \in X$ such that $\|U(t_n,0)u_0\| \geq C \cdot |b_{n+1}|$ for any natural number n. Hence,
$$\sum_{n=0}^{\infty} \overline{\varphi}(\|U(t_n,0)u_0\|) \geq \sum_{n=0}^{\infty} \overline{\varphi}(C \cdot |b_{n+1}|) = \sum_{n=0}^{\infty} \frac{1}{n+1} = \infty.$$
□

3. Weak integral conditions and exponential stability

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup acting on a complex Hilbert space H. When $\mathbf{T}$ is selfadjoint (i.e. $T(t) = T(t)^*$, for every $t \geq 0$), then $\langle T(t)x, x\rangle \geq 0$, for each $t \geq 0$ and every $x \in H$. Indeed, for any $t \geq 0$, we have
$$\langle T(t)x, x\rangle = \langle T(t/2)T(t/2)x, x\rangle = \|T(t/2)x\|^2 \geq 0.$$

In the proof of the next theorem, we use the following inequality of the Cauchy-Buniakovski-Schwartz type. *Let A and B be two selfadjoint operators acting on the complex Hilbert space H. Then,*
$$|\langle ABx, y\rangle|^2 \leq \langle A^2y, y\rangle\langle B^2x, x\rangle, \quad \text{for all } x, y \in H. \tag{3.1}$$

In fact,
$$|\langle ABx, y\rangle|^2 = |\langle Bx, Ay\rangle|^2 \leq \|Bx\|^2\|Ay\|^2 = \langle A^2y, y\rangle\langle B^2x, x\rangle.$$

Theorem 3.1. *Let ϕ be an $\mathcal{R}$-function and let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a selfadjoint, strongly continuous semigroup of bounded linear operators acting on a complex Hilbert space H. If*
$$I(x) := \int_0^{\infty} \phi(\langle T(t)x, x\rangle)dt < \infty, \quad \textit{for all } x \in H \textit{ with } \|x\| = 1, \tag{3.2}$$
then the semigroup $\mathbf{T}$ is uniformly exponentially stable.

Proof. Having in mind that ϕ may be considered a continuous function and by applying the Mean-Value Theorem to the function $t \mapsto \phi(\langle T(2t)x, x\rangle)$ on the interval $[n, n+1]$, we find a real number $t_n(x) \in [n, n+1]$, such that

$$\begin{aligned}\frac{1}{2}I(x) &= \int_0^\infty \phi(\langle T(2t)x, x\rangle)dt \\ &= \sum_{n=0}^\infty \phi(\langle T(2t_n(x)x, x\rangle) \\ &\geq \sum_{n=0}^\infty \phi(\langle T(2t_{4n}(x)x, x\rangle).\end{aligned}$$

Set $s_n \in [2n+1, 2n+2]$ and let y be a unit vector in H. In view of (3.1), successively one has

$$\begin{aligned}|\langle T(2s_n)x, y\rangle|^2 &= |\langle T(2s_n - t_{4n}(x))T(t_{4n}(x))x, y\rangle|^2 \\ &\leq \langle T(4s_n - 2t_{4n}(x))y, y\rangle\langle T(2t_{4n}(x))x, x\rangle \\ &\leq Me^{8\omega}\langle T(2t_{4n}(x))x, x\rangle.\end{aligned}$$

Hence, for any unit vector $x \in H$, one has

$$\sum_{n=0}^\infty \phi\Big(\frac{1}{Me^{8\omega}}\|T(s_n)x\|^4\Big) \leq \sum_{n=0}^\infty \phi(\langle T(2t_{4n}(x))x, x\rangle). \tag{3.3}$$

Clearly $1 \leq s_{n+1} - s_n \leq 3$. By assumption and relation (3.3), we obtain

$$\sum_{n=0}^\infty \phi\Big(\frac{1}{Me^{8\omega}}\|T(s_n)x\|^4\Big) < \infty.$$

Now replacing $\phi(\frac{1}{Me^{8\omega}}t^4)$ by $\varphi(t)$. Obviously, the map φ is an $\mathcal{R}$-function. The assertion follows by Lemma 2.3. □

In the following we recall a concrete example of semigroup verifying (3.2). That the Dirichlet semigroup is exponentially stable is more quite standard. There are several possible ways of showing this. The following is yet another demonstration in addition to this well known fact.

