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THREE SOLUTIONS FOR SINGULAR p-LAPLACIAN TYPE EQUATIONS

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ABSTRACT. In this paper, we consider the singular p-Laplacian type equation

$$\begin{gathered} -\operatorname{div}(|x|^{-\beta}a(x,\nabla u)) = \lambda f(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

where $0 \leq \beta < N - p$, Ω is a smooth bounded domain in $\mathbb{R}^N$ containing the origin, f satisfies some growth and singularity conditions. Under some mild assumptions on a, applying the three critical points theorem developed by Bonanno, we establish the existence of at least three distinct weak solutions to the above problem if f admits some hypotheses on the behavior at $u = 0$ or perturbation property.

1. INTRODUCTION

The three critical points theorem established by Ricceri [6] and extended by Bonanno [2] has been used by several author in the study of nonlinear boundary-value problems; see for example [1, 2, 4, 5, 7, 9]. In particular, Kristály, Lisei and Vargaetc [5] employed Bonanno's theorem to study the p-Laplacian type equation

$$\begin{gathered} -\operatorname{div}(a(x,\nabla u)) = \lambda f(u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies some structural conditions. The simplest case of this problem occurs when $a(x,\xi) = |\xi|^{p-2}\xi$, $p > 1$. In this case (1.1) reduces to an equation involving the p-Laplacian operator. Under the assumptions that the nonlinear term $f(u) : \mathbb{R} \to \mathbb{R}$ is continuous, $(p-1)$-sublinear at infinity and $(p-1)$-superlinear at the origin, Kristály applied Bonanno's variational principle to (1.1) and obtain the existence of three weak solutions.

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In the present paper, we investigate the existence and multiplicity of solutions to the singular p-Laplacian type equation

$$\begin{gathered} -\operatorname{div}(|x|^{-\beta}a(x,\nabla u)) = \lambda f(x,u), \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $0 \le \beta < N-p$, $1 < p < N$ and Ω is a smooth bounded domain in $\mathbb{R}^N$ containing the origin.

In this paper, we use the following notation:

$$\beta_1^* := \frac{N\beta}{N-p}, \quad \beta_2^* := p+\beta, \quad \beta_3^* := N - \frac{N-p-\beta}{p}, \quad p^*(\beta,\alpha) := \frac{(N-\alpha)p}{N-\beta-p}. \tag{1.3}$$

Suppose that the potential $a : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies the assumptions:

Let $A = A(x,\xi) : \Omega\times\mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function, i.e., measurable in x and continuous in ξ, a.e. $x\in\Omega$; $A(x,\xi)$ is of continuous derivative with respect to ξ with $a = \nabla_\xi A$ and satisfies the follows conditions:

- (A1) $A(x,0)=0$ a.e. $x\in\Omega$;
- (A2) there are $p>1$ and a positive constant a_1 such that
$$|a(x,\xi)| \le a_1(1+|\xi|^{p-1}) \quad \text{for a.e. } x\in\Omega \text{ and all } \xi\in\mathbb{R}^N;$$
- (A3) $A(x,\xi)$ is strictly convex in ξ, that is, for $\xi,\eta\in\mathbb{R}^N$ with $\xi\ne\eta$
$$2A\left(x,\frac{\xi+\eta}{2}\right) < A(x,\xi)+A(x,\eta) \quad \text{for a.e. } x\in\Omega;$$
- (A4) $A(x,\xi)$ satisfies the ellipticity condition: There exists a positive constant a_2 such that
$$A(x,\xi)\ge a_2|\xi|^p, \quad \text{for a.e. } x\in\Omega \quad \text{and all } \xi\in\mathbb{R}^N.$$

We suppose the singular nonlinear term $f(x,u)$ fulfils the following hypothesis: Let $f=f(x,u) : \mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function and

- (B1) $f(x,u)$ is subcritical and $(p-1)$-sublinear at infinity, i.e.,
$$\lim_{u\to\infty}\sup_{x\in\Omega}|f(x,u)||u|^{1-p}|x|^{\beta_2^*}=0.$$
- (B2) There exist some α with $\beta_1^*\le\alpha<\beta_2^*$ and a positive continuous function $\mathcal{F}(u)$ with $\mathcal{F}(u)(1+|u|^p)^{-1}\in L^\infty(\mathbb{R})$ such that
$$|F(x,u)|\le\mathcal{F}(u)|x|^{-\alpha} \text{ for a.e. } (x,u)\in\Omega\times\mathbb{R}.$$

In the sequel we consider the weighted space $X=\mathcal{D}^{1,p}(\Omega,|x|^{-\beta}dx)$, which is the completion of $C_0^\infty(\Omega)$ under the norm $(\int_\Omega|\nabla u|^p|x|^{-\beta}dx)^{1/p}$. On X, we define the two functionals

$$\Phi(u)=\int_\Omega A(x,\nabla u)|x|^{-\beta}dx, \quad \Psi(u)=\int_\Omega F(x,u)dx, \tag{1.4}$$

where $F(x,u)=\int_0^u f(x,t)dt$.

