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# MULTI POINT BOUNDARY-VALUE PROBLEMS AT RESONANCE FOR N-ORDER DIFFERENTIAL EQUATIONS: POSITIVE AND MONOTONE SOLUTIONS

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Abstract. In this article, we study a complete $n$-order differential equation subject to the $(p, n-p)$ right focal boundary conditions plus an additional nonlocal constrain. We establish sufficient conditions for the existence of a family of positive and monotone solutions at resonance. The emphasis in this paper is not only that the nonlinearity depends on all higher-order derivatives but mainly that the obtaining solution satisfies the above extra condition. Our approach is based on the Sperner's Lemma, proposing in this way an alternative to the classical methodologies based on fixed point or degree theory and results the introduction of a new set of quite natural hypothesis.

## 1. Introduction

Consider the differential equation

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad 0 < t < \gamma,\ (\gamma > 1) \tag{1.1}$$

subject to the multi-point boundary conditions

$$\begin{gathered} x^{(i)}(0) = 0 \quad \text{for } i = 0, 1, \dots, p-1, \\ x^{(i)}(1) = 0 \quad \text{for } i = p, p+1, \dots, n-1, \\ \sum_{i=1}^{m} \alpha_i x^{(j)}(\xi_i) = 0, \end{gathered} \tag{1.2}$$

where $p \leq j \leq n-1$ and $m \in \mathbb{N}$ are fixed numbers. The boundary value problem

$$\begin{gathered} x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad 0 < t < \gamma, \\ x^{(i)}(0) = 0, \quad \text{for } i = 0, 1, \dots, p-1, \\ x^{(i)}(1) = 0 \quad \text{for } i = p, \dots, n-1, \end{gathered} \tag{1.3}$$

is called the $(p, n-p)$ right focal boundary-value problem [1, 2, 5, 10, 14] and it is a particular case of (1.1)-(1.2). As a matter of fact, (1.3) and its special cases have already been studied by a number of authors (e. g. [4, 17, 19, 22]). In all

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these cases, $f$ depends only on $t$ and $x(t)$, or on $t$ and derivatives of even order: $x(t), x''(t), \dots, x^{(2n-2)}$.

When $\sum_{i=1}^m \alpha_i \neq 0$, the linear operator $Lx(t) = x^{(n)}(t)$, defined in a suitable Banach space, is invertible. This is called the non-resonance case; otherwise, when $\sum_{i=1}^m a_i = 0$, it is called the resonance one. If this is the case, the homogeneous ($f = 0$) boundary value problem, subject to conditions (1.2) has the nontrivial solution $x(t) = c$, where $c \in \mathbb{R}$ is a constant and so $\ker L \neq \{0\}$.

In the recent years, a lot of work have been done on the solvability of multi-point boundary-value problems for second order differential equations at resonance case (see e.g. [11] and the references therein), since these type of boundary-value problems occur frequently in many applications. For example we refer here to the way of determining the speed of a fragellate protozoan in a viscous fluid as well as to the study of perfectly wetting liquids (see [8] for details).

Eloe and Henderson noticed in [8] that for $n = 2$, positive solutions of (1.1)-(1.3) are convex. They also consider a conjugate boundary-value problem (BVP) and using Green functions and the Krasnosel'skii fixed point theorem on a positive cone, they obtain existence results, under sub or superlinearity on the nonlinearity $f$. This convexity defines a vector field in the face-plane of (1.1), the properties of which permit us to verify the assumptions of Sperner's Lemma and then to apply it in order to obtain positive solutions with also positive sign of a number of their first derivatives.

In a similar way, Agarwal and O'Regan in [3], by using inequalities on the Green functions and the nonlinear alternative of Leray-Schauder, obtained existence results of a conjugate or/and a focal boundary-value problem (BVP), under smallness and sign assumptions on $f$, mainly if

$$|f(t, x, x', \dots, x^{(n-1)})| \leq \alpha(t)\psi(|u|)$$

where for a certain $k_0$, $\sup_{c\in(0,\infty)} c/\psi(c) > k_0$.