The state space is $H := L^2([0, \pi], \mathbb{C})$, endowed with the usual inner product and norm, becomes a complex Hilbert space. In addition, the one parameter family $\{T(t)\}_{t\geq 0}$, given by

$$(T(t)x)(\xi) = \frac{2}{\pi}\sum_{n=1}^\infty e^{-tn^2}\sin n\xi\Big(\int_0^\pi x(s)\sin nsds\Big), \quad \xi \in [0, \pi],\ t \geq 0,$$

is a strongly continuous semigroup on H. Moreover, this semigroup solves the following Cauchy Problem with boundary conditions

$$\begin{gathered}\frac{\partial u(t,\xi)}{\partial t} = \frac{\partial^2 u(t,\xi)}{\partial^2 \xi}, \quad t > 0,\ \xi \in [0, \pi] \\ u(t, 0) = u(t, \pi) = 0, \quad t \geq 0 \\ u(0, \xi) = x(\xi),\end{gathered}$$

where $x(\cdot)$ is a given function in H.

More exactly, the unique solution of this Cauchy Problem is given by

$$u(t, \xi, x(\cdot)) = (T(t)x)(\xi). \tag{3.4}$$

The semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is generated by the linear operator A given by $Ax = \ddot{x}$. The maximal domain of A is the set $D(A)$ of all $x \in H$ such that x and $\dot{x}$ are absolutely continuous, $\ddot{x} \in H$ and $x(0) = x(\pi) = 0$. Moreover, $T(t)$ is selfadjoint for all $t \geq 0$. [28, Example 1.3, pp. 178, 198]. Based on Theorem 3.1, we prove that the solution given in (3.4) is exponentially stable; i.e., there exist the positive constants N and ν such that for each $x(\cdot) \in H$, one has

$$\|u(t, \cdot, x(\cdot))\|_2 \leq Ne^{-\nu t}\|x(\cdot)\|_2 \quad \forall t \geq 0.$$

For this to be completed, we remark that for each $x(\cdot) \in H$, one has

$$\begin{aligned}\int_0^\infty \langle T(t)x(\cdot), x(\cdot)\rangle dt &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\|\int_0^\pi x(s)\sin(ns)ds\|^2 \\ &\leq \sum_{n=1}^{\infty}\|\int_0^\pi x(s)\sin(ns)ds\|^2 < \infty\end{aligned}$$

The latter estimate is obtained in the view of Bessel inequality.

Next, we comment on the relationship between the integral conditions (1.3) and (3.2). Clearly, (1.3) implies (3.2), but it is not clear whether (3.2) implies (1.3). However, this happens if we admit that (3.2) holds for all $x \in H$ and the map ϕ is subadditive. This latter fact is a consequence of the well-known formula of polarization.

Our approach in the semigroup case allow us to extend this result to the nonautonomous case of positive and periodic evolution families acting on complex Hilbert spaces. The next result could be known. See for example [3, 7] for its different counterparts. Here we present a new proof.

Theorem 3.2. *Let ϕ be an $\mathcal{R}$-function and let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be a selfadjoint strongly continuous and q-periodic $(q \geq 1)$ evolution family acting on a complex Hilbert space H. If*

$$J(x) := \int_0^\infty \phi(\|U(t,0)x\|)dt < \infty, \quad \textit{for all } x \in H \textit{ with } \|x\| = 1,$$

then $\mathcal{U}$ is uniformly exponentially stable.

Proof. Adopting the technique used in Theorem 3.1, we find some $t_n^x \in [nq, (n+1)q]$ such that

$$\begin{aligned}J(x) &= q\sum_{n=0}^{\infty}\phi(\|U(t_n^x, 0)x\|) \\ &\geq \sum_{n=0}^{\infty}\phi(\|U(t_{2n}^x, 0)x\|).\end{aligned}$$

