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GAIN OF REGULARITY FOR A KORTEWEG - DE VRIES - KAWAHARA TYPE EQUATION

OCTAVIO PAULO VERA VILLAGRÁN

Abstract. We study the existence of local and global solutions, and the gain of regularity for the initial value problem associated to the Korteweg - de Vries - Kawahara (KdVK) equation perturbed by a dispersive term which appears in several fluids dynamics problems. The study of gain of regularity is motivated by the results obtained by Craig, Kappeler and Strauss [8].

1. Introduction

In 1976, Saut and Temam [25] remarked that a solution u of a Korteweg-de Vries type equation cannot gain or lose regularity. They showed that if $u(x,0) = \varphi(x) \in H^s(\mathbb{R})$ for $s \geq 2$, then $u(\cdot,t) \in H^s(\mathbb{R})$ for all $t > 0$. The same result was obtained independently by Bona and Scott [3] through a different method. For the Korteweg-de Vries equation on the line, Kato [17] motivated by work of Cohen [7] showed that if $u(x,0) = \varphi(x) \in L^2_b \equiv H^2(\mathbb{R}) \cap L^2(e^{bx}dx)$ $(b > 0)$ then the solution $u(x,t)$ of the KdV equation becomes C^∞ for all $t > 0$. A main ingredient in the proof was the fact that formally the semi-group $S(t) = e^{-t\partial_x^3}$ in L^2_b is equivalent to $S_b(t) = e^{-t(\partial_x - b)^3}$ in L^2 when $t > 0$. One would be inclined to believe that this was a special property of the KdV equation. This is not however the case. The effect is due to the dispersive nature of the linear part of the equation. Kruzkov and Faminskii [21] proved that for $u(x,0) = \varphi(x) \in L^2$ such that $x^\alpha \varphi(x) \in L^2((0,+\infty))$ the weak solution of the KdV equation has l-continuous space derivatives for all $t > 0$ if $l < 2\alpha$. The proof of this result is based on the asymptotic behavior of the Airy function and its derivatives, and on the smoothing effect of the KdV equation which was found in [17, 21]. Similar work for some special nonlinear Schrödinger equations was done by Hayashi et al. [13, 14] and Ponce [23]. While the proof of Kato appears to depend on special a priori estimates, some of its mystery has been resolved by the result of local gain of finite regularity for various others linear and nonlinear dispersive equations due to Constantin and Saut [11], Sjolin [26], Ginibre and Velo [12] and others. However, all of them require growth conditions on the nonlinear term.

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All the physically significant dispersive equations and systems known to us have linear parts displaying this local smoothing property. To mention only a few, the KdV, Benjamin-Ono, intermediate long wave, various Boussinesq, and Schrödinger equations are included. Continuing with the idea of Craig, Kappeler and Strauss [10] we study a equation of Korteweg - de Vries - Kawahara type (KdVK) which appears in fluids dynamics (see [24] and references therein).

$$u_t + \eta u_{xxxxx} + u_{xxx} + uu_x = 0 \tag{1.1}$$

with $-\infty < x < +\infty$, $t > 0$ and $\eta \in \mathbb{R}$. It is shown that C^∞ solutions $u(x,t)$ are obtained for all $t > 0$ if the initial data $u(x,0)$ decays faster than polynomially on $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and has certain initial Sobolev regularity. In section three we prove the main inequality. In section 4 we prove an important a priori estimate. In section 5 we prove a basic local-in-time existence and uniqueness theorem. In section 6 we prove a basic global existence theorem. In section 7 we develop a series of estimates for solutions of equation (1.1) in weighted Sobolev norms. These provide a starting point for the a priori gain of regularity. In section 8 we prove the following theorem.

Theorem 1.1. *Let $T > 0$ and $u(x,t)$ be a solution of* (1.1) *in the region $\mathbb{R} \times [0,T]$ such that*

$$u \in L^\infty([0,T]; H^5(W_{0L0})) \tag{1.2}$$

for some $L \geq 2$ and all $\sigma > 0$. Then

$$u \in L^\infty([0,T]; H^{5+l}(W_{\sigma,L-l,l})) \bigcap L^2([0,T]; H^{6+l}(W_{\sigma,L-l,l}) \cap H^{7+l}(W_{\sigma,L-l-1,l}))$$

for all $0 \leq l \leq L-1$.

2. Preliminaries

We consider the initial-value problem

$$u_t + \eta u_{xxxxx} + u_{xxx} + uu_x = 0 \tag{2.1}$$

with $-\infty < x < +\infty$, $t \in [0,T]$, T an arbitrary positive time, and $\eta \in \mathbb{R}$ is a constant.

As a notation, we use $\partial = \partial/\partial x, \partial_t = \partial/\partial t$ and $u_j = \partial^j u$, $\partial_j = \partial/\partial u_j$.

Definition. A function $\xi(x,t)$ belongs to the weight class $W_{\sigma\, i\, k}$ if it is a positive C^∞ function on $\mathbb{R} \times [0,T], \xi_x > 0$ and there are constant $c_j, 0 \leq j \leq 5$ such that

$$0 < c_1 \leq t^{-k} e^{-\sigma x} \xi(x,t) \leq c_2 \quad \forall x < -1,\ 0 < t < T, \tag{2.2}$$

$$0 < c_3 \leq t^{-k} x^{-i} \xi(x,t) \leq c_4 \quad \forall x > 1,\ 0 < t < T, \tag{2.3}$$

$$\left(t|\xi_t| + |\partial^j \xi|\right)/\xi \leq c_5 \quad \forall (x,t) \in \mathbb{R} \times [0,T], \forall j \in \mathbb{Z}^+. \tag{2.4}$$

We remark that, we will always consider $\sigma \geq 0$, $i \geq 1$ and $k \geq 0$. For example, let

$$\xi(x) = \begin{cases} 1 + e^{-1/x} & \text{for } x > 0 \\ 1 & \text{for } x \leq 0\,; \end{cases}$$

then $\xi \in W_{0i0}$.

Definition. Let s be a positive integer. We define the space

$$H^s(W_{\sigma ik}) = \{v \colon \mathbb{R} \to \mathbb{R} : \|v\|^2 = \sum_{j=0}^{s} \int_{-\infty}^{+\infty} |\partial^j v(x)|^2 \xi(x,\cdot) dx < +\infty\}$$

with $\xi \in W_{\sigma ik}$ fixed. Note that $H^s(W_{\sigma ik})$ depends on t because $\xi = \xi(x,t)$).

Lemma 2.1 ([6]). *For $\xi \in W_{\sigma i0}$ and $\sigma \geq 0$, $i \geq 0$, there exists a constant $c > 0$ such that, for $u \in H^1(W_{\sigma i0})$,*

$$\sup_{x\in\mathbb{R}} |\xi u^2| \leq c \int_{-\infty}^{+\infty} \left(|u|^2 + |\partial u|^2\right) \xi dx$$

Definition. For fixed $\xi \in W_{\sigma ik}$, we define the spaces

$$L^2([0,T]; H^s(W_{\sigma ik})) = \{v(x,t) : |||v|||^2 = \int_0^T \|v(\cdot,t)\|^2 dt < +\infty\}$$
$$L^\infty([0,T]; H^s(W_{\sigma ik})) = \{v(x,t) : |||v|||_\infty = \operatorname*{ess\,sup}_{t\in[0,T]} \|v(\cdot,t)\| < +\infty\},$$

where s is a positive integer. Note that The usual Sobolev space $H^s(\mathbb{R})$ is $H^s(W_{000})$, i.e., without weight.

We shall derive the a priori estimates assuming that the solution is C^∞, bounded as $x \to -\infty$, and rapidly decreasing together with all of its derivatives as $x \to +\infty$. We consider the following KdVK equation

$$u_t + \eta u_5 + u_3 + uu_1 = 0 \tag{2.5}$$

with $\eta \in \mathbb{R}$ constant. This equation will be studied for $-\infty < x < +\infty$, $t \in [0,T]$ with T an arbitrary positive time.

3. Main Inequality

Lemma 3.1. *Let u be a solution to (2.5) with enough Sobolev regularity (for instance, $u \in H^N(\mathbb{R}), N \geq \alpha + 5$), then*

$$\partial_t \int_{\mathbb{R}} \xi u_\alpha^2 dx + \int_{\mathbb{R}} \mu_1 u_{\alpha+1}^2 dx + \int_{\mathbb{R}} \mu_2 u_{\alpha+2}^2 dx + \int_{\mathbb{R}} \theta u_\alpha^2 dx + \int_{\mathbb{R}} R_\alpha dx \leq 0$$

with

$$\mu_1 = -c_5(5\eta + 3)\xi \quad \text{for} \quad \eta < -3/5 \quad (\text{Natural Condition})$$
$$\mu_2 = -5\eta \partial\xi$$
$$\theta = -\xi_t - \eta\partial^5\xi - \partial^3\xi - \partial(\xi u)$$
$$R_\alpha = O(u_\alpha, \dots)$$

Proof. Taking α-derivatives of (2.5) (for $\alpha \geq 3$) over $x \in \mathbb{R}$

$$\partial_t u_\alpha + \eta u_{\alpha+5} + u_{\alpha+3} + uu_{\alpha+1} + R_\alpha(u_\alpha, u_{\alpha-1}, \dots) = 0 \tag{3.1}$$

Multiply this equation by $2\xi u_\alpha$ and, integrate over $x \in \mathbb{R}$ to have

$$2\int_{\mathbb{R}} \xi u_\alpha \partial_t u_\alpha dx + 2\eta \int_{\mathbb{R}} \xi u_\alpha u_{\alpha+5} dx + 2\int_{\mathbb{R}} \xi u_\alpha u_{\alpha+3} dx$$
$$+2\int_{\mathbb{R}} \xi u u_\alpha u_{\alpha+1} dx + 2\int_{\mathbb{R}} \xi u_\alpha R_\alpha dx = 0$$

Integrating by parts,

$$\partial_t \int_{\mathbb{R}} \xi u_\alpha^2 dx + \int_{\mathbb{R}} (5\eta\partial^3\xi + 3\partial\xi) u_{\alpha+1}^2 dx$$
$$+ \int_{\mathbb{R}} -5\eta\partial\xi u_{\alpha+2}^2 dx + \int_{\mathbb{R}} \theta u_\alpha^2 dx + \int_{\mathbb{R}} R_\alpha dx = 0$$

with $\theta = -\xi_t - \eta\partial^5\xi - \partial^3\xi - \partial(\xi u)$. Using (2.4), it follows for $c_5 > 0$ that

$$\partial_t \int_{\mathbb{R}} \xi u_\alpha^2 dx - c_5(5\eta+3)\int_{\mathbb{R}} \xi u_{\alpha+1}^2 dx - 5\eta\int_{\mathbb{R}} \partial\xi u_{\alpha+2}^2 dx + \int_{\mathbb{R}} \theta u_\alpha^2 dx + \int_{\mathbb{R}} R_\alpha dx \le 0\,,$$

from where we obtain the *main inequality.*

$$\partial_t \int_{\mathbb{R}} \xi u_\alpha^2 dx + \int_{\mathbb{R}} \mu_1 u_{\alpha+1}^2 dx + \int_{\mathbb{R}} \mu_2 u_{\alpha+2}^2 dx + \int_{\mathbb{R}} \theta u_\alpha^2 dx + \int_{\mathbb{R}} R_\alpha dx \le 0 \tag{3.2}$$

with

$$\begin{gathered} \mu_1 = -c_5(5\eta+3)\xi \quad \text{for } \eta < -3/5 \quad \text{(Natural Condition)} \\ \mu_2 = -5\eta\partial\xi \\ \theta = -\xi_t - \eta\partial^5\xi - \partial^3\xi - \partial(\xi u) \\ R_\alpha = O(u_\alpha, \dots) \end{gathered}$$

□

Lemma 3.2. *For $\mu_2 \in W_{\sigma ik}$ an arbitrary weight function and $\eta < -3/5$, there exists $\xi \in W_{\sigma,i+1,k}$ that satisfies*

$$\mu_2 = -5\eta\partial\xi \tag{3.3}$$

Indeed, we have

$$\xi = -\frac{1}{5\eta}\int_{-\infty}^{x} \mu_2(y,t)dy \tag{3.4}$$

Lemma 3.3. *The expression R_α in the inequality of Lemma 3.1 is a sum of terms of the form*

$$\xi u_{\nu_1} u_{\nu_2} u_\alpha \tag{3.5}$$

where $1 \le \nu_1 \le \nu_2 \le \alpha$.