It is not difficult to see that solutions of the problem (1.2) are the critical points of the variational functional $I(u)=\Phi(u)-\lambda\Psi(u)$. Moreover, $I(u)$ is continuous differentiable on the space X, and Fréchet derivation of $I(u)$ can be represented as

$$\langle I'(u),v\rangle=\int_\Omega|x|^{-\beta}a(x,\nabla u)\cdot\nabla v dx-\lambda\int_\Omega f(x,u)v, \quad \forall\, v\in X. \tag{1.5}$$

According to the structural conditions of $a(x,\xi)$ and $f(x,u)$, it is clear that the problem (1.2) is more general than (1.1) since there exists singularity not only in nonlinear term $f(x,u)$, but also in diverge term $\operatorname{div}(|x|^{-\beta}a(x,\nabla u))$, which issues some difficulty. We need some generalized Hardy-Sobolev imbedding result (see Lemma 2.1 below) in proving the P.-S. condition. Since we drop the assumption (Ha) in [5] and replace the usual p-uniform convexity of $A(x,\xi)$ by strict convexity, to show that $I(u)$ is weakly lower semicontinuous on X (Lemma 2.6), we have to give some subtle estimates about the variational functional $I(u)$.

In this paper, when $f(x,u)$ is $(p-1)$-superlinear at the origin, the first main result we establish is:

Theorem 1.1. *Assume* (A1)–(A4), (B1)–(B2) *are satisfied. Let $E = B(x_0, r)$ be a ball contained in Ω, such that for some $K \neq 0$,*

$$\inf_{x\in E} F(x,K) > 0. \tag{1.6}$$

If $F(x,u)$ admits the asymptotic property at the origin:

$$\mathcal{F}(u)|u|^{-p} \to 0 \quad \text{as } u \to 0, \tag{1.7}$$

then, there exists an open interval $\Lambda \subset [0,+\infty)$ and a number $R > 0$ such that for every $\lambda \in \Lambda$, equation (1.2) *has at least three distinct solutions in X, whose X-norms are less than R.*

Note that when $\beta = \alpha = 0$ and $f(x,u) = f(u)$, Theorem 1.1 implies the conclusion in [5, Theorem 2.1].

The conclusion in Theorem 1.1 still holds if the asymptotic property of $f(x,u)$ at the origin is replaced by some other properties. To state the next result, we introduce the following notation:

$$c_2(s) = \inf_{x\in B(x_0,r/2)} \frac{F(x,s)}{1+|s|^p}, \quad c_3(s) = \sup_{|u|\geq s} \mathcal{F}(u)|u|^{-p}, \quad c_4(s) = \sup_{|u|\leq s} \mathcal{F}(u), \tag{1.8}$$

where $B(x_0,r) \subset \Omega$ and $s \geq 0$.

Theorem 1.2. *Assume* (A1)–(A4), (B1)–(B2) *are satisfied. Let $E = B(x_0, r)$ be a ball contained in Ω, such that*

$$F(x,u) \geq 0, \quad \text{for a.e. } x \in E \text{ and all } u \in I, \tag{1.9}$$

where I is either $\mathbb{R}^+$ or $\mathbb{R}^-$. If there exist $L > 0$ and $K \in I$ such that

$$c_2(K)|K|^p \geq Cc_4(L), \quad c_2(K) > C(c_3(L))^{\frac{p}{q}}(c_4(L))^{\frac{q-p}{q}} L^{\frac{p(p-q)}{q}}, \tag{1.10}$$

where $q = p^*(\beta,\alpha)$ and C is a certain positive constant only dependent on p, β, α, N, E, a_1 and a_2. Then the conclusion in Theorem 1.1 remains valid.

Remark 1.3. The above result is new even in the case of $\beta = \alpha = 0$. Moreover, by the method similar to [9], we can show a more general result.

Remark 1.4. If we fix some L and keep $c_2(K)/c_3(L)$ less than a fixed constant, then assumption (1.10) holds when $K > L$ and $c_2(K)$ is large enough.

2. Preliminaries

Firstly, we recall the generalized Hardy-Sobolev imbedding theorem, which can be deduced from Caffarelli-Kohn-Nirenberg inequality (see [3, 8]).

Lemma 2.1. *Suppose that $\beta_1^* \le \widetilde{\alpha} \le \beta_2^*$ and $\beta_1^* \le \widehat{\alpha} < \beta_3^*$. Let U be an arbitrary smooth bounded domain in $\mathbb{R}^N$ containing the origin. We have*

(i) *There exists a constant $S_{\widetilde{\alpha}} > 0$, such that for any $u \in \mathcal{D}^{1,p}(\mathbb{R}^N, |x|^{-\beta}dx)$, there holds*
$$S_{\widetilde{\alpha}} \|u\|^p_{L^{p^*(\beta,\widetilde{\alpha})}(\mathbb{R}^N, |x|^{-\widetilde{\alpha}}dx)} \le \|u\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^N, |x|^{-\beta}dx)} ,$$
where $L^p(U, |x|^{-\alpha}dx)$ is L^p space with $|x|^{-\alpha}$ as weight.

(ii) *For $1 \le \widetilde{q} \le p^*(\beta, \widetilde{\alpha})$, there exists a constant $S_{\widetilde{q},\widetilde{\alpha}} > 0$ such that for any $u \in \mathcal{D}^{1,p}(U, |x|^{-\beta}dx)$, there holds*
$$S_{\widetilde{q},\widetilde{\alpha}} \|u\|^p_{L^{\widetilde{q}}(U,|x|^{-\widetilde{\alpha}}dx)} \le \|u\|^p_{\mathcal{D}^{1,p}(U,|x|^{-\beta}dx)} ,$$
Moreover, $S_{\widetilde{\alpha}} = S_{\widetilde{q},\widetilde{\alpha}}$ is independent of the domain U provided $\widetilde{q} = p^*(\beta, \widetilde{\alpha})$.

