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COMPARISON PRINCIPLE FOR PARABOLIC EQUATIONS IN THE HEISENBERG GROUP

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Abstract. We define two notions of viscosity solutions to parabolic equations in the Heisenberg group, depending on whether the test functions concern only the past or both the past and the future. We then exploit the Heisenberg geometry to prove a comparison principle for a class of parabolic equations and show the sufficiency of considering the test functions that concern only the past.

1. Background and Motivation

In [2], viscosity solutions to a class of fully nonlinear subelliptic equations in the Heisenberg group were introduced and comparison principles were proved by heavily exploiting the geometry of the Heisenberg group. It is natural to adapt the geometric workings of [2] to parabolic equations in the Heisenberg group. For example, such equations have been recently used by Bonk and Capogna to study mean curvature [3].

Our objective is to find the Heisenberg analog of the Euclidean comparison principle for a class of parabolic equations as found in [5, Section 8]. Relying on the Heisenberg geometry, we prove such an analog as our main theorem, Theorem 4.4. In addition, we examine a significant consequence of this comparison principle. Namely, we are able to prove that for this class of parabolic equations, it is sufficient to consider only test functions that refer to the past. This was originally proved in the Euclidean case by Juutinen [12].

Before presenting these two results in Section 4, we begin with a brief introduction to Heisenberg groups in Section 2 and discuss viscosity solutions to parabolic equations in Section 3.

2. The Heisenberg Group

We begin with $\mathbb{R}^{2n+1}$ using the coordinates $(x_1, x_2, \dots, x_{2n}, z)$ and consider the linearly independent vector fields $\{X_i, Z\}$, where the index i ranges from 1 to $2n$,

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defined by

$$X_i = \begin{cases} \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial z} & \text{if } 1 \le i \le n \\ \frac{\partial}{\partial x_i} + \frac{x_{i-n}}{2}\frac{\partial}{\partial z} & \text{if } n < i \le 2n, \end{cases}$$

$$Z = \frac{\partial}{\partial z}.$$

These vector fields obey the relations

$$[X_i, X_j] = \begin{cases} Z & \text{if } j = i+n \\ 0 & \text{otherwise} \end{cases}$$

and for all i,

$$[X_i, Z] = 0.$$

We then have a Lie Algebra denoted h_n that decomposes as a direct sum

$$h_n = V_1 \oplus V_2$$

where V_1 is spanned by the X_i's and V_2 is spanned by Z. We endow h_n with an inner product $\langle \cdot, \cdot \rangle$ and related norm $\| \cdot \|$ so that this basis is orthonormal. The corresponding Lie Group is called the general Heisenberg group of dimension n and is denoted by $\mathbb{H}_n$. The choice of vector fields and their Lie bracket relations forces the exponential map to be the identity and so elements of h_n and $\mathbb{H}_n$ can be identified with each other. Namely,

$$\sum_{i=1}^{2n} x_i X_i + zZ \in h_n \leftrightarrow (x_1, x_2, \dots, x_{2n}, z) \in \mathbb{H}_n.$$

In particular, for any p, q in $\mathbb{H}_n$, written as $p = (x_1, x_2, \dots, x_{2n}, z_1)$ and $q = (y_1, y_2, \dots, y_{2n}, z_2)$ the group multiplication law is given by

$$p \cdot q = \Big(x_1 + y_1, x_2 + y_2, \dots, x_{2n} + y_{2n}, z_1 + z_2 + \frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i)\Big).$$

The natural metric on $\mathbb{H}_n$ is the Carnot-Carathéodory metric given by

$$d_C(p, q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| dt$$

where the set Γ is the set of all curves γ such that $\gamma(0) = p, \gamma(1) = q$ and $\gamma'(t) \in V_1$. By Chow's theorem any two points can be connected by a horizontal curve, which makes $d_C(p, q)$ a left-invariant metric on $\mathbb{H}_n$. (See, for example, [1].) This metric induces a homogeneous norm on $\mathbb{H}_n$, denoted $| \cdot |$, by

$$|p| = d_C(0, p)$$

and we have the estimate

$$|p| \sim \sum_{i=1}^{2n} |x_i| + |z|^{1/2}.$$

This estimate leads us to define the left-invariant gauge $\mathcal{N}$ that is comparable to the Carnot-Carathéodory metric and is given by

$$\mathcal{N}(p) = \Big(\big(\sum_{i=1}^{2n} x_i^2\big)^2 + 16z^2\Big)^{1/4}.$$

We define the Carnot-Carathéodory balls $B(p,r)$ and the gauge balls $B_{\mathcal{N}}(p,r)$ in the obvious way.