$$\nu_1 + \nu_2 = \alpha + 1 \tag{3.6}$$

Proof. Differentiating (2.5) once with respect to x and multiplying by $2\xi u_{\mathbf{1}}$ we have

$$2\xi u_{\mathbf{1}}\partial_t u_1 + 2\eta\xi u_{\mathbf{1}} u_6 + 2\xi u_{\mathbf{1}} u_4 + 2\xi u_{\mathbf{1}} u u_2 + \xi u_{\mathbf{1}} u_1 u_1 = 0\,,$$

where $R_1 \xi u_{\mathbf{1}} = \xi u_1 u_1 u_{\mathbf{1}}$. Taking 2-$x$derivatives of the equation (2.5), and multiplying by $2\xi u_{\mathbf{2}}$ we have

$$2\xi u_{\mathbf{2}}\partial_t u_2 + 2\eta\xi u_{\mathbf{2}} u_7 + 2\xi u_{\mathbf{2}} u_5 + 2\xi u_{\mathbf{2}} u u_3 + 6\xi u_{\mathbf{2}} u_1 u_2 = 0$$

where $R_2 \xi u_{\mathbf{2}} = 6\xi u_1 u_2 u_{\mathbf{2}}$. Taking 3-$x$derivatives of the equation (2.5), and multiplying by $2\xi u_{\mathbf{3}}$ we have

$$2\xi u_{\mathbf{3}}\partial_t u_3 + 2\eta\xi u_{\mathbf{3}} u_8 + 2\xi u_{\mathbf{3}} u_6 + 2\xi u_{\mathbf{3}} u u_4 + 8\xi u_{\mathbf{3}} u_1 u_3 + 6\xi u_{\mathbf{3}} u_2 u_2 = 0$$

where $R_3 \xi u_{\mathbf{3}} = 8\xi u_1 u_3 u_{\mathbf{3}} + 6\xi u_2 u_2 u_{\mathbf{3}}$. Taking 4-$x$derivatives of (2.5), and multiplying by $2\xi u_{\mathbf{4}}$ we have

$$2\xi u_{\mathbf{4}}\partial_t u_4 + 2\eta\xi u_{\mathbf{4}} u_9 + 2\xi u_{\mathbf{4}} u_7 + 2\xi u_{\mathbf{4}} u u_5 + 10\xi u_{\mathbf{4}} u_1 u_4 + 20\xi u_{\mathbf{4}} u_2 u_3 = 0$$

where $R_4 \xi u_{\mathbf{4}} = 10\xi u_1 u_4 u_{\mathbf{4}} + 20\xi u_2 u_3 u_{\mathbf{4}}$. Taking 5-$x$derivatives of (2.5), and multiplying by $2\xi u_{\mathbf{5}}$ we have

$$2\xi u_{\mathbf{5}}\partial_t u_5 + 2\eta\xi u_{\mathbf{5}} u_{10} + 2\xi u_{\mathbf{5}} u_8 + 2\xi u_{\mathbf{5}} u u_6 + 12\xi u_{\mathbf{5}} u_1 u_5 + 30\xi u_{\mathbf{5}} u_2 u_4 + 20\xi u_{\mathbf{5}} u_3 u_3 = 0$$

where $R_5 \xi u_{\mathbf{5}} = 12\xi u_1 u_5 u_{\mathbf{5}} + 30\xi u_2 u_4 u_{\mathbf{5}} + 20\xi u_3 u_3 u_{\mathbf{5}}$. Throw away the first terms in each derivative and the result follows. □

4. An a priori estimate

We show a fundamental *a priori* estimate used for a basic local-in-time existence theorem. We construct a mapping $Z : L^\infty([0,T]; H^s(\mathbb{R})) \to L^\infty([0,T]; H^s(\mathbb{R}))$ with the following property: Given $u^{(n)} = Z(u^{(n-1)})$ and $\|u^{(n-1)}\|_s \le c_0$ then $\|u^{(n)}\|_s \le c_0$, where s and $c_0 > 0$ are constants. This property tells us, in fact, that $Z : \mathbb{B}_{c_0}(0) \to \mathbb{B}_{c_0}(0)$ where $\mathbb{B}_{c_0}(0) = \{v(x,t); \|v(\cdot,t)\|_s \le c_0\}$ is a ball in $L^\infty([0,T]; H^s(\mathbb{R}))$. To guarantee this property, we will appeal to an a priori estimate which is the main object of this section. Differentiating (2.5) four times leads to

$$\partial_t u_4 + \eta u_9 + u_7 + uu_5 + 5u_1u_4 + 10u_2u_3 = 0 \tag{4.1}$$

Let $u = \wedge v$ where $\wedge = (I - \partial^4)^{-1}$. Then $\partial_t u_4 = -v_t + u_t$ by replacing in (4.1) we have

$$-v_t + \eta\wedge v_9 + \wedge v_7 + \wedge v\wedge v_5 + 5\wedge v_1\wedge v_4 + 10\wedge v_2\wedge v_3 - [\eta\wedge v_5 + \wedge v_3 - \wedge v\wedge v_1] = 0 \tag{4.2}$$

The (4.2) is linearized by substituting a new variable w in each coefficient;

$$-v_t + \eta\wedge v_9 + \wedge v_7 + \wedge w\wedge v_5 + 5\wedge w_1\wedge v_4 + 10\wedge w_2\wedge v_3 - [\eta\wedge v_5 + \wedge v_3 - \wedge w\wedge v_1] = 0 \tag{4.3}$$

Equation (4.3) is a linear equation at each iteration which can be solved in any interval of time in which the coefficients are defined. This equation has the form

$$\partial_t v = \eta\wedge v_9^{(n)} + \wedge v_7^{(n)} + b^{(1)}\wedge v_5^{(n)} + b^{(2)}\wedge v_4^{(n)} + b^{(3)} \tag{4.4}$$

We consider the following lemma that will help us setting up the iteration scheme.

Lemma 4.1. *Let $\eta < -3/5$. Given initial data $\varphi \in H^\infty(\mathbb{R}) = \bigcap_{N\ge 0} H^N(\mathbb{R})$ there exists a unique solution of (4.4) where $b^{(1)} = b^{(1)}(\wedge w), b^{(2)} = b^{(2)}(\wedge w_1)$ and $b^{(3)} = b^{(3)}(\wedge w_3, \dots, \wedge w)$ are smooth bounded coefficients with $w \in H^\infty(\mathbb{R})$. The solution is defined in any time interval in which the coefficients are defined.*

Proof. Let $T > 0$ be arbitrary and $M > 0$ a constant. Let

$$\mathcal{L} = 2\xi(\partial_t - \eta\wedge\partial^9 - \wedge\partial^7 - b^{(1)}\wedge\partial^5 - b^{(2)}\wedge\partial^4)$$

where $0 < c_6 \le \xi \le c_7$. We consider the bilinear form $B\colon \mathcal{D} \times \mathcal{D} \mapsto \mathbb{R}$,

$$B(u,v) = \langle u, v\rangle = \int_0^T \int_{\mathbb{R}} e^{-Mt} uv \, dx \, dt$$

where $\mathcal{D} = \{u \in C_0^\infty(\mathbb{R} \times [0,T]) : u(x,0) = 0\}$. We have

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{L}u \cdot u dx = 2\int_{\mathbb{R}} \xi uu_t dx - 2\eta \int_{\mathbb{R}} \xi u\wedge u_9 dx - 2\int_{\mathbb{R}} \xi u\wedge u_7 dx \\ - 2\int_{\mathbb{R}} \xi b^{(1)} u\wedge u_5 dx - 2\int_{\mathbb{R}} \xi b^{(2)} u\wedge u_4 dx\end{aligned}$$

Each term is treated separately. The first term yields

$$2\int_{\mathbb{R}} \xi uu_t dx = \partial_t \int_{\mathbb{R}} \xi u^2 dx - \int_{\mathbb{R}} \xi_t u^2 dx$$

In the second term, by integrating by parts we obtain

$$
\begin{aligned}
&-2\eta\int_{\mathbb{R}}\xi u\wedge u_9dx\\
&=-2\eta\int_{\mathbb{R}}\xi\wedge(I-\partial^4)u\wedge u_9dx-2\eta\int_{\mathbb{R}}\xi\wedge u\wedge u_9dx+2\eta\int_{\mathbb{R}}\xi\wedge u_4\wedge u_9dx\\
&=\eta\int_{\mathbb{R}}\partial^9\xi(\wedge u)^2dx-9\eta\int_{\mathbb{R}}\partial^7\xi(\wedge u_1)^2dx+27\eta\int_{\mathbb{R}}\partial^5\xi(\wedge u_2)^2dx\\
&\quad-30\eta\int_{\mathbb{R}}\partial^3\xi(\wedge u_3)^2dx-\eta\int_{\mathbb{R}}(\partial^5\xi-9\partial\xi)(\wedge u_4)^2dx+5\eta\int_{\mathbb{R}}\partial^3\xi(\wedge u_5)^2dx\\
&\quad-5\eta\int_{\mathbb{R}}\partial\xi(\wedge u_6)^2dx\,.
\end{aligned}
$$

All the others terms are calculated of the same way. We have

$$
\begin{aligned}
&\int_{\mathbb{R}}\mathcal{L}u\cdot udx\\
&=\partial_t\int_{\mathbb{R}}\xi u^2dx-\int_{\mathbb{R}}\xi_tu^2dx+\eta\int_{\mathbb{R}}\partial^9\xi(\wedge u)^2dx-9\eta\int_{\mathbb{R}}\partial^7\xi(\wedge u_1)^2dx\\
&\quad+27\eta\int_{\mathbb{R}}\partial^5\xi(\wedge u_2)^2dx-30\eta\int_{\mathbb{R}}\partial^3\xi(\wedge u_3)^2dx-\eta\int_{\mathbb{R}}(\partial^5\xi-9\partial\xi)(\wedge u_4)^2dx\\
&\quad+5\eta\int_{\mathbb{R}}\partial^3\xi(\wedge u_5)^2dx-5\eta\int_{\mathbb{R}}\partial\xi(\wedge u_6)^2dx+\int_{\mathbb{R}}\partial^7\xi(\wedge u)^2dx-7\int_{\mathbb{R}}\partial^5\xi(\wedge u_1)^2dx\\
&\quad+14\int_{\mathbb{R}}\partial^3\xi(\wedge u_2)^2dx-7\int_{\mathbb{R}}\partial\xi(\wedge u_3)^2dx-\int_{\mathbb{R}}\partial^3\xi(\wedge u_4)^2dx+3\int_{\mathbb{R}}\partial\xi(\wedge u_5)^2dx\\
&\quad+\int_{\mathbb{R}}\partial^5(\xi b^{(1)})(\wedge u)^2dx-5\int_{\mathbb{R}}\partial^3(\xi b^{(1)})(\wedge u_1)^2dx+5\int_{\mathbb{R}}\partial(\xi b^{(1)})(\wedge u_2)^2dx\\
&\quad-\int_{\mathbb{R}}\partial(\xi b^{(1)})(\wedge u_4)^2dx-\int_{\mathbb{R}}\partial^4(\xi b^{(2)})(\wedge u)^2dx+4\int_{\mathbb{R}}\partial^2(\xi b^{(2)})(\wedge u_1)^2dx\\
&\quad-\int_{\mathbb{R}}\xi b^{(2)}(\wedge u_4)^2dx+2\int_{\mathbb{R}}\xi b^{(2)}(\wedge u_4)^2dx\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_{\mathbb{R}}\mathcal{L}u\cdot udx\\
&=\partial_t\int_{\mathbb{R}}\xi u^2dx-5\eta\int_{\mathbb{R}}\partial\xi(\wedge u_6)^2dx+\int_{\mathbb{R}}(5\eta\partial^3\xi+3\partial\xi)(\wedge u_5)^2dx\\
&\quad+\int_{\mathbb{R}}(-\eta\partial^5\xi-\partial^3\xi+9\eta\partial\xi-\partial(\xi b^{(1)})+2\xi b^{(2)})(\wedge u_4)^2dx\\
&\quad+\int_{\mathbb{R}}(-30\eta\partial^3\xi-7\partial\xi)(\wedge u_3)^2dx-\int_{\mathbb{R}}\xi_tu^2dx\\
&\quad+\int_{\mathbb{R}}(27\eta\partial^5\xi+14\partial^3\xi+5\partial^3(\xi b^{(1)})-2\xi b^{(2)})(\wedge u_2)^2dx\\
&\quad+\int_{\mathbb{R}}(-9\eta\partial^7\xi-7\partial^5\xi-5\partial^3(\xi b^{(1)})+4\partial^2(\xi b^{(2)}))(\wedge u)^2dx
\end{aligned}
$$

Using (2.4), $\wedge u_n = (I-(I-\partial^4))\wedge u_{n-4} = \wedge u_{n-4} - u_{n-4}$ for n positive integer and standard estimates it follows that

$$\int_{\mathbb{R}} \mathcal{L}u \cdot u dx \geq \partial_t \int_{\mathbb{R}} \xi u^2 dx - c\int_{\mathbb{R}} \xi u^2 dx \tag{4.5}$$

Multiply this equation by e^{-Mt}, and integrate with respect to t for $t\in[0,T]$ and $u\in\mathcal{D}$.