(iii) *$\mathcal{D}^{1,p}(U, |x|^{-\beta}dx)$ compactly imbeds into $L^{\widehat{q}}(U, |x|^{-\widehat{\alpha}})$ provided $1 \le \widehat{q} < p^*(\beta, \widehat{\alpha})$.*

Remark 2.2. (i) The first assertion in the lemma is a special case of Caffarelli-Kohn-Nirenberg inequality. Particularly, let $\beta = 0, \widetilde{\alpha} = \beta_2^* = p$, one get Hardy inequality; furthermore, let $\beta = \widetilde{\alpha} = \widehat{\alpha} = 0$, the lemma leads to Sobolev theorem.

(ii) There are various forms of description about the imbedding, such as [8] and references therein. We use the form because it looks like a generalization of Hardy-Sobolev imbedding theorem.

For the reader's convenience, we give the proof of the above lemma, which is similar to [8].

Proof of lemma 2.1. Assertion (i) can be directly deduced from main results in [3, Theorem]. In fact, choose the parameters n, p, $\gamma = \beta$, $r = q$, α, a and σ in [3] as N, p, $-\widetilde{\alpha}/p^*(\beta, \widetilde{\alpha})$, $p^*(\beta, \widetilde{\alpha})$, $-\beta/p$, 1 and $-\widetilde{\alpha}/p^*(\beta, \widetilde{\alpha})$, respectively. Then it is not difficult to verify the assumptions in [3] and thus (i) follows.

(ii) Recalling that $\beta_1^* \le \widetilde{\alpha} \le \beta_2^*$ and $1 \le \widetilde{q} \le p^*(\beta, \widetilde{\alpha})$, we have
$$\int_U |u|^{\widetilde{q}} |x|^{-\widetilde{\alpha}} dx \le \Big(\int_U |u|^{p^*(\beta,\widetilde{\alpha})} |x|^{-\widetilde{\alpha}} dx \Big)^{\widetilde{q}/p^*(\beta,\widetilde{\alpha})} \Big(\int_U |x|^{-\widetilde{\alpha}} dx \Big)^{(p^*(\beta,\widetilde{\alpha}) - \widetilde{q})/p^*(\beta,\widetilde{\alpha})} .$$
Since U is bounded and $\widetilde{\alpha} \le \beta_2^* < \beta_3^* < N$, the above inequality and conclusion (i) imply the required result. Employing the scaling method, one can discover that the constant $S_{\widetilde{\alpha}} = S_{\tilde{q},\widetilde{\alpha}}$ is independent of the domain U if $\tilde{q} = p^*(\beta, \tilde{\alpha})$.

(iii) First we prove that $\mathcal{D}^{1,p}(U, |x|^{-\beta}dx)$ imbeds into $L^{\widehat{q}}(U, |x|^{-\widehat{\alpha}}dx)$. According to assertion (ii), it is sufficient to demonstrate the imbedding when $\beta_2^* < \widehat{\alpha} < \beta_3^*$. Indeed, noting that $1 = p^*(\beta, \beta_3^*) < \widehat{q} < p^*(\beta, \beta_2^*) = p$, we calculate
$$\int_U |u|^{\widehat{q}} |x|^{-\widehat{\alpha}} dx \le \Big(\int_U |u|^p |x|^{-\beta_2^*} dx \Big)^{\widehat{q}/p} \Big(\int_U |x|^{-\tau} dx \Big)^{(p-\widehat{q})/p},$$
where $\tau = (\widehat{\alpha} - \beta_2^* \widehat{q}/p)p/(p - \widehat{q})$. Since $\widehat{q} < p^*(\beta, \widehat{\alpha})$, we obtain
$$\widehat{\alpha} < N - \frac{N-\beta-p}{p}\widehat{q}, \quad \tau < \Big(N - \frac{N-\beta-p}{p}\widehat{q} - \frac{\beta+p}{p}\widehat{q}\Big)\frac{p}{p-\widehat{q}} = N,$$

which means that $\mathcal{D}^{1,p}(U,|x|^{-\beta}dx)$ imbeds into $L^{\widehat{q}}(U,|x|^{-\widehat{\alpha}}dx)$.

It remains to prove the imbedding is compact. Assume that the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in $\mathcal{D}^{1,p}(U,|x|^{-\beta}dx)$, it is sufficient to show that there exists a subsequence, still denoted by itself, such that u_n strongly converges to u in $L^{\widehat{q}}(U,|x|^{-\widehat{\alpha}})$ as $n\to\infty$.

In fact, since U is bounded, we observe

$$\begin{aligned}
\|u\|^p_{\mathcal{D}^{1,p}(U)} &= \int_U |\nabla u|^p\,dx\\
&\le (\operatorname{diam} U)^\beta \int_U |\nabla u|^p|x|^{-\beta}dx\\
&\le (\operatorname{diam} U)^\beta \|u\|^p_{\mathcal{D}^{1,p}(U,|x|^{-\beta}dx)}.
\end{aligned}$$

So, $\{u_n\}_{n=1}^{\infty}$ is also bounded in $\mathcal{D}^{1,p}(U)$ and there exists a subsequence, still denoted by itself, weakly converging to some u in $\mathcal{D}^{1,p}(U)$. Remembering that $1<\widehat{q}<p^*(\beta,\widehat{\alpha})\le p^*(\beta,\beta_1^*)=Np/(N-p)$, we conclude that u_n strongly converges to u in $L^{\widehat{q}}(U)$ from the Sobolev theorem.