Given a smooth function $u:\mathbb{H}_n\mapsto\mathbb{R}$, we define the horizontal gradient by

$$\nabla_0 u=(X_1u,X_2u,\dots,X_{2n}u),$$

the full gradient by

$$\nabla u=(X_1u,X_2u,\dots,X_{2n}u,Zu),$$

and the symmetrized horizontal second derivative matrix $(D^2u)^\star$ by

$$((D^2u)^\star)_{ij}=\frac{1}{2}(X_iX_ju+X_jX_iu).$$

A function f is $\mathcal{C}^1$ if X_if is continuous for all i and f is $\mathcal{C}^2$ if f is $\mathcal{C}^1$ and X_iX_jf is continuous for all i and j.

For a more complete treatment of the Heisenberg group, the interested reader is directed to [1], [2], [6], [7] [8], [9], [13], [14] and the references therein.

3. Parabolic Jets and Solutions to Parabolic Equations

In this section, we define and compare various notions of solutions to parabolic equations in the Heisenberg group, in the spirit of [5, Section 8]. We begin by letting $u(p,t)$ be a function in $\mathbb{H}_n\times[0,T]$ for some $T>0$. We consider parabolic equations of the form

$$u_t+F(t,p,u,\nabla u,(D^2u)^\star)=0 \tag{3.1}$$

for continuous and proper $F:[0,T]\times\mathbb{H}_n\times\mathbb{R}\times h_n\times S^{2n}\mapsto\mathbb{R}$. We recall that S^k is the set of $k\times k$ symmetric matrices and the derivatives ∇u and $(D^2u)^\star$ are taken in the space variable p. Examples of parabolic equations include the parabolic P-Laplace equation for $2\le P<\infty$ given by

$$u_t+\Delta_P u=u_t-\operatorname{div}(\|\nabla_0u\|^{P-2}\nabla_0u)=0$$

and the parabolic infinite Laplace equation

$$u_t+\Delta_\infty u=u_t-\langle(D^2u)^\star\nabla_0u,\nabla_0u\rangle=0.$$

For such equations, we define the parabolic superjet of $u(p,t)$ at the point $(p_0,t_0)\in\mathcal{O}_T\equiv\mathcal{O}\times(0,T)$, denoted $P^{2,+}u(p_0,t_0)$, by using triples $(a,\eta,X)\in\mathbb{R}\times h_n\times S^{2n}$ so that $(a,\eta,X)\in P^{2,+}u(p_0,t_0)$ if

$$\begin{aligned}u(p,t)\le u(p_0,t_0)+a(t-t_0)+\langle\eta,p_0^{-1}\cdot p\rangle+\frac{1}{2}\langle X\overline{p_0^{-1}\cdot p},\overline{p_0^{-1}\cdot p}\rangle\\ +o(|t-t_0|+|p_0^{-1}\cdot p|^2)\quad\text{as }(p,t)\to(p_0,t_0).\end{aligned}$$

We recall that $\overline{p_0^{-1}\cdot p}$ is the first $2n$ coordinates of $p_0^{-1}\cdot p$, given by $(x_1-x_1^0,x_2-x_2^0,\dots,x_{2n}-x_{2n}^0)$. This definition is analogous to the superjet definition for subelliptic equations, as detailed in [2]. We define the subjet $P^{2,-}u(p_0,t_0)$ by

$$P^{2,-}u(p_0,t_0)=-P^{2,+}(-u)(p_0,t_0).$$

We define the set theoretic closure of the superjet, denoted $\overline{P}^{2,+}u(p_0,t_0)$, by requiring $(a,\eta,X)\in\overline{P}^{2,+}u(p_0,t_0)$ exactly when there is a sequence

$$(a_n,p_n,t_n,u(p_n,t_n),\eta_n,X_n)\to(a,p_0,t_0,u(p_0,t_0),\eta,X)$$

with the triple $(a_n,\eta_n,X_n)\in P^{2,+}u(p_n,t_n)$. A similar definition holds for the closure of the subjet.

As in the subelliptic case, we may also define jets using the appropriate test functions. Namely, we consider the set $\mathcal{A}u(p_0,t_0)$ by

$$\mathcal{A}u(p_0,t_0)=\{\phi\in\mathcal{C}^2(\mathcal{O}_T): u(p,t)-\phi(p,t)\le u(p_0,t_0)-\phi(p_0,t_0)=0\}$$

consisting of all test functions that touch from above. We define the set of all test functions that touch from below, denoted $\mathcal{B}u(p_0,t_0)$, similarly. The following lemma is proved in the same way as the Euclidean version ([4] and [11]) except we replace the Euclidean distance $|p-p_0|$ with the Heisenberg gauge $\mathcal{N}(p_0^{-1}\cdot p)$.