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} e^{-Mt}\mathcal{L}u\cdot u dx dt \\
&\geq \int_0^T e^{-Mt}\big(\partial_t \int_{\mathbb{R}} \xi u^2 dx\big) dt - c\int_0^T\int_{\mathbb{R}} \xi e^{-Mt} u^2 dx dt \\
&= e^{-Mt}\int_{\mathbb{R}} \xi u^2(x,t)dx|_0^T + M\int_0^T\int_{\mathbb{R}} \xi e^{-Mt}u^2 dx dt \\
&\quad - c\int_0^T\int_{\mathbb{R}} \xi e^{-Mt}u^2 dx dt \\
&= e^{-MT}\int_{\mathbb{R}} \xi(x,T)u^2(x,T)dx + M\int_0^T\int_{\mathbb{R}} \xi e^{-Mt}u^2 dx dt \\
&\quad - c\int_0^T\int_{\mathbb{R}} \xi e^{-Mt}u^2 dx\, dt.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} e^{-Mt}\mathcal{L}u\cdot u dx dt \\
&\geq e^{-MT}\int_{\mathbb{R}} \xi(x,T)u^2(x,T)dx + (M-c)\int_0^T\int_{\mathbb{R}} \xi e^{-Mt}u^2 dx dt \\
&\geq \int_0^T\int_{\mathbb{R}} \xi e^{-Mt}u^2 dx\, dt
\end{aligned}$$

provided M is chosen large enough. Then $\langle \mathcal{L}u,u\rangle \geq \langle u,u\rangle$, for all $u\in\mathcal{D}$. Let $\mathcal{L}^* = 2\xi(-\partial_t + \eta\wedge\partial^9 + \wedge\partial^7 + b^{(1)}\wedge\partial^5 - b^{(2)}\wedge\partial^4)$ be the formal adjoint of $\mathcal{L}$. Let $\mathcal{D}^* = \{w\in C_0^\infty(\mathbb{R}\times[0,T]) : w(x,L)=0\}$. In the same way we prove that

$$\langle \mathcal{L}^* w, w\rangle \geq \langle w,w\rangle \quad \forall w\in\mathcal{D}^* \tag{4.6}$$

From this equation, we have that $\mathcal{L}^*$ is one-one. Therefore $\langle \mathcal{L}^* w, \mathcal{L}^* v\rangle$ is an inner product on $\mathcal{D}^*$. We denote by X the completion of $\mathcal{D}^*$ with respect to this inner product. By the Riesz Representation Theorem, there exists a unique solution $V\in X$, such that for any $w\in\mathcal{D}^*$, $\langle \xi b^{(3)}, w\rangle = \langle \mathcal{L}^*V, \mathcal{L}^*w\rangle$ where we use that $\xi b^{(3)}\in X$. Then if $v=\mathcal{L}^*V$ we have $\langle v,\mathcal{L}^*w\rangle = \langle \xi b^{(3)}, w\rangle$ or $\langle \mathcal{L}^*w, v\rangle = \langle w, \xi b^{(3)}\rangle$. Hence $v=\mathcal{L}^*V$ is a weak solution of $\mathcal{L}v = \xi b^{(3)}$ with $v\in L^2(\mathbb{R}\times[0,T]) \simeq L^2([0,T];L^2(\mathbb{R}))$.
Remark To obtain higher regularity of the solution, we repeat the proof with higher derivatives. It is a standard approximation procedure to obtain a result for general initial data.

The next step is to estimate the corresponding solutions $v=v(x,t)$ of the equation (4.3) via the coefficients of that equation.

Lemma 4.2. *Let $v,w\in C^k([0,+\infty);H^N(\mathbb{R}))$ for all k,N which satisfy (4.3). Let $0<c_8\leq\xi\leq c_9$ and $\eta<-3/5$. For each integer α there exist positive nondecreasing* *functions G and F such that for all $t \geq 0$,*

$$\partial_t \int_{\mathbb{R}} \xi v_\alpha^2 dx \leq G(\|w\|_\lambda)\|v\|_\alpha^2 + F(\|w\|_\alpha) \tag{4.7}$$

where $\|\cdot\|_\alpha$ is the norm in $H^\alpha(\mathbb{R})$ and $\lambda = \max\{1,\alpha\}$.

Proof. Differentiating α-times the equation (4.3), for some $\alpha \geq 0$, we obtain

$$-\partial_t v_\alpha + \eta \wedge v_{\alpha+9} + \wedge v_{\alpha+7} + \sum_{j=6}^{\alpha+5} h^{(j)} \wedge v_j + c_{10} \wedge v_3 \wedge w_{\alpha+2} + p(\wedge w_{\alpha+1}, \dots) = 0 \tag{4.8}$$

where $h^{(j)}$ is a smooth function depending on $\wedge w_i, \dots$ with $i = 5 + \alpha - j$. We multiply equation (4.8) by $2\xi v_\alpha$, and integrate over $\mathbb{R}$,

$$\begin{aligned}
-2\int_{\mathbb{R}} \xi v_\alpha \partial_t v_\alpha dx + 2\eta \int_{\mathbb{R}} \xi v_\alpha \wedge v_{\alpha+9} dx \\
+2\int_{\mathbb{R}} \xi v_\alpha \wedge v_{\alpha+7} dx + 2\sum_{j=6}^{\alpha+5} \int_{\mathbb{R}} \xi h^{(j)} v_\alpha \wedge v_j dx \\
+2c_{10}\int_{\mathbb{R}} \xi v_\alpha \wedge v_3 \wedge w_{\alpha+2} dx + 2\int_{\mathbb{R}} \xi v_\alpha p(\wedge w_{\alpha+1}, \dots) dx = 0
\end{aligned} \tag{4.9}$$

Each of these terms is treated separately. The first term yields

$$-2\int_{\mathbb{R}} \xi v_\alpha \partial_t v_\alpha dx = -\partial_t \int_{\mathbb{R}} \xi v_\alpha^2 dx + \int_{\mathbb{R}} \xi_t v_\alpha^2 dx$$

In the second term we have, by integrating by parts

$$\begin{aligned}
&2\eta \int_{\mathbb{R}} \xi v_\alpha \wedge v_{\alpha+9} dx \\
&= 2\eta \int_{\mathbb{R}} \xi \wedge (I - \partial^4) v_\alpha \wedge v_{\alpha+9} dx \\
&= 2\eta \int_{\mathbb{R}} \xi \wedge v_\alpha \wedge v_{\alpha+9} dx - 2\eta \int_{\mathbb{R}} \xi \wedge v_{\alpha+4} \wedge v_{\alpha+9} dx \\
&= -\eta \int_{\mathbb{R}} \partial^9 \xi (\wedge v_\alpha)^2 dx + 9\eta \int_{\mathbb{R}} \partial^7 \xi (\wedge v_{\alpha+1})^2 dx - 27\eta \int_{\mathbb{R}} \partial^5 \xi (\wedge v_{\alpha+2})^2 dx \\
&\quad + 30\eta \int_{\mathbb{R}} \partial^3 \xi (\wedge v_{\alpha+3})^2 dx + \eta \int_{\mathbb{R}} (\partial^5 \xi - 9\partial \xi)(\wedge v_{\alpha+4})^2 dx \\
&\quad - 5\eta \int_{\mathbb{R}} \partial^3 \xi (\wedge v_{\alpha+5})^2 dx + 5\eta \int_{\mathbb{R}} \partial \xi (\wedge v_{\alpha+6})^2 dx
\end{aligned}$$

The others terms are treated similarly. Replacing the equations obtained, on (4.9), we have

$$
\begin{aligned}
&-\partial_t\int_{\mathbb{R}}\xi v_\alpha^2dx+\int_{\mathbb{R}}\xi_t v_\alpha^2dx-\eta\int_{\mathbb{R}}\partial^9\xi(\wedge v_\alpha)^2dx+9\eta\int_{\mathbb{R}}\partial^7\xi(\wedge v_{\alpha+1})^2dx\\
&-27\eta\int_{\mathbb{R}}\partial^5\xi(\wedge v_{\alpha+2})^2dx+30\eta\int_{\mathbb{R}}\partial^3\xi(\wedge v_{\alpha+2})^2dx-3\delta\int_{\mathbb{R}}\partial^3\xi(\wedge v_{\alpha+3})^2dx\\
&+\eta\int_{\mathbb{R}}(\partial^5\xi-9\partial\xi)(\wedge v_{\alpha+4})^2dx-5\eta\int_{\mathbb{R}}\partial^3\xi(\wedge v_{\alpha+5})^2dx+5\eta\int_{\mathbb{R}}\partial\xi(\wedge v_{\alpha+6})^2dx\\
&-\int_{\mathbb{R}}\partial^7\xi(\wedge v_\alpha)^2dx+7\int_{\mathbb{R}}\partial\xi(\wedge v_{\alpha+1})^2dx-14\int_{\mathbb{R}}\partial^3\xi(\wedge v_{\alpha+2})^2dx\\
&+7\int_{\mathbb{R}}\partial\xi(\wedge v_{\alpha+3})^2dx+\int_{\mathbb{R}}\partial^3\xi(\wedge v_{\alpha+4})^2dx-3\int_{\mathbb{R}}\partial\xi(\wedge v_{\alpha+5})^2dx\\
&-\int_{\mathbb{R}}\partial^5(\xi h^{(\alpha+5)})(\wedge v_\alpha)^2dx-\int_{\mathbb{R}}\partial^3(\xi h^{(\alpha+5)})(\wedge v_{\alpha+1})^2dx\\
&-5\int_{\mathbb{R}}\partial(\xi h^{(\alpha+5)})(\wedge v_{\alpha+2})^2dx+\int_{\mathbb{R}}\partial(\xi h^{(\alpha+5)})(\wedge v_{\alpha+4})^2dx\\
&+2\sum_{j=6}^{\alpha+4}\int_{\mathbb{R}}\xi h^{(j)}v_\alpha\wedge v_jdx+2c_{10}\int_{\mathbb{R}}\xi v_\alpha\wedge v_3\wedge w_{\alpha+2}dx\\
&+2\int_{\mathbb{R}}\xi v_\alpha p(\wedge w_{\alpha+1},\dots)dx=0
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_t\int_{\mathbb{R}}\xi v_\alpha^2dx\\
&=5\eta\int_{\mathbb{R}}\partial\xi(\wedge v_{\alpha+6})^2dx-\int_{\mathbb{R}}(5\eta\partial^3\xi+3\partial\xi)(\wedge v_{\alpha+5})^2dx+\int_{\mathbb{R}}\xi_t(\wedge v_\alpha)^2dx\\
&\quad+\int_{\mathbb{R}}(\eta\partial^5\xi+\partial^3\xi-9\eta\partial\xi+\partial(\xi h^{(\alpha+5)}))(\wedge v_{\alpha+4})^2dx\\
&\quad+\int_{\mathbb{R}}(30\eta\partial^3\xi+7\partial\xi)(\wedge v_{\alpha+3})^2dx\\
&\quad+\int_{\mathbb{R}}(-27\eta\partial^5\xi-14\partial^3\xi-5\partial(\xi h^{(\alpha+5)}))(\wedge v_{\alpha+2})^2dx\\
&\quad+\int_{\mathbb{R}}(9\eta\partial^7\xi+7\partial\xi+\partial^3(\xi h^{(\alpha+5)}))(\wedge v_{\alpha+1})^2dx\\
&\quad+\int_{\mathbb{R}}(-\eta\partial^9\xi-\partial^7\xi-\partial^5(\xi h^{(\alpha+5)}))(\wedge v_\alpha)^2dx+2\sum_{j=6}^{\alpha+4}\xi h^{(j)}v_\alpha\wedge v_jdx\\
&\quad+2c_{10}\int_{\mathbb{R}}\xi v_\alpha\wedge v_3\wedge w_{\alpha+2}dx+2\int_{\mathbb{R}}\xi v_\alpha p(\wedge w_{\alpha+1},\dots)dx
\end{aligned}
$$