Choose a sequence of positive numbers $\{\rho_m\}$ such that $\rho_m\to 0$ as $m\to\infty$ and $\overline{B}_{\rho_m}(0)\subset U$ for all $m\in\mathbb{Z}^+$. Then we deduce

$$\int_{U\setminus\overline{B}_{\rho_m}(0)} |u_n-u|^{\widehat{q}}|x|^{-\widehat{\alpha}}dx \le \rho_m^{-\hat{\alpha}}\|u_n-u\|^{\widehat{q}}_{L^{\widehat{q}}(U\setminus\overline{B}_{\rho_m}(0))} \le C_m\|u_n-u\|^{\widehat{q}}_{L^{\widehat{q}}(U)}.$$

On the other hand, recalling $\widehat{\alpha}<N$, we compute

$$\int_{\overline{B}_{\rho_m}(0)} |u_n-u|^{\widehat{q}}|x|^{-\widehat{\alpha}}dx \le \|u_n-u\|^{\widehat{q}}_{L^{\tau}(U,|x|^{-\widehat{\alpha}}dx)}\Big(\int_{\overline{B}_{\rho_m}(0)}|x|^{-\widehat{\alpha}}dx\Big)^{(\tau-\widehat{q})/\tau},$$

here $\tau=(\widehat{q}+p^*(\beta,\widehat{\alpha}))/2>\widehat{q}$. Combining the above two inequalities, we obtain

$$0\le \int_U |u_n-u|^{\widehat{q}}|x|^{-\widehat{\alpha}}dx \le C_m\|u_n-u\|^{\widehat{q}}_{L^{\widehat{q}}(U)} + C\Big(\int_{\overline{B}_{\rho_m}(0)}|x|^{-\widehat{\alpha}}dx\Big)^{(\tau-\widehat{q}\,)/\tau}.$$

First let $n\to\infty$, then $m\to\infty$, and we derive that u_n strongly converges to u in $L^{\widehat{q}}(U,|x|^{-\widehat{\alpha}})$. $\square$

Secondly, we review Bonanno's three critical points theorem (see [2]), which is the main variational tool in this paper.

Lemma 2.3. *Let $\mathcal{X}$ be a separable and reflexive real Banach space, and let $\phi,\psi:\mathcal{X}\to\mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that*

- (D1) *There exists a function $u_0\in\mathcal{X}$ such that $\phi(u_0)=\psi(u_0)=0$ and $\phi(u)\ge 0$ for every $u\in\mathcal{X}$.*
- (D2) *There exists a function $u_1\in\mathcal{X}$ and a positive number ρ such that*

$$\rho<\phi(u_1),\quad \sup_{\phi(u)<\rho}\psi(u)<\rho\frac{\psi(u_1)}{\phi(u_1)}. \tag{2.1}$$

- (D3) *Further, put*

$$\gamma=\xi\rho\Big[\rho\frac{\psi(u_1)}{\phi(u_1)}-\sup_{\phi(u)<\rho}\psi(u)\Big]^{-1},$$

with $\xi > 1$, and suppose that for every $\lambda \in [0, \gamma]$, the functional $\phi(u) - \lambda\psi(u)$ is sequentially weakly lower semicontinuous, satisfies the P.-S. condition and

$$\lim_{\|u\| \to +\infty} \Big[\phi(u) - \lambda\psi(u)\Big] = +\infty. \tag{2.2}$$

Then, there exists an open interval $\Lambda \subset [0, \gamma]$ and a number $R > 0$ such that, for any $\lambda \in \Lambda$, the equation $\phi'(u) - \lambda\psi'(u) = 0$ admits at least three solutions in $\mathcal{X}$ whose norms are less than R.

In the sequel, by setting $\mathcal{X} = X = \mathcal{D}^{1,p}(\Omega, |x|^{-\beta}dx)$, $\phi(u) = \Phi(u)$, $\psi(u) = \Psi(u)$ and $\xi = +\infty$ we show that the variational functional $I(u)$ satisfies all assumptions in Lemma 2.3.

Lemma 2.4. *Suppose that the assumptions* (B1), (B2) *are satisfied. Then $\Psi(u)$ is weakly continuous on X, i.e., if u_n weakly converges to u in X, $\Psi(u_n)$ converges to $\Psi(u)$.*

Proof. According to assumptions (B1), (B2), it is not difficult to deduce that, for each $\epsilon > 0$, there exists some positive number M_ϵ such that

$$|f(x, u)u| + |F(x, u)| \le \epsilon |u|^p |x|^{-\beta_2^*}, \quad \text{a.e. } x \in \Omega \text{ and all } |u| \in [M_\epsilon, +\infty); \tag{2.3}$$

$$|f(x, u)u| + |F(x, u)| \le \epsilon |u|^p |x|^{-\beta_2^*} + C_\epsilon |u||x|^{-\alpha}, \quad \text{a.e. } x \in \Omega \text{ and all } u \in \mathbb{R}, \tag{2.4}$$

here C_ϵ is a positive number dependent only on ϵ.