Set $s_n = (2n+2)q$ and let x and y be two unit vectors. In view of (3.1), successively one has

$$\begin{aligned}|\langle U(s_n, 0)x, y\rangle|^2 &= |\langle U(s_n, t_{2n}^x)U(t_{2n}^x, 0)x, y\rangle|^2 \\ &\leq \langle U^2(s_n, t_{2n}^x)y, y\rangle\langle U^2(t_{2n}^x, 0)x, x\rangle \\ &\leq M^2e^{4q\omega}\|U(t_{2n}^x, 0)x\|^2.\end{aligned}$$

Hence, for any unit vector $x \in H$, one has

$$\begin{aligned}\sum_{n=0}^{\infty} \phi(\|U(t_{2n}^x, 0)x\|) &\geq \sum_{n=0}^{\infty} \phi(\frac{1}{Me^{2q\omega}} |\langle U(s_n, 0)x, x\rangle|) \\ &= \sum_{n=0}^{\infty} \phi(\frac{1}{Me^{2q\omega}} \|U((n+1)q, 0)x\|^2).\end{aligned} \tag{3.5}$$

By the assumption and relation (3.5), we obtain

$$\sum_{n=0}^{\infty} \phi\Big(\frac{1}{Me^{2q\omega}} \|U((n+1)q, 0)x\|^2\Big) < \infty,$$

for all unit vectors $x \in H$. Now replace $\phi(\frac{1}{Me^{2q\omega}} t^2)$ by $\varphi(t)$. The assertion follows by Lemma 2.3. □

For positive evolution families we can give even a weak version of the Rolewicz's theorem.

Theorem 3.3. *Let ϕ be an $\mathcal{R}$-function and let $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ be a positive strongly continuous and q-periodic ($q \geq 1$) evolution family acting on a complex Hilbert space H. If*

$$J(x) := \int_0^{\infty} \phi(\langle U(t,0)x, x\rangle) dt < \infty, \quad \textit{for all } x \in H \textit{ with } \|x\| = 1,$$

then $\mathcal{U}$ is uniformly exponentially stable.

Proof. Again using the technique used in Theorem 3.1, we find some $t_n^x \in [nq, (n+1)q]$ such that

$$J(x) = q \sum_{n=0}^{\infty} \phi(\langle U(t_n^x, 0)x, x\rangle) \geq \sum_{n=0}^{\infty} \phi(\langle U(t_{2n}^x, 0)x, x\rangle)).$$

Set $s_n = (2n+2)q$. In view of (3.1), successively one has

$$\begin{aligned}|\langle U^{1/2}(s_n, 0)x, y\rangle|^2 &= |\langle U^{1/2}(s_n, t_{2n}^x) U^{1/2}(t_{2n}^x, 0)x, y\rangle|^2 \\ &\leq \langle U(s_n, t_{2n}^x)y, y\rangle \langle U(t_{2n}^x, 0)x, x\rangle \\ &\leq Me^{2q\omega} \langle U(t_{2n}^x, 0)x, x\rangle.\end{aligned}$$

Hence, for any unit vector $x \in H$, one has

$$\begin{aligned}\sum_{n=0}^{\infty} \phi(\langle U(t_{2n}^x, 0)x, x\rangle) &\geq \sum_{n=0}^{\infty} \phi(\frac{1}{Me^{2q\omega}} |\langle U^{1/2}(s_n, 0)x, x\rangle|^2) \\ &= \sum_{n=0}^{\infty} \phi\Big(\frac{1}{Me^{2q\omega}} |\langle U((n+1)q, 0)x, x\rangle|^2\Big).\end{aligned} \tag{3.6}$$

By the assumption and relation (3.6), we obtain

$$\sum_{n=0}^{\infty} \phi\Big(\frac{1}{Me^{2q\omega}} |\langle U((n+1)q, 0)x, x\rangle|^2\Big) < \infty,$$

for all unit vectors $x \in H$. In particular for $n+1 = 2m$, we have

$$\sum_{m=0}^{\infty} \phi\Big(\frac{1}{Me^{2q\omega}} \|U(mq, 0)x\|^4\Big) < \infty,$$

Now replace $\phi(\frac{1}{M^2 e^{4q\omega}} t^4)$ by $\varphi(t)$ and apply Lemma 2.3. □

We leave open the question whether Theorem 3.3 remains valid for periodic (selfadjoint) evolution families.

Acknowledgements. The authors want to thank the anonymous referees for their careful reading and helpful comments.

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