Using (2.4) we have that the first and the second term in the right hand side of the above expression are nonpositive. Hence

$$\begin{aligned}\partial_t \int_{\mathbb{R}} \xi v_\alpha^2 dx \le & \int_{\mathbb{R}} (\eta\partial^5\xi + \partial^3\xi - 9\eta\partial\xi + \partial(\xi h^{(\alpha+5)}))(\wedge v_{\alpha+4})^2 dx + \int_{\mathbb{R}} \xi_t (\wedge v_\alpha)^2 dx \\ & + \int_{\mathbb{R}} (30\eta\partial^3\xi + 7\partial\xi)(\wedge v_{\alpha+3})^2 dx \\ & + \int_{\mathbb{R}} (-27\eta\partial^5\xi - 14\partial^3\xi - 5\partial(\xi h^{(\alpha+5)}))(\wedge v_{\alpha+2})^2 dx \\ & + \int_{\mathbb{R}} (9\eta\partial^7\xi + 7\partial\xi + \partial^3(\xi h^{(\alpha+5)}))(\wedge v_{\alpha+1})^2 dx \\ & + \int_{\mathbb{R}} (-\eta\partial^9\xi - \partial^7\xi - \partial^5(\xi h^{(\alpha+5)}))(\wedge v_\alpha)^2 dx + 2\sum_{j=6}^{\alpha+4} \xi h^{(j)} v_\alpha \wedge v_j dx \\ & + 2c_{10}\int_{\mathbb{R}} \xi v_\alpha \wedge v_3 \wedge w_{\alpha+2} dx + 2\int_{\mathbb{R}} \xi v_\alpha p(\wedge w_{\alpha+1}, \dots) dx\end{aligned}$$

Using that $\wedge v_n = \wedge v_{n-4} - v_{n-4}$ and standard estimates, the Lemma follows. □

5. Uniqueness and Existence of a Local Solution

In this section, we study the uniqueness and the existence of local strong solutions in the Sobolev space $H^N(\mathbb{R})$ for $N \ge 5$ for the problem (2.5). To establish the existence of strong solutions for (2.5) we use the a priori estimate together with an approximation procedure.

Theorem 5.1 (Uniqueness)**.** *Let $\eta < -3/5, \varphi \in H^N(\mathbb{R})$ with $N \ge 5$ and $0 < T < +\infty$. Then there is at most one strong solution $u \in L^\infty([0,T]; H^N(\mathbb{R}))$ of (2.5) with initial data $u(x,0) = \varphi(x)$.*

Proof. Assume that $u, v \in L^\infty([0,T]; H^N(\mathbb{R}))$ are two solutions of (2.5) with $u_t, v_t \in L^\infty([0,T]; H^{N-5}(\mathbb{R}))$ and with the same initial data. Then

$$(u-v)_t + \eta(u-v)_5 + (u-v)_3 + uu_1 - vv_1 = 0 \tag{5.1}$$

with $(u-v)(x,0) = 0$. By (5.1),

$$(u-v)_t + \eta(u-v)_5 + (u-v)_3 + (u-v)u_1 + (u-v)_1 v = 0\,. \tag{5.2}$$

Multiplying (5.2) by $2\xi(u-v)$ and integrating with respect to x over $\mathbb{R}$,

$$\begin{aligned} & 2\int_{\mathbb{R}} \xi(u-v)(u-v)_t dx + 2\eta\int_{\mathbb{R}} \xi(u-v)(u-v)_5 dx \\ & + 2\int_{\mathbb{R}} \xi(u-v)(u-v)_3 dx + 2\int_{\mathbb{R}} \xi u u_1 (u-v)^2 dx \\ & + 2\int_{\mathbb{R}} \xi v(u-v)(u-v)_1 dx = 0\end{aligned} \tag{5.3}$$

Each term is treated separately. In the first term we obtain

$$2\int_{\mathbb{R}} \xi(u-v)(u-v)_t dx = \partial_t \int_{\mathbb{R}} \xi(u-v)^2 dx - \int_{\mathbb{R}} \xi_t (u-v)^2 dx$$

In the others terms, we also integrate by parts,

$$\begin{aligned}2\eta\int_{\mathbb{R}}\xi(u-v)(u-v)_5dx &= -\eta\int_{\mathbb{R}}\partial^5\xi(u-v)^2dx+5\eta\int_{\mathbb{R}}\partial^3\xi(u-v)_1^2dx\\ &= -5\eta\int_{\mathbb{R}}\partial\xi(u-v)_2^2dx\\ 2\int_{\mathbb{R}}\xi(u-v)(u-v)_3dx &= -\int_{\mathbb{R}}\partial^3\xi(u-v)^2dx+3\int_{\mathbb{R}}\partial\xi(u-v)_1^2dx\\ 2\int_{\mathbb{R}}\xi v(u-v)(u-v)_1dx &= -\int_{\mathbb{R}}\partial(\xi v)(u-v)^2dx\end{aligned}$$

Replacing these expression in (5.3), we have

$$\begin{aligned}&\partial_t\int_{\mathbb{R}}\xi(u-v)^2dx-\int_{\mathbb{R}}\xi_t(u-v)^2dx-\eta\int_{\mathbb{R}}\partial^5\xi(u-v)^2dx+5\eta\int_{\mathbb{R}}\partial^3\xi(u-v)_1^2dx\\ &-5\eta\int_{\mathbb{R}}\partial\xi(u-v)_2^2dx-\int_{\mathbb{R}}\partial^3\xi(u-v)dx+3\int_{\mathbb{R}}\partial\xi(u-v)_1^2dx\\ &+2\int_{\mathbb{R}}\xi u_1(u-v)^2dx-\int_{\mathbb{R}}\partial(\xi v)(u-v)^2dx=0\end{aligned}$$

then

$$\begin{aligned}&\partial_t\int_{\mathbb{R}}\xi(u-v)^2dx+\int_{\mathbb{R}}(5\eta\partial^3\xi+3\partial\xi)(u-v)_1^2dx-5\eta\int_{\mathbb{R}}\partial\xi(u-v)_2^2dx\\ &+\int_{\mathbb{R}}(-\xi_t-\eta\partial^5\xi-\partial^3\xi+2\xi u_1-\partial(\xi v))(u-v)^2dx=0\end{aligned}$$

By using (2.4), we obtain for $c_5>0$ and $\eta<-3/5$ that

$$\begin{aligned}&\partial_t\int_{\mathbb{R}}\xi(u-v)^2dx-c_5\int_{\mathbb{R}}(5\eta+3)\xi(u-v)_1^2dx-5\eta\int_{\mathbb{R}}\partial\xi(u-v)_2^2dx\\ &\le\int_{\mathbb{R}}(\xi_t+\eta\partial^5\xi+\partial^3\xi-2\xi u_1+\partial(\xi v))(u-v)^2dx\end{aligned}$$

and using Gagliardo-Nirenberg's inequality and standard estimates, we have

$$\partial_t\int_{\mathbb{R}}\xi(u-v)^2dx\le c\int_{\mathbb{R}}\xi(u-v)^2dx$$

By Gronwall's inequality and the fact that $(u-v)$ vanishes at $t=0$, it follows that $u=v$. This proves the uniqueness of the solution. □

We construct the mapping $Z\colon L^\infty([0,T];H^s(\mathbb{R}))\to L^\infty([0,T];H^s(\mathbb{R}))$ by

$$\begin{gathered}u^{(0)}=\varphi(x)\\ u^{(n)}=Z(u^{(n-1)})\quad n\ge 1,\end{gathered}$$

where $u^{(n-1)}$ is in place of w in equation (4.3) and $u^{(n)}$ is in place of v which is the solution of equation (4.3). By Lemma 4.1, $u^{(n)}$ exists and is unique in $C((0,+\infty);H^N(\mathbb{R}))$. A choice of c_0 and the use of the a priori estimate in §4 show that $Z\colon\mathbb{B}_{c_0}(0)\to\mathbb{B}_{c_0}(0)$ where $\mathbb{B}_{c_0}(0)$ is a bounded ball in $L^\infty([0,T];H^s(\mathbb{R}))$ □

Theorem 5.2 (Local solution). *Let $\eta<-3/5$ and N an integer ≥ 5. If $\varphi\in H^N(\mathbb{R})$, then there is $T>0$ and u such that u is a strong solution of* (2.5), *$u\in L^\infty([0,T];H^N(\mathbb{R}))$, and $u(x,0)=\varphi(x)$*

Proof. We prove that for $\varphi \in H^\infty(\mathbb{R}) = \bigcap_{k\geq 0} H^k(\mathbb{R})$ there exists a solution $u \in L^\infty([0,T]; H^N(\mathbb{R}))$ with initial data $u(x,0) = \varphi(x)$ which time of existence $T > 0$ only depends on the norm of φ. We define a sequence of approximations to equation (4.3) as

$$\begin{aligned} &- v_t^{(n)} + \eta \wedge v_9^{(n)} + \wedge v_7^{(n)} + \wedge v^{(n-1)} \wedge v_5^{(n)} - \eta \wedge v_5^{(n)} \\ &+ 5\wedge v_1^{(n-1)} \wedge v_4^{(n)} + O(\wedge v_3^{(n-1)}, \wedge v_1^{(n-1)}, \dots) = 0 \end{aligned} \tag{5.4}$$

where the initial condition $v^{(n)}(x,0) = \varphi(x) - \partial^4\varphi(x)$. The first approximation is given by $v^{(0)}(x,0) = \varphi(x) - \partial^4\varphi(x)$. Equation (5.4) is a linear equation at each iteration which can be solved in any interval of time in which the coefficients are defined. This is shown in Lemma 4.1. By Lemma 4.2, it follows that

$$\partial_t \int_{\mathbb{R}} \xi [v_\alpha^{(n)}]^2 dx \leq G(\|v^{(n-1)}\|_\lambda) \|v^{(n)}\|_\alpha^2 + F(\|v^{(n-1)}\|_\alpha) \tag{5.5}$$

Choose $\alpha = 1$ and let $c \geq \|\varphi - \partial^4\varphi\|_1 \geq \|\varphi\|_5$. For each iterate $n, \|v^{(n)}(\cdot,t)\|$ is continuous in $t \in [0,T]$ and $\|v^{(n)}(\cdot,0)\| \leq c$. Define $c_0 = \frac{c_9}{2c_8}c^2 + 1$. Let $T_0^{(n)}$ be the maximum time such that $\|v^{(k)}(\cdot,t)\|_1 \leq c_3$ for $0 \leq t \leq T_0^{(n)}$, $0 \leq k \leq n$. Integrating (5.5) over $[0,t]$ we have for $0 \leq t \leq T_0^{(n)}$ and $j = 0,1$.

$$\int_0^t \big(\partial_s \int_{\mathbb{R}} \xi [v_j^{(n)}]^2 dx\big) ds \leq \int_0^t G(\|v^{(n-1)}\|_1) \|v^{(n)}\|_j^2 ds + \int_0^t F(\|v^{(n-1)}\|_j) ds$$

It follows that

$$\begin{aligned} &\int_{\mathbb{R}} \xi(x,t) [v_j^{(n)}(x,t)]^2 dx \\ &\leq \int_{\mathbb{R}} \xi(x,0) [v_j^{(n)}(x,0)]^2 dx + \int_0^t G(\|v^{(n-1)}\|_1) \|v^{(n)}\|_j^2 ds + \int_0^t F(\|v^{(n-1)}\|_j) ds \end{aligned}$$

hence

$$\begin{aligned} c_8 \int_{\mathbb{R}} [v_j^{(n)}]^2 dx &\leq \int_{\mathbb{R}} \xi [v_j^{(n)}]^2 dx \\ &\leq \int_{\mathbb{R}} \xi(x,0) [v_j^{(n)}(x,0)]^2 dx + \int_0^t G(\|v^{(n-1)}\|_1) \|v^{(n)}\|_j^2 ds \\ &\quad + \int_0^t F(\|v^{(n-1)}\|_j) ds \end{aligned}$$

and

$$\int_{\mathbb{R}} [v_j^{(n)}]^2 dx \leq \frac{c_9}{c_8} \int_{\mathbb{R}} [v_j^{(n)}(x,0)]^2 dx + \frac{G(c_3)}{c_8} c_3^2 t + \frac{F(c_3)}{c_8} t$$

and we obtain for $j = 0,1$ that

$$\|v^{(n)}\|_1 \leq \frac{c_9}{c_8} c^2 + \frac{G(c_0)}{c_8} c_0^2 t + \frac{F(c_0)}{c_8} t$$