Assume that u_n converges weakly to u in X, then for any $\epsilon \ge 0$, we conclude

$$\begin{aligned} |F(x, u_n) - F(x, u)| &\le |f(x, \theta u + (1-\theta)u_n)||u_n - u| \\ &\le (\epsilon |u|^{p-1}|x|^{-\beta_2^*} + \epsilon |u_n|^{p-1}|x|^{-\beta_2^*} + \overline{C}_\epsilon |x|^{-\alpha})|u_n - u|, \end{aligned}$$

where $0 < \theta < 1$. The definition of $\Psi(u)$ thus implies that

$$\begin{aligned} |\Psi(u_n) - \Psi(u)| &\le \int_\Omega |F(x, u_n) - F(x, u)| dx \\ &\le \int_\Omega \Big(\epsilon \frac{|u|^{p-1} + |u_n|^{p-1}}{|x|^{\beta_2^*}} + \frac{\overline{C}_\epsilon}{|x|^\alpha}\Big)|u_n - u| \, dx \\ &\le C\epsilon(\|u_n\|_X^p + \|u\|_X^p) + \overline{C}_\epsilon \|u_n - u\|_{L^1(\Omega; |x|^{-\alpha}dx)}. \end{aligned}$$

Since X compactly imbeds into $L^1(\Omega; |x|^{-\alpha}dx)$, taking $n \to \infty$, we obtain

$$\limsup_{n \to \infty} |\Psi(u_n) - \Psi(u)| \le C\epsilon \|u\|_X^p.$$

Let $\epsilon \to 0^+$ in the above inequality, and the conclusion in the lemma follows. $\square$

Lemma 2.5. *Suppose that the assumptions* (A1)–(A4), (B1)–(B2) *are satisfied. Then $I(u)$ is weakly lower semicontinuous on X.*

Proof. Owing to previous lemma, it suffice to show weakly lower semicontinuity of $\Phi(u)$ on X. We argue by contradiction, assume that $\{u_n\}$ is a function sequence weakly converging to u in X, but there is a subsequence u_{n_k} such that $\lim_{k\to\infty} \Phi(u_{n_k}) > \Phi(u)$. Without loss of generalization, one can assume that

$$\Phi(u_{n_k}) > \Phi(u) + \delta, \quad \text{for } k = 1, 2, \dots,$$

where δ is a positive number.

In view of Mazur theorem, there exists a sequence $\{v_m\}$ strongly converging to u in X, where v_m is a convex combination of finitely many u_{n_k}; i.e., for any $m \in \mathbb{Z}^+$,

$$v_m = \sum_{i=1}^{m} \alpha_{mi} u_{n_{k_i}}, \quad \text{with } \alpha_{mi} > 0, \ \sum_{i=1}^{m} \alpha_{mi} = 1.$$

Since $A(x,\xi)$ is convex with respect to ξ, we then derive

$$\begin{aligned} \Phi(v_m) &\ge \sum_{i=1}^{m} \alpha_{mi} \int_\Omega A(x, \nabla u_{n_{k_i}})|x|^{-\beta} dx \\ &= \sum_{i=1}^{m} \alpha_{mi} \Phi(u_{n_{k_i}}) > \Phi(u) + \delta, \quad \text{for } m = 1, 2, \dots, \end{aligned}$$

which contradicts that $\{v_m\}$ strongly converges to u in X. □

Lemma 2.6. *Suppose that the assumptions* (A1)–(A4), (B1)–(B2) *are satisfied. Then $I(u)$ satisfies the P.-S. condition.*

Proof. Suppose that $\{u_n\} \subset X$ is a P.-S. sequence for $I(u)$, that is, $\{I(u_n)\}$ is bounded, and $\|I'(u_n)\|_{X^*} \to 0$ as $n \to 0$, where X^* is the dual space of X.

We claim that $\{u_n\}$ admits a strongly convergent subsequence. Firstly, we show that $\{u_n\}$ is bounded in X. In fact, combining assumption (A4), (2.4) and Lemma 2.1, we calculate

$$\begin{aligned} C \ge I(u_n) &= \int_\Omega A(x, \nabla u_n)|x|^{-\beta} dx - \lambda \int_\Omega F(x, u_n) dx \\ &\ge a_2 \int_\Omega |\nabla u_n|^p |x|^{-\beta} dx - \lambda \int_\Omega (\epsilon |u|^p |x|^{-\beta_2^*} + C_\epsilon |x|^{-\alpha}) dx \\ &\ge (a_2 - \lambda S_{\beta_2^*}^{-1} \epsilon) \|u_n\|_X^p - \overline{C}_\epsilon. \end{aligned}$$

Fix $\epsilon > 0$ small enough that $a_2 - \lambda S_{\beta_2^*}^{-1}\epsilon \ge a_2/2$, then we discover that $\{u_n\}$ is bounded in X. There thus exists a subsequence of $\{u_n\}$, still denoted by itself, such that $\{u_n\}$ weakly converges to u in X. Moreover, without loss of generalization, one can assume that $f(x, u_n)$ weakly converges to $f(x, u)$ in X^*.