Lemma 3.1.

$$P^{2,+}u(p_0,t_0)=\{(\phi_t(p_0,t_0),\nabla\phi(p_0,t_0),(D^2\phi(p_0,t_0))^\star):\phi\in\mathcal{A}u(p_0,t_0)\}.$$

We may now relate the traditional Euclidean parabolic jets found in [5] to the Heisenberg parabolic jets via the following lemma.

Lemma 3.2. *Let DL_{p_0} be the differential of the left multiplication map at the point p_0, let $P^{2,+}_{eucl}u(p_0,t_0)$ be the traditional Euclidean parabolic superjet of u at the point (p_0,t_0) and let $(a,\eta,X)\in\mathbb{R}\times\mathbb{R}^{2n+1}\times S^{2n+1}$. Then,*

$$(a,\eta,X)\in\overline{P}^{2,+}_{eucl}u(p_0,t_0)$$

gives the element

$$\Big(a,DL_{p_0}\eta,(DL_{p_0}\;X\;(DL_{p_0})^T)_{2n}\Big)\in\overline{P}^{2,+}u(p_0,t_0)$$

with the convention that for any matrix M, M_m is the $m\times m$ principal minor.

Proof. This proof is similar to the corresponding result for subelliptic jets as found in [2]. We then highlight the main details.

We may assume that $u(p_0,t_0)=0$. We first consider the case when p_0 is the origin. Let $(a,\eta,X)\in P^{2,+}_{\mathrm{eucl}}u(0,t_0)$. Then we have

$$u(p,t)\le a(t-t_0)+\langle\eta,p\rangle_{\mathrm{eucl}}+\langle Xp,p\rangle_{\mathrm{eucl}}+o(|t-t_0|+\|p\|^2_{\mathrm{eucl}})$$

for (p,t) near $(0,t_0)$. Suppose that α is $o(|t-t_0|+\|p\|^2_{\mathrm{eucl}})$. Then we have

$$\begin{aligned}\frac{\alpha}{|t-t_0|+|p|^2}&=\frac{\alpha}{|t-t_0|+\|p\|^2_{\mathrm{eucl}}}\times\frac{|t-t_0|+\|p\|^2_{\mathrm{eucl}}}{|t-t_0|+|p|^2}\\&\le\frac{\alpha}{|t-t_0|+\|p\|^2_{\mathrm{eucl}}}\times\big(1+\frac{\|p\|^2_{\mathrm{eucl}}}{|p|^2}\big).\end{aligned}$$

We thus conclude that α is $o(|t-t_0|+|p|^2)$. Using the fact that $\langle\eta,p\rangle_{\mathrm{eucl}}=\langle\eta,p\rangle$ at the origin, we obtain

$$u(p,t)\le a(t-t_0)+\langle\eta,p\rangle+\langle Xp,p\rangle_{\mathrm{eucl}}+o(|t-t_0|+|p|^2).$$

We next observe that

$$\langle Xp,p\rangle_{\mathrm{eucl}}=\langle(X)_{2n}\overline{p},\overline{p}\rangle+o(|p|^2)$$

where $(X)_{2n}$ is the $2n\times 2n$ principal minor and $\overline{p}$ is as above. We therefore obtain the inequality

$$u(p,t)\le a(t-t_0)+\langle\eta,p\rangle+\langle(X)_{2n}\overline{p},\overline{p}\rangle+o(|t-t_0|+|p|^2).$$

The general case follows from left translation of p_0. $\square$

We then use these jets to define subsolutions and supersolutions to Equation (3.1) in the usual way.

Definition 3.3. Let $(p_0,t_0)\in\mathcal{O}_T$ be as above. The upper semicontinuous function u is a *viscosity subsolution* in $\mathcal{O}_T$ if for all $(p_0,t_0)\in\mathcal{O}_T$ we have $(a,\eta,X)\in P^{2,+}u(p_0,t_0)$ produces

$$a+F(t_0,p_0,u(p_0,t_0),\eta,X)\le 0.$$

A lower semicontinuous function u is a *viscosity supersolution* in $\mathcal{O}_T$ if for all $(p_0,t_0)\in\mathcal{O}_T$ we have $(b,\nu,Y)\in P^{2,-}u(p_0,t_0)$ produces

$$b+F(t_0,p_0,u(p_0,t_0),\nu,Y)\ge 0.$$

A continuous function u is a *viscosity solution* in $\mathcal{O}_T$ if it is both a viscosity subsolution and viscosity supersolution.