Claim: $T_0^{(n)}$ does not approach 0

On the contrary, assume that $T_0^{(n)} \to 0$. Since $\|v^{(n)}(\cdot,t)\|$ is continuous for $t \geq 0$, there exists $\tau \in [0,T]$ such that $\|v^{(k)}(\cdot,\tau)\|_1 = c_0$ for $0 \leq \tau \leq T_0^{(n)}, 0 \leq k \leq n$. Then

$$c_0^2 \leq \frac{c_9}{c_8} c^2 + \frac{G(c_0)}{c_8} c_0^2 T_0^{(n)} + \frac{F(c_0)}{c_8} T_0^{(n)}.$$

as $n \to +\infty$, we have

$$\Big(\frac{c_9}{2c_8}c^2+1\Big)^2 \le \frac{c_9}{c_8}c^2 \quad \Longrightarrow \quad \frac{c_9^2}{4c_8^2}c^4+1\le 0$$

which is a contradiction. Consequently $T_0^{(n)} \not\to 0$. Choosing $T = T(c)$ sufficiently small, and T not depending on n, one concludes that

$$\|v^{(n)}\|_1 \le C \tag{5.6}$$

for $0 \le t \le T$. This shows that $T_0^{(n)} \ge T$. Hence from (5.6) we imply that there exists a subsequence $v^{(n_j)} := v^{(n)}$ such that

$$v^{(n)} \overset{*}{\rightharpoonup} v \quad \text{weakly on} \quad L^\infty([0,T];H^1(\mathbb{R})) \tag{5.7}$$

Claim: $u = \wedge v$ is a solution.
In the linearized equation (5.4) we have

$$\wedge v_9^{(n)} = \wedge(I-(I-\partial^4))v_5^{(n)} = \wedge v_5^{(n)} - v_5^{(n)} = \partial^4(\underbrace{\wedge v_1^{(n)}}_{\in L^2(\mathbb{R})}) - \underbrace{\partial^4(v_1^{(n)})}_{\in H^{-4}(\mathbb{R})}$$

Since $\wedge = (I-\partial^4)^{-1}$ is bounded in $H^1(\mathbb{R})$ so $\wedge v_9^{(n)}$ belongs to $H^{-4}(\mathbb{R})$. $v^{(n)}$ is still bounded in $L^\infty([0,T];H^1(\mathbb{R})) \hookrightarrow L^2([0,T];H^1(\mathbb{R}))$ and since $\wedge : L^2(\mathbb{R}) \to H^4(\mathbb{R})$ is a bounded operator,

$$\|\wedge v_1^{(n)}\|_{H^4(\mathbb{R})} \le c_{11}\|v_1^{(n)}\|_{L^2(\mathbb{R})} \le c_{12}\|v_1^{(n)}\|_{H^1(\mathbb{R})}.$$

Consequently $\wedge v_1^{(n)}$ is bounded in $L^2([0,T];H^4(\mathbb{R})) \hookrightarrow L^2([0,T];L^2(\mathbb{R}))$. It follows that $\partial^4(\wedge v_1^{(n)})$ is bounded in $L^2([0,T];H^{-4}(\mathbb{R}))$, and

$$\wedge v_9^{(n)} \quad \text{is bounded in} \quad L^2([0,T];H^{-4}(\mathbb{R})) \tag{5.8}$$

Similarly, the other terms are bounded. By (5.4), $v_t^{(n)}$ is a sum of terms each of which is the product of a coefficient, uniformly bounded on n and a function in $L^2([0,T];H^{-4}(\mathbb{R}))$ uniformly bounded on n such that $v_t^{(n)}$ is bounded in $L^2([0,T];H^{-4}(\mathbb{R}))$. On the other hand, $H^1_{\rm loc}(\mathbb{R}) \overset{c}{\hookrightarrow} H^{1/2}_{\rm loc}(\mathbb{R}) \hookrightarrow H^{-4}(\mathbb{R})$. By Lions-Aubin's compactness Theorem [22] there is a subsequence $v^{(n_j)} := v^{(n)}$ such that $v^{(n)} \to v$ strongly on $L^2([0,T];H^{1/2}_{\rm loc}(\mathbb{R}))$. Hence, for a subsequence $v^{(n_j)} := v^{(n)}$, we have $v^{(n)} \to v$ a. e. in $L^2([0,T];H^{1/2}_{\rm loc}(\mathbb{R}))$. Moreover, from (5.8), $\wedge v_9^{(n)} \rightharpoonup \wedge v_9$ weakly in $L^2([0,T];H^{-4}(\mathbb{R}))$.

Similarly, $\wedge v_5^{(n)} \rightharpoonup \wedge v_5$ weakly in $L^2([0,T];H^{-4}(\mathbb{R}))$. Since $\|\wedge v^{(n)}\|_{H^5(\mathbb{R})} \le c_{13}\|v^{(n)}\|_{H^1(\mathbb{R})} \le c_{14}\|v^{(n)}\|_{H^{1/2}(\mathbb{R})}$ and $v^{(n)} \to v$ strongly on $L^2([0,T];H^{1/2}_{\rm loc}(\mathbb{R}))$ then $\wedge v^{(n)} \to \wedge v$ strong in $L^2([0,T];H^5_{\rm loc}(\mathbb{R})) \hookrightarrow L^2([0,T];H^4_{\rm loc}(\mathbb{R}))$. Thus the fourth term on the right hand side of (5.4), $\wedge v^{(n-1)}\wedge v_5^{(n)} \rightharpoonup \wedge v\wedge v_5$ weakly in $L^2([0,T];L^1_{\rm loc}(\mathbb{R}))$ as $\wedge v_5^{(n)} \rightharpoonup \wedge v_5$ weakly in $L^2([0,T];H^{-4}(\mathbb{R}))$ and $\wedge v^{(n-1)} \to \wedge v$ strongly on $L^2([0,T];H^4_{\rm loc}(\mathbb{R}))$. Similarly, the other terms in (5.4) converge to their limits, implying $v_t^{(n)} \rightharpoonup v_t$ weakly in $L^2([0,T];L^1_{\rm loc}(\mathbb{R}))$. Passing to the limit

$$\begin{aligned} v_t &= \partial^4(\eta\wedge v_5 + \wedge v_3 + \wedge v\wedge v_1) - (\eta\wedge v_5 + \wedge v_3 + \wedge v\wedge v_1) \\ &= -(I-\partial^4)(\eta\wedge v_5 + \wedge v_3 + \wedge v\wedge v_1) \end{aligned}$$

thus $v_t + (I-\partial^4)(\eta\wedge v_5 + \wedge v_3 + \wedge v\wedge v_1) = 0$. This way, we have that (2.5) for $u = \wedge v$. Now, we prove that there exists a solution to (2.5) with $u \in L^\infty([0,T];H^N(\mathbb{R}))$ and

$N \geq 6$, where T depends only on the norm of φ in $H^5(\mathbb{R})$. We already know that there is a solution $u \in L^\infty([0,T];H^5(\mathbb{R}))$. It is suffices to show that the approximating sequence $v^{(n)}$ is bounded in $L^\infty([0,T];H^{N-4}(\mathbb{R}))$. Taking $\alpha = N-2$ and considering (5.5) for $\alpha \geq 2$, we define $c_{N-5} = \frac{c_9}{2c_8}\|\varphi(\cdot)\|_N + 1$. Let $T^{(n)}_{N-5}$ be the largest time such that $\|v^{(k)}(\cdot,t)\|_\alpha \leq c_{N-5}$ for $0 \leq t \leq T^{(n)}_{N-5}, 0 \leq k \leq n$. Integrating (5.5) over $[0,t]$, for $0 \leq t \leq T^{(n)}_{N-5}$, we have

$$\int_0^t \Big(\partial_s \int_{\mathbb{R}} \xi [v_\alpha^{(n)}]^2 dx\Big) ds \leq \int_0^t G(\|v^{(n-1)}\|_\alpha)\|v^{(n)}\|_\alpha^2 ds + \int_0^t F(\|v^{(n-1)}\|_\alpha) ds.$$

It follows that

$$\begin{aligned}
&\int_{\mathbb{R}} \xi(x,t)[v_\alpha^{(n)}(x,t)]^2 dx \\
&\leq \int_{\mathbb{R}} \xi(x,0)[v_\alpha^{(n)}(x,0)]^2 dx + \int_0^t G(\|v^{(n-1)}\|_\alpha)\|v^{(n)}\|_\alpha^2 ds + \int_0^t F(\|v^{(n-1)}\|_\alpha) ds
\end{aligned}$$

hence

$$\begin{aligned}
c_8 \int_{\mathbb{R}} [v_\alpha^{(n)}]^2 dx &\leq \int_{\mathbb{R}} \xi [v_\alpha^{(n)}]^2 dx \\
&\leq \int_{\mathbb{R}} \xi(x,0)[v_\alpha^{(n)}(x,0)]^2 dx + \int_0^t G(\|v^{(n-1)}\|_\alpha)\|v^{(n)}\|_\alpha^2 ds \\
&\quad + \int_0^t F(\|v^{(n-1)}\|_\alpha) ds.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}} [v_\alpha^{(n)}]^2 dx &\leq \frac{c_9}{c_8}\int_{\mathbb{R}} [v_\alpha^{(n)}(x,0)]^2 dx + \frac{G(c_{N-5})}{c_8} c_{N-5}^2 t + \frac{F(c_{N-5})}{c_8} t \\
&\leq \frac{c_9}{c_8}\|v^{(n)}(\cdot,0)\|_\alpha^2 + \frac{G(c_{N-5})}{c_8} c_{N-5}^2 t + \frac{F(c_{N-5})}{c_8} t \\
&\leq \frac{c_9}{c_8}\|\varphi(\cdot,0)\|_N^2 + \frac{G(c_{N-5})}{c_8} c_{N-5}^2 t + \frac{F(c_{N-5})}{c_8} t
\end{aligned}$$

and we obtain

$$\|v^{(n)}(\cdot,t)\|_\alpha^2 \leq \frac{c_9}{c_8}\|\varphi(\cdot,0)\|_N^2 + \frac{G(c_3)}{c_8} c_3^2 t + \frac{F(c_3)}{c_8} t\,.$$

Claim: $T^{(n)}_{N-5}$ does not approach 0.

On the contrary, assume that $T^{(n)}_{N-5} \to 0$. Since $\|v^{(n)}(\cdot,t)\|$ is continuous for $t \geq 0$, there exists $\tau \in [0, T_{N-5}]$ such that $\|v^{(k)}(\cdot,\tau)\|_\alpha = c_{N-5}$ for $0 \leq \tau \leq T^{(n)}$, $0 \leq k \leq n$. Then

$$c_{N-5}^2 \leq \frac{c_9}{c_8}\|\varphi(\cdot,0)\|_N^2 + \frac{G(c_{N-5})}{c_8} c_{N-5}^2 T^{(n)}_{N-5} + \frac{F(c_{N-5})}{c_8} T^{(n)}_{N-5}.$$

as $n \to +\infty$ we have

$$\big(\frac{c_9}{2c_8}\|\varphi(\cdot,0)\|_N^2 + 1\big)^2 \leq \frac{c_9}{c_8}\|\varphi(\cdot,0)\|_N^2 \Longrightarrow \frac{c_9^2}{4c_8^2}\|\varphi(\cdot,0)\|_N^4 + 1 \leq 0$$

which is a contradiction. Then $T^{(n)}_{N-5} \not\to 0$. By choosing $T_{N-5} = T_{N-5}(\|\varphi(\cdot,0)\|_N^2)$ sufficiently small, and T_{N-5} not depending on n, we conclude that

$$\|v^{(n)}(\cdot,t)\|_\alpha^2 \leq c_{N-5}^2 \quad \text{for all} \quad 0 \leq t \leq T_{N-5}. \tag{5.9}$$

This shows that $T^{(n)}_{N-5} \geq T_{N-5}$. Thus,

$$v \in L^\infty([0,T_{N-5}];H^\alpha(\mathbb{R})) \equiv v \in L^\infty([0,T_{N-5}];H^{N-4}(\mathbb{R})).$$

Now, denote by $0 \leq T^*_{N-5} \leq +\infty$ the maximal number such that for all $0 < t \leq T^*_{N-5}$, $u = \wedge v \in L^\infty([0,t];H^N(\mathbb{R}))$. In particular $T_{N-5} \leq T^*_{N-5}$ for all $N \geq 6$. Thus, T can be chosen depending only on the norm of φ in $H^5(\mathbb{R})$. Approximating φ by $\{\varphi_j\} \in C_0^\infty(\mathbb{R})$ such that $\|\varphi - \varphi_j\|_{H^N(\mathbb{R})} \to 0$ as $j \to +\infty$. Let u_j be a solution of (2.5) with $u_j(x,0) = \varphi_j(x)$. According to the above argument, there exists T which is independent on n but depending only on $\sup_j \|\varphi_j\|$ such that u_j exists on $[0,T]$ and a subsequence $u_j \overset{j\to+\infty}{\to} u$ in $L^\infty([0,T];H^N(\mathbb{R}))$. □

As a consequence of Theorems 5.1 and 5.2 and its proof, one obtains the following result.