We next demonstrate that there exists a subsequence of $\{u_n\}$, still denoted by itself, such that

$$\lim_{n\to\infty} \nabla u_n = \nabla u \quad \text{a.e. in } \Omega. \tag{2.5}$$

Indeed, the facts that $\{u_n\}$ is bounded in X and $\|I'(u_n)\|_{X^*} \to 0$ as $n \to \infty$ implies that

$$\langle I'(u_n) - I'(u), u_n - u\rangle = \langle I'(u_n), u_n - u\rangle - \langle I'(u), u_n - u\rangle = o(1), \quad \text{as } n \to \infty. \tag{2.6}$$

Furthermore, repeat the argument in the proof of Lemma 2.4, and it is easy to deduce

$$\begin{aligned} J(u, u_n) &:= \int_\Omega [f(x, u_n) - f(x, u)][u_n - u] dx \\ &= \int_\Omega f(x, u_n)(u_n - u)\, dx - \int_\Omega f(x, u)(u_n - u)\, dx = o(1), \end{aligned} \tag{2.7}$$

as $n \to \infty$. On the other hand,

$$\langle I'(u_n) - I'(u), u_n - u\rangle = \int_\Omega H(x, u, u_n)|x|^{-\beta} dx - \lambda J(u, u_n), \tag{2.8}$$

where

$$H(x,u,u_n) := [a(x,\nabla u_n) - a(x,\nabla u)]\cdot[\nabla u_n - \nabla u].$$

Combining (2.6), (2.7) and (2.8), we obtain

$$\lim_{n\to\infty}\int_\Omega H(x,u,u_n)|x|^{-\beta}dx = 0. \tag{2.9}$$

Notice that $H(x,u,u_n)\geq 0$ since $A(x,\xi)$ is convex in ξ. So, (2.9) implies that there exists a subsequence of $\{u_n\}$, still denoted by itself, such that $H(x,u,u_n)\to 0$ a.e. in Ω as $n\to\infty$. Hence, (2.5) follows from the strict convexity of $A(x,\xi)$.

Then, we prove that there exists a subsequence of $\{u_n\}$, still denoted by itself, such that

$$\lim_{n\to\infty}\int_\Omega |x|^{-\beta}a(x,\nabla u_n)\cdot\nabla u_n\,dx = \int_\Omega |x|^{-\beta}a(x,\nabla u)\cdot\nabla u\,dx. \tag{2.10}$$

According to the growth condition (A2) and (2.5), we can assume that $a(x,\nabla u_n)$ weakly converges to $a(x,\nabla u)$ in X^*, maybe a subsequence of $\{u_n\}$. Recalling that $f(x,u_n)$ weakly converges to $f(x,u)$ in X^*, we infer that $I'(u_n)$ weakly converges to $I'(u)$ in X^*. Hence, as $n\to\infty$, we deduce

$$\begin{aligned}
o(1) &= \langle I'(u_n), u_n - u\rangle - \langle I'(u_n) - I'(u), u\rangle\\
&= \langle I'(u_n), u_n\rangle - \langle I'(u), u\rangle\\
&= \int_\Omega |x|^{-\beta}[a(x,\nabla u_n)\cdot\nabla u_n - a(x,\nabla u)\cdot\nabla u]dx\\
&\quad - \lambda\int_\Omega [f(x,u_n)u_n - f(x,u)u]\,dx.
\end{aligned}$$

Repeating the procedure as in the proof of (2.7), we can achieve (2.10).

On the other hand, since $A(x,\xi)$ is convex with $A(x,0)=0$ and satisfies elliptic condition, we observe

$$a(x,\xi)\cdot\xi \geq A(x,\xi)\geq a_2|\xi|^p, \quad \text{for all } \xi\in\mathbb{R}^N,$$

which implies $a_2|\nabla u_n|^p$ and $a_2|\nabla u|^p$ being dominated by $a(x,\nabla u_n)\cdot\nabla u_n$, $a(x,\nabla u)\cdot\nabla u$, respectively. Combining (2.5), (2.10) and the dominated convergence theorem, we conclude that ∇u_n converges to ∇u in $L^p(\Omega,|x|^{-\beta}dx)$, that is u_n strongly converges to u in X. $\square$

3. Proof of the main results

To prove Theorems 1.1 and 1.2, we set notation as follows:

$$\Pi(F;M) = M^{p-q}\Big(\frac{\rho}{a_2 S_\alpha}\Big)^{q/p}\sup_{|u|\geq M}\mathcal{F}(u)|u|^{-p} + \mu(\Omega)\sup_{|u|\leq M}\mathcal{F}(u), \tag{3.1}$$

where $\mu(\Omega) := \int_\Omega |x|^{-\alpha}\,dx$ and $q = p^*(\beta,\alpha)$ as defined in (1.3). One can establish the next result.

Lemma 3.1. *Suppose that the hypothesis* (B2) *and* (A4) *are satisfied. For every $u\in X$ with $\Phi(u)\leq\rho$, we have*

$$\Psi(u)\leq \Pi(F;M).$$

Proof. According to assumption (A4) and Lemma 2.1, for every $u \in \Phi^{-1}(-\infty, \rho]$, we have

$$\|u\|_X^p \le \frac{\Phi(u)}{a_2} \le \frac{\rho}{a_2}, \quad \|u\|_\alpha^q \le \frac{\|u\|_X^q}{S_\alpha^{q/p}} \le \Big(\frac{\rho}{a_2 S_\alpha}\Big)^{q/p}, \tag{3.2}$$

where $\|u\|_\alpha^q := \int_\Omega |u|^q|x|^{-\alpha}dx$. By setting $\Omega_M := \{x \in \Omega : |u(x)| \ge M\}$, we can deduce

$$\mu(\Omega_M) \le M^{-q} \int_{\Omega_M} |u|^q|x|^{-\alpha}\,dx \le M^{-q}\|u\|_\alpha^q\,. \tag{3.3}$$