We also wish to define what [12] refers to as parabolic viscosity solutions. We first need to consider the set

$$\mathcal{A}^-u(p_0,t_0)=\{\phi\in\mathcal{C}^2(\mathcal{O}_T):u(p,t)-\phi(p,t)\le u(p_0,t_0)-\phi(p_0,t_0)=0\text{ for }t<t_0\}$$

consisting of all functions that touch from above only when $t<t_0$. Note that this set is larger than $\mathcal{A}u$ and corresponds physically to the past alone playing a role in determining the present. We define $\mathcal{B}^-u(p_0,t_0)$ similarly. We then have the following definition.

Definition 3.4. An upper semicontinuous function u on $\mathcal{O}_T$ is a *parabolic viscosity subsolution* in $\mathcal{O}_T$ if $\phi\in\mathcal{A}^-u(p_0,t_0)$ produces

$$\phi_t(p_0,t_0)+F(t_0,p_0,u(p_0,t_0),\nabla\phi(p_0,t_0),(D^2\phi(p_0,t_0))^\star)\le 0.$$

An lower semicontinuous function u on $\mathcal{O}_T$ is a *parabolic viscosity supersolution* in $\mathcal{O}_T$ if $\phi\in\mathcal{B}^-u(p_0,t_0)$ produces

$$\phi_t(p_0,t_0)+F(t_0,p_0,u(p_0,t_0),\nabla\phi(p_0,t_0),(D^2\phi(p_0,t_0))^\star)\ge 0.$$

A continuous function is a *parabolic viscosity solution* if it is both a parabolic viscosity supersolution and subsolution.

It is easily checked that parabolic viscosity sub(super-)solutions are viscosity sub(super-)solutions. The reverse implication will be a consequence of the comparison principle proved in the next section.

4. Comparison Principle

To prove the comparison principle, we need to chose the proper penalty function that has the desired properties. We consider the function $\varphi:\mathbb{H}_n\times\mathbb{H}_n\mapsto\mathbb{R}$ given by

$$\varphi(p,q)=\frac{1}{2}\sum_{i=1}^{2n}(x_i-y_i)^2+\frac{1}{2}\Big(z_1-z_2+\frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i-x_iy_{n+i})\Big)^2.$$

We observe that we have

$$\varphi(p,q)=\frac{1}{2}\|q^{-1}\cdot p\|^2.$$

This choice gives us desired properties as detailed in the next lemma.

Lemma 4.1. *Let the vector η be given by*

$$\eta = q^{-1}\cdot p = \begin{pmatrix} (x_1-y_1) \\ (x_2-y_2) \\ \vdots \\ (x_{2n}-y_{2n}) \\ z_1-z_2+\frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_i y_{n+i}) \end{pmatrix}.$$

Recall that the differential of left multiplication with respect to p, denoted DL_p, is given by

$$\begin{pmatrix} I_{2n\times 2n} & \mathcal{P} \\ 0_{1\times 2n} & 1 \end{pmatrix}$$

with the $2n\times 1$ vector $\mathcal{P}$ given by

$$(-\frac{1}{2}x_{n+1}, -\frac{1}{2}x_{n+2}, \dots, -\frac{1}{2}x_{2n}, \frac{1}{2}x_1, \frac{1}{2}x_2, \dots, \frac{1}{2}x_n)^T$$

with a similar definition for DL_q using the vector $\mathcal{Q}$. Denoting Euclidean differentiation with respect to the point r by D_r, then have the following properties:

(1) $D_p\varphi(p,q) = DL_q\eta$
(2) $D_q\varphi(p,q) = -DL_p\eta$
(3) $D_p\eta = DL_q^T$
(4) $D_q\eta = -DL_p^T$
(5) $DL_pDL_q = DL_qDL_p$
(6) $DL_pD_p\varphi(p,q) = -DL_qD_q\varphi(p,q) \equiv \Upsilon(p,q)$
(7)
$$\begin{aligned} &DL_p(D_{pp}\varphi(p,q)DL_p^T + D_{pq}\varphi(p,q)DL_q^T) \\ &= \frac{1}{2}(z_1-z_2+\frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_iy_{n+i}))\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix} \end{aligned}$$
(8)
$$\begin{aligned} &DL_q(D_{qq}\varphi(p,q)DL_q^T + D_{qp}\varphi(p,q)DL_p^T) \\ &= \frac{1}{2}(z_1-z_2+\frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_iy_{n+i}))\begin{pmatrix} 0_{n\times n} & I_{n\times n} & 0_{n\times 1} \\ -I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix} \end{aligned}$$
(9) *Let $v\in h_n$ be a vector. Then*
$$\begin{aligned} &\langle D_{pp}\varphi(p,q)DL_p^Tv, DL_p^Tv\rangle + \langle D_{pq}\varphi(p,q)DL_q^Tv, DL_p^Tv\rangle \\ &+ \langle D_{qp}\varphi(p,q)DL_p^Tv, DL_q^Tv\rangle + \langle D_{qq}\varphi(p,q)DL_q^Tv, DL_q^Tv\rangle = 0 \end{aligned}$$
(10) *Let $v\in h_n$ be a vector. We recall that $\overline{v}$ is the first $2n$ coordinates of v. Then*
$$\begin{aligned} &\|\begin{pmatrix} D_{pp}\varphi(p,q) & D_{pq}\varphi(p,q) \\ D_{qp}\varphi(p,q) & D_{qq}\varphi(p,q) \end{pmatrix}\begin{pmatrix} DL_p^Tv \\ DL_q^Tv \end{pmatrix}\|^2 \\ &= \frac{1}{2}\|\overline{v}\|^2(z_1-z_2+\frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_iy_{n+i}))^2. \end{aligned}$$