Corollary 5.3. *Let $\varphi \in H^N(\mathbb{R})$ with $N \geq 5$ such that $\varphi^{(\gamma)} \to \varphi$ in $H^N(\mathbb{R})$. Let u and $u^{(\gamma)}$ be the corresponding unique solutions given by Theorems 5.1 and 5.2 in $L^\infty([0,T];H^N(\mathbb{R}))$ with T depending only on $\sup_\gamma \|\varphi^{(\gamma)}\|_{H^5(\mathbb{R})}$ then*

$$\begin{gathered} u^{(\gamma)} \overset{*}{\rightharpoonup} u \quad \textit{weakly on } L^\infty([0,T];H^N(\mathbb{R})), \\ u^{(\gamma)} \to u \quad \textit{strongly on } L^2([0,T];H^{N+1}(\mathbb{R})), \\ u^{(\gamma)} \to u \quad \textit{strongly on } L^2([0,T];H^{N+2}(\mathbb{R})) \end{gathered}$$

6. Existence of Global Solutions

Here, we will try to extend the local solution $u \in L^\infty([0,T];H^N(W_{0i0}))$ of (2.5) obtained in Theorem 5.2 to $t \geq 0$. A standard way to obtain these extensions consists into deducing global estimations for the $H^N(W_{0i0})$-norm of u in terms of the $H^N(W_{0i0})$-norm of $u(x,0) = \varphi(x)$. These estimations are frecuently based on conservation laws which contain the L^2-norm of the solution and their spatial derivatives. It is not possible to do the same to give a solution of the problem of global existence because the difficulty here is that the weight depends on the variables x and t variables. To solve our problem we follow a different method using the Leibniz rule like in the proof of Theorem 3.1 of Bona and Saut, cf. [5].

Theorem 6.1. *For $\eta < -3/5$ there exists a global solution to (2.5) in the space $H^s(\mathbb{R}) \bigcap H^N(W_{0i0})$ with N integer ≥ 5 and $s \geq 2$.*

Proof. The first part was proved in [1], with $N \geq 5$ and a nonegative integer i. Taking ∂^α derivatives of the equation (2.5)

$$\partial_t u_\alpha + \eta u_{\alpha+5} + u_{\alpha+3} + (uu_1)_\alpha = 0. \tag{6.1}$$

We multiply (6.1) by $2\xi u_\alpha$ and integrate over $\mathbb{R}$.

$$2\int_{\mathbb{R}} \xi u_\alpha \partial_t u_\alpha dx + 2\eta \int_{\mathbb{R}} \xi u_\alpha u_{\alpha+5} dx + 2\int_{\mathbb{R}} \xi u_\alpha u_{\alpha+3} dx + 2\int_{\mathbb{R}} \xi u_\alpha (uu_1)_\alpha dx = 0. \tag{6.2}$$

Each term is treated separately. The first term yields

$$2\int_{\mathbb{R}}\xi u_\alpha\partial_t u_\alpha dx=\partial_t\int_{\mathbb{R}}\xi u_\alpha^2dx-\int_{\mathbb{R}}\xi_t u_\alpha^2dx.$$

In the second and third term, integrating by parts, we obtain

$$\begin{gathered}2\eta\int_{\mathbb{R}}\xi u_\alpha u_{\alpha+5}dx=-\eta\int_{\mathbb{R}}\partial^5\xi u_\alpha^2dx+5\eta\int_{\mathbb{R}}\partial^3\xi u_{\alpha+1}^2dx-5\eta\int_{\mathbb{R}}\partial\xi u_{\alpha+2}^2dx\,,\\ 2\int_{\mathbb{R}}\xi u_\alpha u_{\alpha+3}dx=-\int_{\mathbb{R}}\partial^3\xi u_\alpha^2dx+3\int_{\mathbb{R}}\partial\xi u_{\alpha+1}^2dx.\end{gathered}$$

In the last term, using the Leibniz rule, we obtain

$$\begin{aligned}&2\int_{\mathbb{R}}\xi u_\alpha(uu_1)_\alpha dx\\ &=2\int_{\mathbb{R}}\xi uu_\alpha u_{\alpha+1}dx+2\alpha\int_{\mathbb{R}}\xi u_1u_\alpha^2dx+2\frac{\alpha(\alpha-1)}{2}\int_{\mathbb{R}}\xi u_2u_{\alpha-1}u_\alpha dx\\ &\quad+2\frac{\alpha!}{3!(\alpha-3)!}\int_{\mathbb{R}}\xi u_3u_{\alpha-2}u_\alpha dx+2\frac{\alpha!}{4!(\alpha-4)!}\int_{\mathbb{R}}\xi u_4u_{\alpha-3}u_\alpha dx\\ &\quad+\dots+2\int_{\mathbb{R}}\xi u_1u_\alpha^2dx.\end{aligned}$$

Integrating by parts it follows that

$$\begin{aligned}&2\int_{\mathbb{R}}\xi u_\alpha(uu_1)_\alpha dx\\ &=-\int_{\mathbb{R}}\partial(\xi u)u_\alpha^2dx+2\alpha\int_{\mathbb{R}}\xi u_1u_\alpha^2dx-\frac{\alpha(\alpha-1)}{2}\int_{\mathbb{R}}\partial(\xi u_2)u_{\alpha-1}^2dx\\ &\quad+2\frac{\alpha!}{3!(\alpha-3)!}\int_{\mathbb{R}}\xi u_3u_{\alpha-2}u_\alpha dx+2\frac{\alpha!}{4!(\alpha-4)!}\int_{\mathbb{R}}\xi u_4u_{\alpha-3}u_\alpha dx\\ &\quad+\dots+2\int_{\mathbb{R}}\xi u_1u_\alpha^2dx.\end{aligned}$$

Substituting in (6.2), we have

$$\begin{aligned}&\partial_t\int_{\mathbb{R}}\xi u_\alpha^2dx-\int_{\mathbb{R}}\xi_tu_\alpha^2dx-\eta\int_{\mathbb{R}}\partial^5\xi u_\alpha^2dx+5\eta\int_{\mathbb{R}}\partial^3\xi u_{\alpha+1}^2dx\\ &-5\eta\int_{\mathbb{R}}\partial\xi u_{\alpha+2}^2dx-\int_{\mathbb{R}}\partial^3\xi u_\alpha^2dx+3\int_{\mathbb{R}}\partial\xi u_{\alpha+1}^2dx-\int_{\mathbb{R}}\partial(\xi u)u_\alpha^2dx\\ &\quad+2\alpha\int_{\mathbb{R}}\xi u_1u_\alpha^2dx-\frac{\alpha(\alpha-1)}{2}\int_{\mathbb{R}}\partial(\xi u_2)u_{\alpha-1}^2dx+2\frac{\alpha!}{3!(\alpha-3)!}\int_{\mathbb{R}}\xi u_3u_{\alpha-2}u_\alpha dx\\ &\quad+2\frac{\alpha!}{4!(\alpha-4)!}\int_{\mathbb{R}}\xi u_4u_{\alpha-3}u_\alpha dx+\dots+2\int_{\mathbb{R}}\xi u_1u_\alpha^2dx=0\end{aligned}$$

hence

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}} \xi u_\alpha^2 dx + \int_{\mathbb{R}} (5\eta\partial^3\xi + 3\partial\xi) u_{\alpha+1}^2 dx - 5\eta \int_{\mathbb{R}} \partial\xi u_{\alpha+2}^2 dx \\
&+ \int_{\mathbb{R}} (-\xi_t - \eta\partial^5\xi - \partial^3\xi - \partial(\xi u) + 2\alpha\xi u_1) u_\alpha^2 dx \\
&- \frac{\alpha(\alpha-1)}{2} \int_{\mathbb{R}} \partial(\xi u_2) u_{\alpha-1}^2 dx + 2\frac{\alpha!}{3!(\alpha-3)!} \int_{\mathbb{R}} \xi u_3 u_{\alpha-2} u_\alpha dx \\
&+ 2\frac{\alpha!}{4!(\alpha-4)!} \int_{\mathbb{R}} \xi u_4 u_{\alpha-3} u_\alpha dx + \dots + 2\int_{\mathbb{R}} \xi u_1 u_\alpha^2 dx = 0
\end{aligned}$$

then using (2.4), Gagliardo - Nirenberg's inequality and standard estimates we get

$$\partial_t \int_{\mathbb{R}} \xi u_\alpha^2 dx + \int_{\mathbb{R}} (5\eta + 3)\xi u_{\alpha+1}^2 dx - 5\eta \int_{\mathbb{R}} \partial\xi u_{\alpha+2}^2 dx \le c \int_{\mathbb{R}} \xi u_\alpha^2 dx\,. \tag{6.3}$$

Integrating (6.3) in $t \in [0, T_{\max} = T]$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \xi u_\alpha^2 dx + \int_0^t \int_{\mathbb{R}} (5\eta+3)\xi u_{\alpha+1}^2 dxds - 5\eta \int_0^t \int_{\mathbb{R}} \partial\xi u_{\alpha+2}^2 dxds \\
&\le \|\varphi\|_\alpha^2 + \int_0^t \big(c \int_{\mathbb{R}} \xi u_\alpha^2 dx\big) ds\,,
\end{aligned}$$

where

$$\int_{\mathbb{R}} \xi u_\alpha^2 dx \le \|\varphi\|_\alpha^2 + \int_0^t \big(c \int_{\mathbb{R}} \xi u_\alpha^2 dx\big) ds.$$

Using Gronwall's inequality

$$\int_{\mathbb{R}} \xi u_\alpha^2 dx \le \|\varphi\|_\alpha^2 e^{ct} \le \|\varphi\|_\alpha^2 e^{cT}$$

it follows that

$$\int_{\mathbb{R}} \xi u_\alpha^2 dx \le C = C(T, \|\varphi\|).$$

Then for any $T = T_{\max} > 0$, there exists $C = C(T, \|\varphi\|)$ such that

$$\|u\|_\alpha^2 + \int_0^t \int_{\mathbb{R}} (5\eta+3)\xi u_{\alpha+1}^2 dxds - 5\eta \int_0^t \int_{\mathbb{R}} \partial\xi u_{\alpha+2}^2 dxds \le C.$$

This concludes the proof. □

7. Persistence Theorem

As a starting point for the a priori gain of regularity results that will be discussed in the next section, we need to develop some estimates for solutions of the equation (2.5) in weighted Sobolev norms. The existence of these weighted estimates is often called the *persistence* of a property of the initial data φ. We show that if $\varphi \in H^5(\mathbb{R}) \bigcap H^L(W_{0i0})$ for $L \ge 0, i \ge 1$ then the solution $u(\cdot, t)$ evolves in $H^L(W_{0i0})$ for $t \in [0, T]$. The time interval of such persistence is at least as long as the interval guaranteed by the existence Theorem 5.2.

Theorem 7.1 (Persistence). *Let $i \geq 1$ and $L \geq 0$ be non-negative integers, $0 < T < +\infty$. Assume that u is the solution to* (2.5) *in $L^\infty([0,T]; H^5(\mathbb{R}))$ with initial data $\varphi(x) = u(x,0) \in H^5(\mathbb{R})$. If $\varphi(x) \in H^L(W_{0i0})$ then*

$$u \in L^\infty([0,T]; H^5(\mathbb{R}) \bigcap H^L(W_{0i0})) \tag{7.1}$$

$$\int_0^T \int_{\mathbb{R}} |\partial^{L+1} u(x,t)|^2 \mu_1 dxdt < +\infty \tag{7.2}$$

$$\int_0^T \int_{\mathbb{R}} |\partial^{L+2} u(x,t)|^2 \mu_2 dxdt < +\infty\,, \tag{7.3}$$

where σ is arbitrary, $\mu_1 \in W_{\sigma,i,0}$ and $\mu_2 \in W_{\sigma,i-1,0}$ for $i \geq 1$.