By assumption (B2), for every $u \in \Phi^{-1}(-\infty, \rho]$, we have the following estimate:

$$\begin{aligned}\Psi(u) = \int_\Omega F(x,u)\,dx &\le \sup_{|u|\ge M} \mathcal{F}(u)|u|^{-p}\int_{\Omega_M}|u|^p|x|^{-\alpha} + \int_{\Omega\setminus\Omega_M} F(x,u)\,dx\\ &\le \sup_{|u|\ge M}\mathcal{F}(u)|u|^{-p}\|u\|_\alpha^p\mu(\Omega_M)^{1-p/q} + \sup_{|u|\le M}\mathcal{F}(u)\mu(\Omega).\end{aligned}$$

Combining (3.2) and (3.3), we obtain $\Psi(u) \le \Pi(F; M)$ for every $u \in \Phi^{-1}(-\infty, \rho]$. □

Proof of Theorem 1.1. To apply Bonanno's three critical points theorem, we have to verify all conditions in Lemma 2.3.

Recalling the definition of $\Phi(u), \Psi(u)$, we conclude that $\Phi(0) = \Psi(0) = 0$ and $\Phi(u) \ge 0$ for all $u \in X$, which is the condition **(D1)** in Lemma 2.3.

Put $\gamma = +\infty$, then Lemma 2.5 and Lemma 2.6 imply that the functional $I(u) = \Phi(u) - \lambda\Psi(u)$ is sequentially weakly lower semicontinuous and satisfies the P.-S. condition. Moreover, using (A4), (2.4) and Lemma 2.1, we compute

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &\ge a_2\|u\|_X^p - \lambda\int_\Omega(\epsilon|u|^p|x|^{-\beta_2^*} + C_\epsilon|u||x|^{-\alpha})\,dx\\ &\ge a_2\|u\|_X^p - \epsilon\lambda S_{\beta_2^*}^{-1}\|u\|_X^p - C_\epsilon\lambda S_{1,\alpha}^{-1/p}\|u\|_X,\end{aligned}$$

fix a positive ϵ less than $a_2\lambda^{-1}S_{\beta_2^*}/2$, then (2.2) is obvious and we manifest assumption (D3).

In the following, we verify the condition (D2), or equivalently, (2.1). In fact, we can define a function the same as in [5]:

$$u_\sigma(x) = \begin{cases} 0, & x \in \mathbb{R}^N \setminus E;\\ K, & x \in B(x_0, \sigma r);\\ \frac{K}{r(1-\sigma)}(r - |x - x_0|), & x \in E \setminus B(x_0, \sigma r),\end{cases} \tag{3.4}$$

where $0 < \sigma < 1$ to be determined later. Owing to assumption (1.6) and (B2), we observe that

$$\begin{aligned}\Psi(u_\sigma) &= \int_E F(x, u_\sigma)dx\\ &\ge \int_{E\cap\{u_\sigma(x)=K\}} F(x,u_\sigma)dx - \max_{|u|\le|K|}\mathcal{F}(u)\int_{E\cap\{|u_\sigma(x)|<|K|\}}|x|^{-\alpha}\,dx\\ &\ge \inf_{x\in E} F(x,K)\int_{B(x_0,\sigma r)}dx - \max_{|u|\le|K|}\mathcal{F}(u)\int_{E\setminus B(x_0,\sigma r)}|x|^{-\alpha}\,dx.\end{aligned}$$

As $\sigma \to 1^-$, the first term on the right hand side of the above inequality tends to the positive constant $\omega r^N \inf_E F(x,K)$, here ω is the volume of the unit ball, and the second term goes to zero. We thus pick up some σ and u_σ such that $\Psi(u_\sigma) > 0$. Furthermore, from assumption (A4), we see that $\Phi(u_\sigma) \geq a_2 \|u_\sigma\|_X^p > 0$.

According to the Lemma 3.1, to verify (2.1), it suffice to turn up two positive numbers M and ρ, such that

$$0 < \rho < \Phi(u_\sigma) \quad \text{and} \quad \frac{\Pi(F;M)}{\rho} < \frac{\Psi(u_\sigma)}{\Phi(u_\sigma)}. \tag{3.5}$$

Indeed, in view of assumption (1.7) and (B2), we see that, for any $\varepsilon > 0$, there exist some positive constant M such that $\mathcal{F}(u) \leq \varepsilon |u|^p$, for all $u \in [-M, M]$ and $\mathcal{F}(u)|u|^{-p} \leq C$ for all $u \in \mathbb{R}$, where C is independent of M. Put $\rho = \delta^p M^p$ with δ is a positive number to be determined later, then we deduce

$$\frac{\Pi(F;M)}{\rho} \leq C\delta^{q-p}\Big(\frac{1}{a_2 S_\alpha}\Big)^{q/p} + \varepsilon \delta^{-p}\mu(\Omega)$$

One can first fix $\delta > 0$ small enough, then choose $\varepsilon > 0$ so small that the right hand side of the above inequality is less than $\Psi(u_\sigma)/\Phi(u_\sigma)$, finally choose M and ρ satisfy (3.5), which yields condition (2.1). Hence, we testify all the conditions in Lemma 2.3 and the desired conclusion follows. ☐