Proof. The first five properties are elementary calculations and left to the reader. The sixth follows from the fifth and the first two. We therefore turn our attention to the last four. Let M_{pq} be the left-hand side of (7). Then,

$$\begin{aligned} M_{pq} &= DL_p\Big(D_p(DL_q\eta)DL_p^T + D_q(DL_q\eta)DL_q^T\Big)\\ &= DL_p\Big(DL_qD_p\eta DL_p^T + D_q(DL_q)\eta DL_q^T + DL_qD_q\eta DL_q^T\Big)\\ &= DL_p\Big(DL_qDL_q^TDL_p^T + D_q(DL_q)\eta DL_q^T - DL_qDL_p^TDL_q^T\Big)\\ &= DL_p\Big(D_q(DL_q)\eta DL_q^T\Big) \end{aligned}$$

and so we are left to compute only the derivative of the matrix DL_q. Knowing the definition of η above and the formula for DL_q as given above, we see that when $1 \le i \le n$, we have $D_{y_i}(DL_q)$ is a matrix with every entry 0 except for the $(i+n, 2n+1)$ entry, which is $\frac{1}{2}$. When $n < i < 2n$, we have $D_{y_i}(DL_q)$ has all entries 0 except for the $(i-n, 2n+1)$ entry, which is $-\frac{1}{2}$. Cleary, $D_{z_2}(DL_q)$ is the 0 matrix. We then compute

$$D_q(DL_q)\eta = \frac{1}{2}(z_1 - z_2 + \frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_iy_{n+i}))\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1}\\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1}\\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}.$$

We then have

$$\begin{aligned} &\begin{pmatrix} I_{2n\times 2n} & \mathcal{P}\\ 0_{1\times 2n} & 1\end{pmatrix}\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1}\\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1}\\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}\begin{pmatrix} I_{2n\times 2n} & 0_{2n\times 1}\\ \mathcal{Q}^T & 1\end{pmatrix}\\ &= \begin{pmatrix} I_{2n\times 2n} & \mathcal{P}\\ 0_{1\times 2n} & 1\end{pmatrix}\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1}\\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1}\\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix} = \begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1}\\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1}\\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix} \end{aligned}$$

and Property (7) follows. To prove (8), we let M_{pq} be the left-hand side of (8). Then

$$\begin{aligned} M_{pq} &= DL_q\Big(-D_q(DL_p\eta)DL_q^T + D_p(-DL_p\eta)DL_p^T\Big)\\ &= DL_q\Big(-DL_pD_q\eta DL_q^T - D_p(DL_p)\eta DL_p^T - DL_pD_p\eta DL_p^T\Big)\\ &= DL_q\Big(DL_pDL_p^TDL_q^T - D_p(DL_p)\eta DL_p^T - DL_pDL_q^TDL_p^T\Big)\\ &= -DL_q\Big(D_p(DL_p)\eta DL_p^T\Big) \end{aligned}$$

and we compute $D_p(DL_p)\eta$ in the same way as the above computation for $D_q(DL_q)\eta$ and arrive at Property (8).

To prove Property (9), we note that the right hand side can be written as

$$\begin{aligned} &\langle DL_p(D_{pp}\varphi(p,q)DL_p^T + D_{pq}\varphi(p,q)DL_q^T)v, v\rangle\\ &+ \langle DL_q(D_{qq}\varphi(p,q)DL_q^T + D_{qp}\varphi(p,q)DL_p^T)v, v\rangle \end{aligned}$$

Using Properties (7) and (8) we see this is zero. Property (9) then follows.