Proof. We use induction on α. Let

$$u \in L^\infty([0,T]; H^5(\mathbb{R}) \bigcap H^\alpha(W_{0i0})) \quad \text{for} \quad 0 \leq \alpha \leq L.$$

We derive formally some a priori estimate for the solution where the bound, involves only the norms of u in $L^\infty([0,T]; H^5(\mathbb{R}))$ and the norms of φ in $H^5(W_{0i0})$. We do this by approximating $u(x,t)$ through smooth solutions, and the weight functions by smooth bounded functions. By Theorem 5.2, we have

$$u(x,t) \in L^\infty([0,T]; H^N(\mathbb{R})) \quad \text{with} \quad N = \max\{L, 5\}.$$

In particular $u_j(x,t) \in L^\infty([0,T] \times \mathbb{R})$ for $0 \leq j \leq N-1$. To obtain (7.1)-(7.2) and (7.3) there are two ways of approximation perform. We approximate general solutions by smooth solutions, and we approximate general weight functions by bounded weight functions. The first of these procedures has already been discussed, so we will concentrate on the second.

Given a smooth weight function $\mu_2(x) \in W_{\sigma,i-1,0}$ with $\sigma > 0$, we take a sequence $\mu_2^\beta(x)$ of smooth bounded weight functions approximating $\mu_2(x)$ from below, uniformly on any half line $(-\infty, c)$. Define the weight functions for the α-th induction step as

$$\xi_\beta(x,t) = -\frac{1}{5\eta}\Big(1 + \int_{-\infty}^x \mu_2^\beta(y,t)dy\Big)$$

then the ξ_β are bounded weight functions which approximate a desired weight function $\xi \in W_{0i0}$ from below, uniformly on a compact set. For $\alpha = 0$, multiplying (2.5) by $2\xi_\beta u$, and integrating over $x \in \mathbb{R}$.

$$2\int_{\mathbb{R}} \xi_\beta u u_t dt + 2\eta \int_{\mathbb{R}} \xi_\beta u u_5 dx + 2\int_{\mathbb{R}} \xi_\beta u u_3 dx + 2\int_{\mathbb{R}} \xi_\beta u^2 u_1 dx = 0. \tag{7.4}$$

Each term is treated separately. In the first term we have

$$2\int_{\mathbb{R}} \xi_\beta u u_t dx = \partial_t \int_{\mathbb{R}} \xi_\beta u^2 dx - \int_{\mathbb{R}} \partial_t \xi_\beta u^2 dx.$$

For the others terms, using integration by parts, we have

$$2\eta \int_{\mathbb{R}} \xi_\beta u u_5 dx = -\eta \int_{\mathbb{R}} \partial^5 \xi_\beta u^2 dx + 5\eta \int_{\mathbb{R}} \partial^3 \xi_\beta u_1^2 dx - 5\eta \int_{\mathbb{R}} \partial \xi_\beta u^2 dx.$$

$$2\int_{\mathbb{R}} \xi_\beta u u_3 dx = -\int_{\mathbb{R}} \partial^3 \xi_\beta u^2 dx + 3\int_{\mathbb{R}} \partial \xi_\beta u_1^2 dx\,,$$

$$2\int_{\mathbb{R}} \xi_\beta u^2 u_1 dx = -\frac{2}{3}\int_{\mathbb{R}} \partial \xi_\beta u^3 dx.$$

Replacing in (7.4), we obtain

$$\begin{aligned}\partial_t \int_{\mathbb{R}} \xi_\beta u^2 dx - \int_{\mathbb{R}} \partial_t \xi_\beta u^2 dx - \eta \int_{\mathbb{R}} \partial^5 \xi_\beta u^2 dx + 5\eta \int_{\mathbb{R}} \partial^3 \xi_\beta u_1^2 dx \\ - 5\eta \int_{\mathbb{R}} \partial \xi_\beta u^2 dx - \int_{\mathbb{R}} \partial^3 \xi_\beta u^2 dx + 3 \int_{\mathbb{R}} \partial \xi_\beta u_1^2 dx - \frac{2}{3} \int_{\mathbb{R}} \partial \xi_\beta u^3 dx = 0\end{aligned}$$

then

$$\begin{aligned}\partial_t \int_{\mathbb{R}} \xi_\beta u^2 dx + \int_{\mathbb{R}} (5\eta \partial^3 \xi_\beta + 3\partial \xi_\beta) u_1^2 dx - 5\eta \int_{\mathbb{R}} \partial \xi_\beta u_2^2 dx \\ + \int_{\mathbb{R}} (-\partial_t \xi_\beta - \eta \partial^5 \xi_\beta - \partial^3 \xi_\beta - \frac{2}{3} \xi_\beta u) u^2 dx = 0\,.\end{aligned}$$

Using (2.4), for $c_5 > 0$ ($\eta < -3/5$),

$$\begin{aligned}\partial_t \int_{\mathbb{R}} \xi_\beta u^2 dx - c_5(5\eta + 3) \int_{\mathbb{R}} \xi_\beta u_1^2 dx - 5\eta \int_{\mathbb{R}} \partial \xi_\beta u_2^2 dx \\ + \int_{\mathbb{R}} (-\partial_t \xi_\beta - \eta \partial^5 \xi_\beta - \partial^3 \xi_\beta - \frac{2}{3} \xi_\beta u) u^2 dx \le 0\,.\end{aligned}$$

Using again (2.4) and Gagliardo-Nirenberg's inequality, we obtain

$$\partial_t \int_{\mathbb{R}} \xi_\beta u^2 dx - c_5(5\eta + 3) \int_{\mathbb{R}} \xi_\beta u_1^2 dx - 5\eta \int_{\mathbb{R}} \partial \xi_\beta u_2^2 dx \le c \int_{\mathbb{R}} \xi_\beta u^2 dx$$

thus

$$\partial_t \int_{\mathbb{R}} \xi_\beta u^2 dx \le c \int_{\mathbb{R}} \xi_\beta u^2 dx.$$

We apply Gronwall's lemma to conclude

$$\int_{\mathbb{R}} \xi_\beta u^2 dx \le C = C(T, \|\varphi\|) \tag{7.5}$$

for $0 \le t \le T$ and c not depending on $\beta > 0$, the weighted estimate remains true for $\beta \to 0$.

Now, we assume that the result is true for $(\alpha - 1)$ and we prove that it is true for α. To prove this, we start from the main inequality (3.2) with μ_1, μ_2 and ξ given by μ_1^β, μ_2^β and ξ_β respectively.

$$\partial_t \int_{\mathbb{R}} \xi_\beta u_\alpha^2 dx + \int_{\mathbb{R}} \mu_1^\beta u_{\alpha+1}^2 dx + \int_{\mathbb{R}} \mu_2^\beta u_{\alpha+2}^2 dx + \int_{\mathbb{R}} \theta_\beta u_\alpha^2 dx + \int_{\mathbb{R}} R_\alpha dx \le 0$$

with

$$\begin{gathered}\mu_1^\beta = -c_5(5\eta + 3)\xi_\beta \quad \text{for } \eta < -3/5 \text{ (Natural Condition)} \\ \mu_2^\beta = -5\eta \partial \xi_\beta \\ \theta_\beta = -\partial_t \xi_\beta - \eta \partial^5 \xi_\beta - \partial^3 \xi_\beta - \partial(\xi_\beta u) \\ R_\alpha = O(u_\alpha, \dots)\end{gathered}$$

then

$$\begin{aligned}\partial_t \int_{\mathbb{R}} \xi_\beta u_\alpha^2 dx + \int_{\mathbb{R}} \mu_1^\beta u_{\alpha+1}^2 dx + \int_{\mathbb{R}} \mu_2^\beta u_{\alpha+2}^2 dx &\le -\int_{\mathbb{R}} \theta_\beta u_\alpha^2 dx - \int_{\mathbb{R}} R_\alpha dx \\ &\le \Big| -\int_{\mathbb{R}} \theta_\beta u_\alpha^2 dx - \int_{\mathbb{R}} R_\alpha dx \Big| \\ &\le \int_{\mathbb{R}} |\theta_\beta| u_\alpha^2 dx + \int_{\mathbb{R}} |R_\alpha| dx\,.\end{aligned}$$

Using (2.4) and Gagliardo-Nirenberg in the first term of the right side we obtain

$$\int_{\mathbb{R}} |\theta_\beta| dx \le c \int_{\mathbb{R}} \xi_\beta u_\alpha^2 dx$$

Thus

$$\partial_t \int_{\mathbb{R}} \xi_\beta u_\alpha^2 dx + \int_{\mathbb{R}} \mu_1^\beta u_{\alpha+1}^2 dx + \int_{\mathbb{R}} \mu_2^\beta u_{\alpha+2}^2 dx \le c \int_{\mathbb{R}} \xi_\beta u_\alpha^2 dx + \int_{\mathbb{R}} |R_\alpha| dx.$$

According to (3.5), $\int_{\mathbb{R}} R_\alpha dx$ contains a term of the form

$$\int_{\mathbb{R}} \xi_\beta u_{\nu_1} u_{\nu_2} u_\alpha dx. \tag{7.6}$$

Let $\nu_2 \le \alpha - 2$. Integrating (7.6) by parts and using Hölder's inequality we obtain

$$c\Big[\Big(\int_{\mathbb{R}} \xi_\beta u_{\nu_2+1}^2 dx\Big)^{1/2} + \Big(\int_{\mathbb{R}} \xi_\beta u_{\nu_2}^2 dx\Big)^{1/2}\Big]\Big(\int_{\mathbb{R}} \xi_\beta u_{\alpha-1}^2 dx\Big)^{1/2} \tag{7.7}$$

where (7.7) is bounded by hypothesis. Now suppose that $\alpha - 1 = \nu_1 = \nu_2$, then in (7.6) we obtain

$$\Big|\int_{\mathbb{R}} \xi_\beta u_{\alpha-1} u_{\alpha-1} u_\alpha dx\Big| \le \|u_{\alpha-1}\|_{L^\infty(\mathbb{R})} \Big(\int_{\mathbb{R}} \xi_\beta u_{\alpha-1}^2 dx\Big)^{1/2} \Big(\int_{\mathbb{R}} \xi_\beta u_\alpha^2 dx\Big)^{1/2}$$

where $\|u_{\alpha-1}\|_{L^\infty(\mathbb{R})}$ is bounded by hypothesis, and the estimate is complete. Finally, for $\nu_1 = \alpha - 2; \nu_2 = \alpha - 1$ we have

$$\begin{aligned}\Big|\int_{\mathbb{R}} \xi_\beta u_{\alpha-2} u_{\alpha-1} u_\alpha dx\Big| &= \Big|\int_{\mathbb{R}} \sqrt{\xi_\beta} u_{\alpha-2} u_{\alpha-1} \sqrt{\xi_\beta} u_\alpha dx\Big| \\ &\le \big\|\sqrt{\xi_\beta} u_{\alpha-2}\big\|_{L^\infty(\mathbb{R})} \Big|\int_{\mathbb{R}} u_{\alpha-1} \sqrt{\xi_\beta} u_\alpha dx\Big| \\ &\le \big\|\sqrt{\xi_\beta} u_{\alpha-2}\big\|_{L^\infty(\mathbb{R})} \big\|u_{\alpha-1}\big\|_{L^2(\mathbb{R})} \Big(\int_{\mathbb{R}} \xi_\beta u_\alpha^2 dx\Big)^{1/2} \\ &\le c\, \|u_{\alpha-1}\|_{L^2(\mathbb{R})} \Big(\int_{\mathbb{R}} \xi_\beta u_\alpha^2 dx\Big)^{1/2}.\end{aligned}$$

Using these estimates in (7.5), and applying the Gronwall's argument, we obtain for $0 \le t \le T$,

$$\partial_t \int_{\mathbb{R}} \xi_\beta u_\alpha^2 dx + \int_{\mathbb{R}} \mu_1^\beta u_{\alpha+1}^2 dx + \int_{\mathbb{R}} \mu_2^\beta u_{\alpha+2}^2 dx \le c_0 e^{c_1 t}\Big(\int_{\mathbb{R}} \xi_\beta \varphi_\alpha^2(x) dx + 1\Big)$$

where c_0 and c_1 are independent β such that letting the parameter $\beta \to 0$ the desired estimates (7.2) and (7.3) are obtained. $\square$

8. Main Theorem

In this section we state and prove our main Theorem, which states that if the initial data $u(x,0)$ decays faster than polinomially on $\mathbb{R}^+=\{x\in\mathbb{R}; x>0\}$ and possesses certain initial Sobolev regularity, then the solution $u(x,t)\in C^\infty$ for all $t>0$. For the main Theorem, we take $6\le\alpha\le L+4$. For $\alpha\le L+4$, we take

$$\mu_1\in W_{\sigma,L-\alpha+5,\alpha-5} \quad\Longrightarrow\quad \xi\in W_{\sigma,L-\alpha+5,\alpha-5} \tag{8.1}$$

$$\mu_2\in W_{\sigma,L-\alpha+4,\alpha-5} \quad\Longrightarrow\quad \xi\in W_{\sigma,L-\alpha+5,\alpha-5} \tag{8.2}$$

Lemma 8.1 (Estimate of error terms)**.** *Let $6\le\alpha\le L+4$ and the weight functions be chosen as in (8.1)-(8.2), then*

$$\Big|\int_0^T\int_{\mathbb{R}}\big(\theta u_\alpha^2+R_\alpha\big)\,dxdt\Big|\le c\,, \tag{8.3}$$

where c depends only on the norms of u in

$$L^\infty([0,T];H^\beta(W_{\sigma,L-\beta+5,\beta-5}))\bigcap L^2([0,T];H^{\beta+1}(W_{\sigma,L-\beta+5,\beta-5})\\ \bigcap H^{\beta+2}(W_{\sigma,L-\beta+4,\beta-5}))$$

for $5\le\beta\le\alpha-1$, and the norms of u in $L^\infty([0,T];H^5(W_{0L0}))$.