Chyan and Henderson [7], consider the (between conjugate and focal) BVP

$$\begin{gathered}
(-1)^{n-k} x^{(n)} = \lambda a(t) f(t, x), \\
x^{(i)}(0) = 0,\ 0 \leq i \leq k-1 \\
x^{(l)}(1) = 0,\ j \leq l \leq j+n-k-1
\end{gathered}$$

for a (fixed) $1 \leq j \leq k-1$ and established values of $\lambda$ to get positive solutions to that problem, assuming similar conditions. A general overview off much of the work which has been done on these subjects and the methods used is given in the book of Agarwal, O'Regan and Wong [5].

Consider the $2n$-order nonlinear scalar differential equation

$$x^{(2n)}(t) = f(t, x(t), x'(t), \dots x^{(2n-1)}(t)), \quad 0 \leq t \leq \gamma, \tag{1.4}$$

and further the associated $(2k, 2(n-k))$ multi-point focal boundary value problem defined by:

$$\begin{gathered}
x^{(i)}(0) = 0,\ 0 \leq j \leq 2k-1 \\
x^{(j)}(1) = 0,\ 2k \leq j \leq 2n-1, \\
\sum_{i=1}^m a_i x^{(2p)}(\xi_j) = 0
\end{gathered} \tag{1.5}$$

where $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ is continuous, $m \geq 2$, $n \geq 2$ are integers, $1 \leq k \leq n-1$, $p \in \{k, k+1, \dots, n-1\}$, $\alpha_i \in \mathbb{R}$ $(i = 1, 2, \dots, m)$ with $\sum_{i=1}^{m} a_i = 0$ and $0 \leq \xi_1 < \xi_2 < \dots < \xi_m \leq 1$ are fixed. Consider the cone

$$\begin{aligned}\mathbb{K}_0 = \big\{&(x, x', \dots, x^{(2n-1)}) \in \mathbb{R}^{2n}/\{0\} : x^{(i)} \geq 0,\ 0 \leq i \leq 2k-1 \\ &\text{and } (-1)^j x^{(j)} \geq 0,\ 2k \leq j \leq 2n-1\big\}.\end{aligned}$$

We assume throughout this paper that the function $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ is continuous and positive:

$$f(t, x, x', x'', \dots, x^{(2n-1)}) > 0, \text{ on the cone } \mathbb{K}_0\,. \tag{1.6}$$

Further assume that $f$ is:

(1) Nondecreasing on every of its last $2(n-k)$ variables and (strictly) increasing in (at least) one of $n-k-1$ even-order derivatives $x^{(2k)}, x^{(2k+2)}, \dots, x^{(2n-2)}$
(2) Bounded at $-\infty$ on every of its last $n-k-1$ odd-order derivatives $x^{(2k+1)}$, $\dots$, $x^{(2n-1)}$, uniformly for

$$\big(t, x, x', \dots, x^{(2k-2)}, x^{(2k-1)}, x^{(2k)}, x^{(2k+2)}, x^{(2k+4)}, \dots, x^{(2n-2)}\big) \in W,$$

where $W$ is any compact subset of $[0,1] \times \mathbb{R}^{n-k}$:

$$\lim_{x^{(2\rho+1)} \to -\infty} f(t, x(t), x'(t), \dots x^{(2n-1)}(t)) \leq K,\ (\rho = k, k+1, \dots, n-1). \tag{1.7}$$

In a recent paper (published in this journal) Liu and Ge [13] based on the *coincidence degree method* of Gaines and Mawhin [12] proved that the boundary-value problem at resonance (without any extra condition) (1.1)–(1.8), where

$$\begin{gathered} x^{(i)}(0) = 0 \quad \text{for } i = 0, 1, \dots, p-1, \\ x^{(i)}(1) = 0 \quad \text{for } i = p+1, \dots, n-1, \\ \sum_{i=1}^{m} \alpha_i x^{(p)}(\xi_i) = 0, \end{gathered} \tag{1.8}$$

has at least one solution, under the following assumptions:

(A1) There is $M > 0$ such that for any $x \in \operatorname{dom} L/\ker L$, with $|x^{(p)}(t)| > M$ for all $t \in (0, \frac{1}{2})$, it follows that

$$\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^{1} \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s, x(s), x'(s), \dots, x^{(n-1)}(s))ds \neq 0$$

(A2) There is a function $a \in C^0[0,1]$ and positive numbers $a_i$ and $\beta_i \in [0,1)$ $(i = 0, 1, \dots, n-1)$ such that

$$|f(t, x_0, x_1, \dots, x_{n-1})| \leq a(t) + \sum_{i=0}^{n-1} a_i |x_i|^{\beta_i}$$

for $t \in [0,1]$ and $(x_0, x_1, \dots, x_{n-1}) \in \mathbb{R}^n$

(A3) There is $M^* > 0$ such that for any $c \in \mathbb{R}$ then either

$$c \sum_{i=1}^{m} \alpha_i \int_{\xi_i}^{1} \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s, cs^p, cps^{p-1}, \dots, cp!, 0, \dots, 0)ds < 0,$$

or

$$c\sum_{i=1}^{m}\alpha_i\int_{\xi_i}^{1}\frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!}f(s,cs^p,cps^{p-1},\dots,cp!,0,\dots,0)ds>0,$$

for any $c\in\mathbb{R}$ with $|c|>M^*$.

Motivated and inspired by [13] see also [9, 11, 15, 21], we establish in this paper sufficient conditions for the existence of at least one solution of (1.1)–1.2 at resonance. Our principal tool for the analysis of trajectories of (1.4) will be a theorem of combinatorial topology, namely the Knaster-Kuratowski-Mazurkiewicz's principle or (as it is known) Sperner's Lemma (cf. [6]), as we did in [20]. It is worth noticing that the use of Sperner's Lemma in this study, gives an alternative to the usual considerations of topological methods, such as fixed point theories, and results to more strongly conclusions under possibly weaker but in any case much different assumptions. In addition, the most known results on focal or conjugate BVP are based on conditions which are expressed in terms of Green's functions a fact making the resulting criteria too complicated for practical use. In this paper, conditions are of simple form which contain no explicit reference to the Green's functions. Moreover we get a whole $(n-p-1)-$ parametric family of solutions of BVP (1.4)-(1.5) any member of which satisfies properties as the above (see Remark 2.8 at the end of paper).

We study first two special cases of (1.4). More precisely, for the $(2n)$-order differential equation (1.4) and the $(2k,2(n-k))$ multi-point focal boundary value conditions (1.5) *we need only natural assumptions like monotonicity, a kind of boundednes of $f$ and a sign property* (see (1.6)-(1.7), so we do not assume any growth or separation constraint on $f$ ). Then the obtaining solution $x(t)$ is not only positives, but it has its firth $2k$ derivatives also positive on the interval $(0,1]$, i.e.

$$x^{(i)}(t)>0,\quad 0<t\le 1,\ 0\le i\le 2k-1.$$

Furthermore we prove that

$$x^{(2\rho)}(t)>0,\quad x^{(2\rho+1)}(t)<0,\quad 0<t<1,\ k\le\rho<n-1.$$

A similar result is obtaining for the $(2p+1,2(n-p))$ focal boundary-value problem

$$x^{(2n+1)}(t)=f(t,x(t),x'(t),\dots x^{(2n)}(t)),\quad 0\le t\le 1,$$

$$x^{(j)}(0)=0,\quad 0\le j\le 2p,$$

$$x^{(j)}(1)=0,\quad 2p+1\le j\le 2n,\quad \sum_{i=1}^{m}\alpha_i x^{(j)}(\xi_i)=0;$$

i.e., obtaining solution $x=x(t)$, $t\in[0,1]$ satisfying

$$x^{(i)}(t)>0,\quad 0<t<1,\quad 0\le\ i\le 2p$$

and further

$$x^{(2\rho+1)}(t)\ge 0,\quad x^{(2\rho+2)}(t)\le 0,\quad 0<t<1,\ \ p\le\rho<n-1.$$

As a last result of this work, we indicate how one can study the general boundary-value problem (1.1)-(1.2) by applying our Theorem 2.7.