Using the proofs of Properties (7) and (8), we have

$$\begin{aligned}
&\begin{pmatrix} D_{pp}\varphi(p,q) & D_{pq}\varphi(p,q) \\ D_{qp}\varphi(p,q) & D_{qq}\varphi(p,q) \end{pmatrix} \begin{pmatrix} DL_p^T v \\ DL_q^T v \end{pmatrix} \\
&= \begin{pmatrix} (D_{pp}\varphi(p,q)DL_p^T + D_{pq}\varphi(p,q)DL_q^T)v \\ (D_{qp}\varphi(p,q)DL_p^T + D_{qq}\varphi(p,q)DL_q^T)v \end{pmatrix} \\
&= \frac{1}{2}(z_1 - z_2 + \frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_i y_{n+i})) \begin{bmatrix} \begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix} v \\ \begin{pmatrix} 0_{n\times n} & I_{n\times n} & 0_{n\times 1} \\ -I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix} v \end{bmatrix}.
\end{aligned}$$

We then see that

$$\begin{aligned}
&\| \begin{pmatrix} D_{pp}\varphi(p,q) & D_{pq}\varphi(p,q) \\ D_{qp}\varphi(p,q) & D_{qq}\varphi(p,q) \end{pmatrix} \begin{pmatrix} DL_p^T v \\ DL_q^T v \end{pmatrix} \|^2 \\
&= \frac{1}{2}\|\overline{v}\|^2 (z_1 - z_2 + \frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_i y_{n+i}))^2
\end{aligned}$$

and Property (10) is proved. □

Using the penalty function φ we next need to show the existence of Heisenberg jet elements when considering subsolutions and supersolutions in $\mathbb{H}_n$. This theorem is based on [5, Thm. 8.2], which details the Euclidean case.

Theorem 4.2. *Let u be a viscosity subsolution to Equation* (3.1) *and v be a viscosity supersolution to Equation* (3.1) *in the bounded parabolic set $\Omega \times (0,T)$ where Ω is a (bounded) domain. Let τ be a positive real parameter and let $\varphi(p,q)$ be as above. Suppose the local maximum of*

$$M_\tau(p,q,t) \equiv u(p,t) - v(q,t) - \tau\varphi(p,q)$$

occurs at the interior point (p_τ, q_τ, t_τ) of the parabolic set $\Omega \times \Omega \times (0,T)$. Then, for each $\tau > 0$, there are elements $(a, \tau\Upsilon, \mathcal{X}^\tau) \in \overline{P}^{2,+}u(p_\tau, t_\tau)$ and $(a, \tau\Upsilon, \mathcal{Y}^\tau) \in \overline{P}^{2,-}v(q_\tau, t_\tau)$ so that if

$$\lim_{\tau\to\infty} \tau\varphi(p_\tau, q_\tau) = 0,$$

then we have $\mathcal{X}^\tau \le \mathcal{Y}^\tau + \mathcal{R}^\tau$ with $\mathcal{R}^\tau \to 0$ as $\tau \to \infty$.

Proof. We first need to check that Condition 8.5 in [5] is satisfied, namely that there exists an $r > 0$ so that for each M, there exists a C so that $b \le C$ when $(b,\eta,X) \in P^{2,+}_{\text{eucl}}u(p,t)$, $|p - p_\tau| + |t - t_\tau| < r$, and $|u(p,t)| + \|\eta\| + \|X\| \le M$ with a similar statement holding for $-v$. If this condition is not met, then for each $r > 0$, we have an M so that for all C, $b > C$ when $(b,\eta,X) \in P^{2,+}_{\text{eucl}}u(p,t)$. By Lemma 3.2 we would have

$$(b, DL_p\eta, (DL_p\ X\ DL_p^T)_{2n}) \in P^{2,+}u(p,t)$$

contradicting the fact that u is a subsolution. A similar conclusion is reached for $-v$ and so we conclude that this condition holds. We may then apply Theorem 8.3

of [5] and obtain, by our choice of φ,

$$(a, \tau D_p\varphi(p_\tau, q_\tau), X^\tau) \in \overline{P}^{2,+}_{\mathrm{eucl}} u(p_\tau, t_\tau)$$
$$(a, -\tau D_q\varphi(p_\tau, q_\tau), Y^\tau) \in \overline{P}^{2,-}_{\mathrm{eucl}} v(q_\tau, t_\tau)$$

and by Lemma 3.2 we have

$$(a, \tau\Upsilon(p_\tau, q_\tau), \mathcal{X}^\tau) \in \overline{P}^{2,+} u(p_\tau, t_\tau)$$
$$(a, \tau\Upsilon(p_\tau, q_\tau), \mathcal{Y}^\tau) \in \overline{P}^{2,-} v(q_\tau, t_\tau)$$

where

$$\mathcal{X}^\tau = (DL_{p_\tau}\ X^\tau\ DL^T_{p_\tau})_{2n}$$
$$\mathcal{Y}^\tau = (DL_{q_\tau}\ Y^\tau\ DL^T_{q_\tau})_{2n}.$$