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, denote u_σ as (3.4) and fix $\sigma = 1/2$. Owing to assumptions (1.9) and (1.10), it is clear that

$$\Psi(u_\sigma) \geq \int_{E\cap\{u_\sigma(x)=K\}} F(x, u_\sigma)dx \geq c_2(K)(1 + |K|^p)\int_{B(x_0,r/2)} dx.$$

Moreover, recalling assumptions (A4) and (A2), we have

$$\begin{gathered}
\Phi(u_\sigma) \geq a_2 \int_E |\nabla u_\sigma|^p |x|^{-\beta}\,dx \geq a_2\Big(\frac{2|K|}{r}\Big)^p \int_{E\setminus B(x_0,r/2)} |x|^{-\beta}\,dx, \\
\Phi(u_\sigma) \leq a_1 \int_E (|\nabla u_\sigma| + |\nabla u_\sigma|^p)|x|^{-\beta}\,dx \leq a_1\Big(\frac{2|K|}{r} + \Big(\frac{2|K|}{r}\Big)^p\Big)\int_E |x|^{-\beta}\,dx.
\end{gathered} \tag{3.6}$$

We thus get

$$\rho\frac{\Psi(u_\sigma)}{\Phi(u_\sigma)} \geq \delta c_2(K)\rho, \tag{3.7}$$

where δ is a positive constant dependent only on p, β, N, E and a_1.

On the other hand, let $M = L$ in (3.1), according to the definition in (1.8), we obtain

$$\Pi(F;L) \leq c_3(L)L^{p-q}\Big(\frac{\rho}{a_2 S_\alpha}\Big)^{q/p} + c_4(L)\mu(\Omega). \tag{3.8}$$

Denote by

$$\rho_1 = \Big(\frac{\delta c_2(K)L^{q-p}(a_2 S_\alpha)^{q/p}}{2c_3(L)}\Big)^{\frac{p}{q-p}}, \quad \rho_2 = \frac{\Phi(u_\sigma)}{2}.$$

Let $\rho = \min\{\rho_1, \rho_2\}$. When $\rho = \rho_1$, in view of (3.7), (3.8) and assumption (1.10), we compute

$$
\begin{aligned}
\rho\frac{\Psi(u_\sigma)}{\Phi(u_\sigma)} - \Pi(F;L) &\geq \delta c_2(K)\rho_1 - \Pi(F;L) \\
&\geq \frac{\delta}{2}c_2(K)\rho_1 - c_4(L)\mu(\Omega) \\
&= \delta^*(c_2(K))^{\frac{q}{q-p}}L^p(c_3(L))^{\frac{p}{p-q}} - c_4(L)\mu(\Omega) \\
&\geq \delta^* C^{\frac{q}{q-p}}c_4(L) - c_4(L)\mu(\Omega) > 0,
\end{aligned}
$$

where δ^* and C are constants dependent only on $p, \beta, \alpha, N, E, \Omega, a_1$ and a_2.

In the other case of $\rho = \rho_2$, owing to (3.6), (3.7), (3.8) and assumption (1.10), we deduce

$$
\begin{aligned}
\rho\frac{\Psi(u_\sigma)}{\Phi(u_\sigma)} - \Pi(F;L) &\geq \frac{\delta}{2}c_2(K)\rho_2 - c_4(L)\mu(\Omega) \\
&\geq \delta^{**}c_2(K)|K|^p - c_4(L)\mu(\Omega) \\
&\geq \delta^{**}Cc_4(L) - c_4(L)\mu(\Omega) > 0,
\end{aligned}
$$

where δ^{**}, C are constants dependent only on $p, \beta, \alpha, N, E, \Omega, a_1$ and a_2. So, we achieve assumption (2.1) in any cases and the conclusion in the theorem is derived from Lemma 2.3. $\square$

In the following, we give two simple examples:

Example 3.2. Consider the mean curvature equation

$$
\begin{gathered}
-\operatorname{div}(|x|^{-\beta}(1+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u) = \lambda|u|^{m+\frac{p-m}{|u|+1}}|x|^{-\alpha}, \quad x \in \Omega, \\
u = 0, \quad x \in \partial\Omega\,.
\end{gathered}
\tag{3.9}
$$

Employing Theorem 1.1, we can get the following result: If $2 \leq p < N$, $m < p-1, 0 \leq \beta < N-p$, $\beta_1^* \leq \alpha < \beta_2^*$, then (3.9) admits at least three distinct weak solutions.

Example 3.3. Consider the p-Laplacian equation involving singular weight:

$$
\begin{gathered}
-\operatorname{div}(|x|^{-\beta}|\nabla u|^{p-2}\nabla u) = \lambda|x|^{-\alpha}g(u), \quad x \in \Omega, \\
u = 0, \quad x \in \partial\Omega,
\end{gathered}
\tag{3.10}
$$

where

$$
g(u) = \begin{cases} e^u, & u \in [-t,t], \\ e^t, & u \in [t,\infty), \\ e^{-t}, & u \in (-\infty,-t]. \end{cases}
$$

Applying Theorem 1.2, we conclude that: If $1 < p < N$, $0 \leq \beta < N-p$ and $\beta_1^* \leq \alpha < \beta_2^*$, then (3.10) admits at least three distinct weak solutions provided t is sufficiently large.

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