For the sequel, we need some preliminary material and a classical result. Let $p_0, p_1, \dots, p_{n-k}$ be $n-k+1$ points of the $(n-k)$-dimensional Euclidean space $\mathbb{R}^{n-k}$. Then the *simplex* $S = [p_0 p_1 \dots p_{n-k}]$ is defined by

$$S := \{p \in \mathbb{R}^{n-k} : \exists\, \lambda_i \geq 0 \text{ with } \sum_{i=0}^{n-k} \lambda_i = 1 \text{ and } p = \sum_{i=0}^{n-k} \lambda_i p_i\}.$$

The points $p_0, p_1, \dots, p_{n-k}$ are called *vertices* of the simplex and the simplex $[p_{i_0} p_{i_1} \dots p_{i_k}]$, $0 \leq k \leq n-k-1$, is a *face* of $S$. If the vectors $p_0, p_1, \dots, p_{n-k}$ are linearly independent, then $S$ is an $n-k$-dimensional simplex spanned by these points.

Our principle is based on the following result from combinatorial analysis, known as Sperner's lemma ([6] or [16, Theorem 5, p. 310]).

**Lemma 1.1.** *Let $S$ be a closed $n$-simplex with vertices $\{e^0, e^1, \dots, e^{n-k}\}$ and let $\{E_1, E_2, \dots, E_{n-k}\}$ be a closed covering of $S$ such that any closed face $[e^{i_0} e^{i_1} \dots e^{i_k}]$ of $S$ is containing in the union $E_{i_0} \cup E_{i_1} \cup \dots \cup E_{i_k}$. Then the intersection $\cap_{i=0}^{n-k} E_i$ is nonempty.*

## 2. Main Result

To study the general boundary value problem (1.4)-(1.5), we assume first that both $n$ and $p$ are even integers. In this way we form the particular case,

$$x^{(2n)}(t) = f(t, x(t), x'(t), \dots x^{(2n-1)}(t)), \quad 0 \leq t \leq \gamma,\ \gamma > 1 \tag{2.1}$$

associated wit the $(2k, 2(n-k)$ multipoint focal value problem

$$\begin{gathered} x^{(i)}(0) = 0, \quad 0 \leq i \leq 2k-1, \\ x^{(j)}(1) = 0, \quad 2k \leq j \leq 2n-1, \\ \sum_{i=1}^{m} a_i x^{(2p)}(\xi_j) = 0. \end{gathered} \tag{2.2}$$

It will be convenient to represent (2.1) as a second order system of the form

$$X''(t) = f(t, X, X')$$

where, for notational reasons, we set $X = (Y, Z)$, if

$$\begin{gathered} Y = (y_0, y_1, \dots, y_{2n_1-1}) = (x, x^{''}, \dots, x^{(2k-2)}) \in \mathbb{R}^k, \\ Z = (z_0, z_1, \dots, z_{2(n-k)}) = (x^{(2k)}, x^{(2k+2)}, \dots, x^{(2n-2)}) \in \mathbb{R}^{(n-k)}. \end{gathered}$$

Then the boundary conditions at (2.2) take the form

$$Y(0) = Y'(0) = 0 \quad \text{and} \quad Z(1) = Z'(1) = 0.$$

For a (fixed) $\alpha > 0$, solutions of (2.1) are defined by trajectories of the initial value problems

$$\begin{gathered} x^{(i)}(0) = 0, \quad 0 \leq i \leq 2k-1, \\ x^{(2j)}(0) = \alpha, \\ x^{(2j+1)}(0) = -\lambda_{j+1} \leq 0, \quad k \leq j \leq n-1; \end{gathered} \tag{2.3}$$

i.e. (2.3) can be written in vector notation as

$$Y(0) = 0 = Y'(0) \quad \text{and} \quad Z(0) = \alpha(1, 1, \dots, 1),\ Z'(0) = v, \tag{2.4}$$