Given a vector $v = (v_1, v_2, \dots, v_{2n})$, we consider the extension $\hat{v}$ to all of h_n by $\hat{v} = (v, 0)$. We then have

$$\begin{aligned}\langle \mathcal{X}^\tau v, v\rangle - \langle \mathcal{Y}^\tau v, v\rangle &= \langle DL_{p_\tau}\ X^\tau\ DL^T_{p_\tau}\hat{v}, \hat{v}\rangle - \langle DL_{q_\tau}\ Y^\tau\ DL^T_{q_\tau}\hat{v}, \hat{v}\rangle \\ &\le \tau\langle (A^2 + A)\Big(DL^T_{p_\tau}\hat{v} \oplus DL^T_{q_\tau}\hat{v}\Big), \Big(DL^T_{p_\tau}\hat{v} \oplus DL^T_{q_\tau}\hat{v}\Big)\rangle\end{aligned}$$

where the matrix A is given by

$$A = \begin{pmatrix} D_{pp}\varphi(p_\tau, q_\tau) & D_{pq}\varphi(p_\tau, q_\tau) \\ D_{qp}\varphi(p_\tau, q_\tau) & D_{qq}\varphi(p_\tau, q_\tau) \end{pmatrix}.$$

We may then combine Properties (9) and (10) of Lemma 4.1 and the fact that $\tau\varphi(p_\tau, q_\tau) \to 0$ as $\tau \to \infty$ to obtain the matrix estimate. $\square$

Using the vector Υ, we may define a class of parabolic equations to which we shall prove a comparison principle.

Definition 4.3. We say the continuous, proper function

$$F : [0, T] \times \overline{\Omega} \times \mathbb{R} \times h_n \times S^{2n} \mapsto \mathbb{R}$$

is *admissable* if for each $t \in [0, T]$, there is the same function $\omega : [0, \infty] \mapsto [0, \infty]$ with $\omega(0+) = 0$ so that F satisfies

$$F(t, q, r, \tau\Upsilon, Y) - F(t, p, r, \tau\Upsilon, X) \le \omega(d_C(p, q) + \tau\|\Upsilon(p, q)\|^2 + \|X - Y\|). \quad (4.1)$$

Note that $\|\Upsilon(p, q)\|^2 \sim \varphi(p, q)$ by the calculations above.

We now formulate the comparison principle for the problem

$$\begin{aligned} u_t + F(t, p, u, \nabla u, (D^2u)^\star) &= 0 \quad \text{in } (0, T) \times \Omega && \text{(E)} \\ u(p, t) &= g(p, t) \quad p \in \partial\Omega,\ t \in [0, T) && \text{(BC)} \\ u(p, 0) &= \psi(p) \quad p \in \overline{\Omega} && \text{(IC)} \end{aligned} \qquad (4.2)$$

Here, $\psi \in C(\overline{\Omega})$ and $g \in C(\overline{\Omega} \times [0, T))$. Note that this is the Heisenberg version of the problem considered in [5]. We also adopt their definition that a subsolution $u(p, t)$ to Problem (4.2) is a viscosity subsolution to (E), $u(p, t) \le g(p, t)$ on $\partial\Omega$ with $0 \le t < T$ and $u(p, 0) \le \psi(p)$ on $\overline{\Omega}$. Supersolutions and solutions are defined in an analogous matter.

Theorem 4.4. *Let Ω be a bounded domain in $\mathbb{H}_n$. Let F be admissible. If u is a viscosity subsolution and v a viscosity supersolution to Problem* (4.2) *then $u \leq v$ on $[0,T) \times \Omega$.*