Proof. We must estimate both R_α and θ. We begin with a term of R_α of the form

$$\xi u_{\nu_1}u_{\nu_2}u_\alpha \tag{8.4}$$

assuming that $\nu_1\le\alpha-2$. By the induction hypothesis, u is bounded in $L^\infty([0,T];H^\beta(W_{\sigma,L-(\beta-5)^+,(\beta-5)^+}))$ for $0\le\beta\le\alpha-1$. By Lemma 2.1,

$$\sup_{t>0}\sup_{x\in\mathbb{R}}\zeta u_\beta^2<+\infty \tag{8.5}$$

for $0\le\beta\le\alpha-2$ and $\zeta\in W_{\sigma,L-(\beta-4)^+,(\beta-4)^+}$. We estimate u_{ν_1} using (8.5). We estimate u_{ν_2} and u_α using the weighted L^2 bounds

$$\int_0^T\int_{\mathbb{R}}\zeta u_{\nu_2}^2dxdt<+\infty\quad\text{for }\zeta\in W_{\sigma,L-(\nu_2-5)^+,(\nu_2-6)^+} \tag{8.6}$$

and the same with ν_2 replaced by α. It suffices to check the powers of t, the powers of x as $x\to+\infty$ and the exponential of x as $x\to-\infty$.
For $x>1$. In the term (8.4), the factor ξ constributed according to (8.1)-(8.2)

$$\xi(x,t)=t^{(\alpha-5)}x^{(L-\alpha+5)}t^{-(\alpha-5)}x^{-(L-\alpha+5)}\xi(x,t)\le c_2t^{(\alpha-5)}x^{(L-\alpha+5)}$$

by (2.3). Then $\xi u_{\nu_1}u_{\nu_2}u_\alpha\le c_2t^{(\alpha-5)}x^{(L-\alpha+5)}u_{\nu_1}u_{\nu_2}u_\alpha$. Moreover

$$\begin{aligned}u_{\nu_1}u_{\nu_2}u_\alpha&=t^{\frac{(\nu_1-4)^+}{2}}x^{\frac{L-(\nu_1-4)^+}{2}}t^{\frac{-(\nu_1-4)^+}{2}}x^{\frac{-(L-(\nu_1-4)^+)}{2}}\\&\quad\times u_{\nu_1}t^{\frac{(\nu_2-6)^+}{2}}x^{\frac{L-(\nu_2-5)^+}{2}}t^{\frac{-(\nu_2-6)^+}{2}}x^{\frac{-(L-(\nu_2-5)^+)}{2}}u_{\nu_2}\\&\quad\times t^{\frac{(\alpha-6)^+}{2}}x^{\frac{L-(\alpha-5)^+}{2}}t^{\frac{-(\alpha-6)^+}{2}}x^{\frac{-(L-(\alpha-5)^+)}{2}}u_\alpha.\end{aligned}$$

It follows that

$$\begin{aligned}&\xi u_{\nu_1}u_{\nu_2}u_\alpha\\&\le c_2t^Mx^Tt^{\frac{(\nu_1-4)^+}{2}}x^{\frac{L-(\nu_1-4)^+}{2}}u_{\nu_1}t^{\frac{(\nu_2-6)^+}{2}}x^{\frac{L-(\nu_2-5)^+}{2}}u_{\nu_2}t^{\frac{(\alpha-6)^+}{2}}x^{\frac{L-(\alpha-5)}{2}}u_\alpha\end{aligned} \tag{8.7}$$

where $M = \alpha - 5 - \frac{1}{2}(\nu_1 - 4)^+ - \frac{1}{2}(\nu_2 - 6)^+ - \frac{1}{2}(\alpha - 6)^+$ and

$$T = (T - \alpha + 5) - \frac{1}{2}(T - (\alpha - 5)^+) - \frac{1}{2}(T - (\nu_2 - 5)^+) - \frac{1}{2}(T - (\nu_1 - 4)^+).$$

Claim $M \geq 0$ is large enough, that the extra power of t can be omitted.

$$\begin{aligned}
2M &= 2\alpha - 10 - (\nu_1 - 4)^+ - (\nu_2 - 6)^+ - (\alpha - 6)^+ \\
&= \alpha - 4 - (\nu_1 - 4)^+ - (\nu_2 - 6)^+ \\
&= \alpha - 4 - \nu_1 + 4 - \nu_2 + 6 \\
&= \alpha + 6 - (\nu_1 + \nu_2) \\
&= \alpha + 6 - (\alpha + 1) = 5 \geq 0.
\end{aligned}$$

Claim $T \leq 0$ is such that the extra power x^T can be bounded as $x \to +\infty$.

$$T = L - \alpha + 5 - \frac{1}{2}(L - (\alpha - 5)^+) - \frac{1}{2}(L - (\nu_2 - 5)^+) - \frac{1}{2}(L - (\nu_1 - 4)^+).$$

Thus

$$\begin{aligned}
2T &= 2L - 2\alpha + 10 - (L - (\alpha - 5)^+) - L + (\nu_2 - 5)^+ - L + (\alpha - 4)^+ \\
&= -L - \alpha + \nu_1 + \nu_2 - 4 \\
&= -L - \alpha + \alpha + 1 - 4 \\
&= -(L + 3) \leq 0.
\end{aligned}$$

Now, we study the behavior as $x \to -\infty$. Since each factor $u_{\nu_j} (j = 1, 2)$ must grow slower than an exponential $e^{\sigma^{'} |x|}$ and ξ decays as an exponential $e^{-\sigma|x|}$, we simply need to choose the appropriate relationship between σ and $\sigma^{'}$ at each induction step. The analysis of all the terms of R_α will be completed with the case of $\nu_1 \geq \alpha - 1$. Then in (3.6) if $2(\alpha + 1) \leq \alpha + 1$, $\alpha \leq 3$, but $\alpha \geq 5$ so this possibility is impossible. For $x < 1$ the estimate is similar, except for an exponential weight. This completes the estimate of R_α.

Now we estimate the term θu_α^2 where θ is given in (3.2). We have that θ involves derivatives of u only up to order one and hence θu_α^2 is a sum of terms of the same type which we have already encountered in R_α. So, its integral can be bounded in the same manner. Indeed (3.2) shows that θ depends on $\xi_t, \partial^5 \xi$ and derivatives of lower order. By using (3.3) we have the claim. $\square$

Theorem 8.2 (Main Theorem). *Let $T > 0$ and $u(x,t)$ be a solution of* (2.5) *in the region $\mathbb{R} \times [0,T]$ such that*

$$u \in L^\infty([0,T]; H^5(W_{0L0})) \tag{8.8}$$

for some $L \geq 2$ and all $\sigma > 0$. Then u is in $L^\infty([0,T]; H^{5+l}(W_{\sigma,L-l,l}))$ $\cap\, L^2([0,T]; H^{6+l}(W_{\sigma,L-l,l}) \cap H^{7+l}(W_{\sigma,L-l-1,l}))$ for all $0 \leq l \leq L - 1$.

Remark 8.3. If the assumption (8.8) holds for all $L \geq 2$, the solution is infinitely differentiable in the x-variable. From (2.5) we have that the solution is C^∞ in both variables.

Proof. We use induction on α. For $\alpha = 5$, let u be a solution of (2.5) satisfying (8.8). Therefore, $u_t \in L^\infty([0,T]; L^2(W_{0L0}))$ where $u \in L^\infty([0,T]; H^5(W_{0L0}))$ and $u_t \in L^\infty([0,T]; L^2(W_{0L0}))$. Then $u \in C([0,T]; L^2(W_{0L0})) \bigcap C_w([0,T]; H^5(W_{0L0}))$. Hence $u \colon [0,T] \mapsto H^5(W_{0L0})$ is a weakly continuous function. In particular, $u(\cdot, t) \in H^5(W_{0L0})$ for all t. Let $t_0 \in (0,T)$ and $u(\cdot, t_0) \in H^5(W_{0L0})$, then there are $\{\varphi^{(n)}\} \subset$

$C_0^\infty(\mathbb{R})$ such that $\varphi^{(n)}(\cdot) \to u(\cdot, t_0)$ in $H^5(W_{0L0})$. Let $u^{(n)}(x,t)$ be a unique solution of (2.5) with $u^{(n)}(x,t_0) = \varphi^{(n)}(x)$. Then by Theorems 5.1 and 5.2, there exists in a time interval $[t_0, t_0+\delta]$ where $\delta > 0$ does not depend on n and u is a unique solution of (2.5) $u^{(n)} \in L^\infty([t_0, t_0+\delta]; H^5(W_{0L0}))$ with $u^{(n)}(x,t_0) \equiv \varphi^{(n)}(x) \to u(x,t_0) \equiv \varphi(x)$ in $H^5(W_{0L0})$. Now, by Theorem 7.1, we have

$$u^{(n)} \in L^\infty([t_0, t_0+\delta]; H^5(W_{0L0})) \bigcap L^2([t_0, t_0+\delta]; H^6(W_{\sigma L0}) \cap H^7(W_{\sigma, L-1,0}))$$

with a bound that depends only on the norm of $\varphi^{(n)}$ in $H^5(W_{0L0})$. Furthermore, Theorem 7.1 guarantees the non-uniform bounds

$$\sup_{[t_0,t_0+\delta]} \sup_x (1+|x_+|)^k |\partial^\alpha u^{(n)}(x,t)| < +\infty$$

for each n, k and α. The main inequality (3.2) and the estimate (8.3) are therefore valid for each $u^{(n)}$ in the interval $[t_0, t_0+\delta]$. μ_2 may be chosen arbitrarily in its weight class (8.1) and then ξ is defined by (3.4) and the constant c_1, c_2, c_3, c_4 are independent of n. From (3.2) and (8.1)-(8.2) we have

$$\sup_{[t_0,t_0+\delta]} \int_{\mathbb{R}} \xi [u_\alpha^{(n)}]^2 dx + \int_{t_0}^{t_0+\delta} \int_{\mathbb{R}} \mu_1 [u_{\alpha+1}^{(n)}]^2 dx dt + \int_{t_0}^{t_0+\delta} \int_{\mathbb{R}} \mu_2 [u_{\alpha+2}^{(n)}]^2 dx dt \le c \quad (8.9)$$

where by (8.3), c is independ of n. Estimate (8.9) is proved by induction for $\alpha = 5, 6, \dots$ Thus $u^{(n)}$ is also bounded in

$$\begin{aligned} L^\infty([t_0, t_0+\delta]; H^\alpha(W_{\sigma, L-\alpha+5, \alpha-5})) \bigcap L^2([t_0, t_0+\delta]; H^{\alpha+1}(W_{\sigma, L-\alpha+5, \alpha-5})) \\ \bigcap L^2([t_0, t_0+\delta]; H^{\alpha+2}(W_{\sigma, L-\alpha+4, \alpha-5})) \end{aligned} \quad (8.10)$$

for $\alpha \ge 5$. Since $u^{(n)} \longrightarrow u$ in $L^\infty([t_0, t_0+\delta]; H^5(W_{0L0}))$. By Corollary 5.3 it follows that u belongs to the space (8.10). Since δ is fixed, this result is valid over the whole interval $[0, T]$. □

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Facultad de Ingeniería, Universidad Católica de la Santísima Concepción, Paicaví 3000, Concepción - Chile

E-mail address: `overa@ucsc.cl` `Fax: (41)735300`