where $v = -(\lambda_k, \lambda_{k+1}, \dots, \lambda_{n-k}) \in \mathbb{R}^{n-k}$. A solution of the initial value problem (2.1)-(2.4) will be denoted by

$$X = X(t;v) = (Y(t,v), Z(t,v)) \quad \text{or simply } x = x(t;v).$$

Assume that $\mathbb{K}$ denotes the closed positive cone of $\mathbb{R}^{n-k}$ and let $\partial\mathbb{K}$ be its boundary, which consists of the hyperplanes

$$H_i = \{x \in \mathbb{R}^{n-k} : x_i = 0,\ x_j \geq 0,\ j \neq i\} \quad (i = k, k+1, \dots, n-1).$$

**Definition 2.1.** The trajectory $X(t,v)$ egresses from $\mathbb{K}$ through $H_i$, whenever there exists $0 < t_1 \leq 1$ such that $z_i(t,v) = x^{(2i)}(t,v) \geq 0$, $z_i(t_1,v) = 0$ and $z_j(t,v) = x^{(2j)}(t,v) > 0$ for $0 \leq t \leq t_1$ $(j \neq i)$.

If moreover there exists an $\varepsilon > 0$ such that $z_i(t,v) < 0$, $t_1 < t \leq t_1 + \varepsilon$, then $X(t,v)$ egresses strictly from the cone $\mathbb{K}$ through $H_i$.

We also consider the modification

$$F(t,X,X') = F(t,Y,Y',Z,Z') := \begin{cases} f(t,Y,Y',Z,Z_0'), & \text{if } Z' \not\leq 0 \\ f(t,Y,Y',Z,Z'), & \text{otherwise,} \end{cases}$$

(we replace by 0 only the positive coordinates of $Z'$, so $Z_0' \geq 0$, where the inequality $Z_0' \geq 0$ must be regarded components-wise) as well as the associating equation

$$X'' = F(t,X,X') \tag{2.5}$$

The next lemma shows that, if a trajectory satisfies a certain initial condition of type (2.3) or (2.4), then it egresses from $\mathbb{K}$ through any hyperplane $H_i$, $(i = k, k+1, \dots, n-1)$ for some $t_1 \leq 1$. Let $e^i := (\delta_{i1}, \delta_{i2}, \dots, \delta_{i(n-k)}) \in \mathbb{R}^{n-k}$, where

$$\delta_{ij} := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

is the well-known Kronecker delta function.

**Lemma 2.2.** *Assume that $f$ is continuous and satisfies* (1.6)-(1.7)*. Then for each $\rho = k, k+1, \dots, n-1$ there exists a $\lambda_\rho > 0$ such that for any $\lambda > \lambda_\rho$, any trajectory $X(t,v)$ egresses from $\mathbb{K}$ through $H_\rho$, for some $t_1 \leq 1$.*

*Proof.* Taking into account the assumption (1.6) we may write for all $0 \leq t \leq \gamma$,

$$F(t,x,x',\dots x^{(2n-1)}) > 0, \quad Y \geq 0, \quad Y' \geq 0, \quad Z > 0, \quad Z' \leq 0\,. \tag{2.6}$$

Consider now, for any (fixed) $\rho = k, \dots, n-1$, a solution $X(t, \lambda e^\rho)$ of the differential equation (2.5), satisfying the initial conditions

$$\begin{gathered} y_j(0,\lambda e^\rho) = 0, \quad y_j'(0,\lambda e^\rho) = 0 \quad (j = 0,1,\dots,2k-1), \\ z_j(0,\lambda e^\rho) = \alpha, \quad z_j'(0,\lambda e^\rho) = 0,\ k \leq j \leq n-1,\ (j \neq \rho), \\ z_\rho(0,\lambda e^\rho) = \alpha, \quad z_\rho'(0,\lambda e^\rho) = -\lambda. \end{gathered} \tag{2.7}$$