Proof. Our proof follows that of [5][Thm. 8.2] and so we discuss only the main parts. For $\varepsilon > 0$, we substitute $\tilde{u} = u - \frac{\varepsilon}{T-t}$ for u and prove the theorem for

$$
\begin{gathered}
u_t + F(t,p,u,\nabla u,(D^2u)^\star) \leq -\frac{\varepsilon}{T^2} < 0 \\
\lim_{t\uparrow T} u(p,t) = -\infty \quad \text{uniformly on } \overline{\Omega}
\end{gathered}
$$

and take limits to obtain the desired result. Assume the maximum occurs at $(p_0,t_0) \in \Omega \times (0,T)$ with

$$u(p_0,t_0) - v(p_0,t_0) = \delta > 0.$$

Let

$$M_\tau = u(p_\tau,t_\tau) - v(q_\tau,t_\tau) - \tau\varphi(p_\tau,q_\tau)$$

with (p_τ,q_τ,t_τ) the maximum point in $\overline{\Omega} \times \overline{\Omega} \times [0,T)$ of $u(p,t) - v(q,t) - \tau\varphi(p,q)$. Using the same proof as in [2, Lemma 5.2] we conclude that

$$\lim_{\tau\to\infty} \tau\varphi(p_\tau,q_\tau) = 0.$$

If $t_\tau = 0$, we have

$$0 < \delta \leq M_\tau \leq \sup_{\overline{\Omega}\times\overline{\Omega}} (\psi(p) - \psi(q) - \tau\varphi(p,q))$$

leading to a contradiction for large τ. We therefore conclude $t_\tau > 0$ for large τ. Since $u \leq v$ on $\partial\Omega \times [0,T)$ by Equation (BC) of Problem (4.2), we conclude that for large τ, we have (p_τ,q_τ,t_τ) is an interior point. That is, $(p_\tau,q_\tau,t_\tau) \in \Omega\times\Omega\times(0,T)$. Using the previous theorem, we obtain

$$
\begin{gathered}
(a,\tau\Upsilon(p_\tau,q_\tau),\mathcal{X}^\tau) \in \overline{P}^{2,+} u(p_\tau,t_\tau) \\
(a,\tau\Upsilon(p_\tau,q_\tau),\mathcal{Y}^\tau) \in \overline{P}^{2,-} v(q_\tau,t_\tau)
\end{gathered}
$$

that satisfy the equations

$$
\begin{gathered}
a + F(t_\tau,p_\tau,u(p_\tau,t_\tau),\tau\Upsilon(p_\tau,q_\tau),\mathcal{X}^\tau) \leq -\frac{\varepsilon}{T^2} \\
a + F(t_\tau,q_\tau,v(q_\tau,t_\tau),\tau\Upsilon(p_\tau,q_\tau),\mathcal{Y}^\tau) \geq 0.
\end{gathered}
$$

Using the fact that F is proper and that $u(p_\tau,t_\tau) \geq v(q_\tau,t_\tau)$ (otherwise $M_\tau < 0$), we have

$$
\begin{aligned}
0 < \frac{\varepsilon}{T^2} &\leq F(t_\tau,q_\tau,v(q_\tau,t_\tau),\tau\Upsilon(p_\tau,q_\tau),\mathcal{Y}^\tau) - F(t_\tau,p_\tau,u(p_\tau,t_\tau),\tau\Upsilon(p_\tau,q_\tau),\mathcal{X}^\tau) \\
&\leq \omega(d_C(p_\tau,q_\tau) + C\tau\varphi(p_\tau,q_\tau) + \|\mathcal{R}_\tau\|).
\end{aligned}
$$

We arrive at a contradiction as $\tau \to \infty$. □

We then have the following corollary, showing the equivalence of parabolic viscosity solutions and viscosity solutions.

Corollary 4.5. *For admissable F, we have the parabolic viscosity solutions are exactly the viscosity solutions.*

Proof. We showed above that parabolic viscosity sub(super-)solutions are viscosity sub(super-)solutions. To prove the converse, we will follow the proof of the subsolution case found in [12], highlighting the main details. Assume that u is not a parabolic viscosity subsolution. Let $\phi \in \mathcal{A}^- u(p_0, t_0)$ have the property that

$$\phi_t(p_0, t_0) + F(t_0, p_0, \phi(p_0, t_0), \nabla\phi(p_0, t_0), (D^2\phi(p_0, t_0))^\star) \geq \epsilon > 0$$

for a small parameter ϵ. We may assume p_0 is the origin. Let $r > 0$ and define $S_r = B_{\mathcal{N}}(r) \times (t_0 - r, t_0)$ and let ∂S_r be its parabolic boundary. Then the function

$$\tilde{\phi}_r(p, t) = \phi(p, t) + (t_0 - t)^8 - r^8 + (\mathcal{N}(p))^8$$

is a classical supersolution for sufficiently small r. We then observe that $u \leq \tilde{\phi}_r$ on ∂S_r but $u(0, t_0) > \tilde{\phi}(0, t_0)$. Thus, the comparison prinicple, Theorem 4.4, does not hold. Thus, u is not a viscosity subsolution. The supersolution case is identical and omitted. $\square$

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