Since $z_\rho(0;\lambda e^\rho) = \alpha > 0$, by the above representation of $X = (Y,Z)$, there exists a $\bar{t} > 0$ such that for $0 < t < \bar{t}$ the coordinates of $X$ satisfy $y_j(t) > 0$ and $y_j'(0,\lambda e^\rho) > 0$ when $j = 0,1,\dots,k-1$ and $z_j(t) > 0$ and $z_j'(t) > 0$ for $j = k,\dots,\rho-1$. By the modification $F$ of $f$ and (2.6), it follows also that $z_j(t) > 0$ and $z_j'(t) > 0$, $0 < t < \bar{t}$ for $j = \rho+1,\dots,n-1$. Furthermore each component $y_j(t), y_j'(t)$, $z_j(t)$ and $z_j'(t)$ is an increasing function as long as $z_\rho(t,\lambda e^\rho) > 0$. As a result, if some component has a zero in $(0,1]$, the first such zero must be in $z_\rho(t,\lambda e^\rho)$.

Therefore, we must show that for $\lambda$ sufficiently large the tranjectory $X(t;\lambda e^\rho)$ egresses (strictly) from the positive cone $\mathbb{K}$ for some $t_1 \le 1$.

Assume that this is not the case, i.e. for every $\lambda \in \mathbb{R}$,

$$x^{(2\rho)}(t;\lambda e^\rho) \ge 0, \quad 0 < t \le 1. \tag{2.8}$$

Then by the differential equation (2.5) and the Taylor's formula, for any $t \le 1$ we get a point $\hat{t} \le t$ such that (set for simplicity $x(t) = x(t;v) = x(t;\lambda e^\rho)$)

$$x^{(2\rho)}(t) = \sum_{j=2\rho}^{2n-1} \frac{t^j}{j!} x^{(j)}(0) + \frac{t^{2n}}{(2n)!} x^{(2n)}(\hat{t}).$$

Thus, in view of (2.7) and the choice $v = \lambda e^\rho$, we get

$$x^{(2\rho)}(t) = \alpha \sum_{j=\rho}^{n-1} \frac{t^{2j}}{(2j)!} - \frac{t^{2\rho+1}}{(2\rho+1)!}\lambda + \frac{t^{2n}}{(2n)!} F(t, X(\hat{t}), X'(\hat{t})). \tag{2.9}$$

If we prove that there is $M > 0$ independent of $\hat{t}$ and $\lambda$ such that

$$|F(\hat{t}, X(\hat{t}), X'(\hat{t}))| \le M, \tag{2.10}$$

then letting $\lambda \to +\infty$, we obviously obtain $x^{(2\rho)}(t) < 0$, i.e. a contradiction to (2.8). By the above analysis, the solution $x(t)$ egresses from the cone $\mathbb{K}$ through $H_\rho$ and this certainly means that

$$z_\rho(t,\lambda e^\rho) > 0, \quad 0 \le t < t_1,\ z_\rho(t_1,\lambda e^\rho) = 0,$$
$$z_\rho(t,\lambda e^\rho) < 0, \quad t_1 < t < t_1 + \varepsilon$$

for some points $t_1 \le 1$ and $\varepsilon > 0$. So let us set

$$\lambda_\rho := \max\{\lambda \ge 0 : z_\rho(t,\lambda e^\rho) = x^{(2\rho)}(t,\lambda e^\rho) \ge 0,\ 0 \le t \le 1\}. \tag{2.11}$$

Now, to show (2.10), we note first that for $\lambda > \lambda_0 \ge 0$ and since

$$x^{(2\rho)}(0,\lambda e^\rho) = \alpha \quad \text{and} \quad x^{(2\rho+1)}(0,\lambda e^\rho) = -\lambda$$
$$x^{(2\rho)}(0,\lambda_0 e^\rho) = \alpha\ and \quad x^{(2\rho+1)}(0,\lambda_0 e^\rho) = -\lambda_0,$$

by continuity of $z_\rho(.,\lambda e^\rho) = x^{(2\rho)}(.,\lambda e^\rho)$, we get a number $0 < \tau \le 1$ such that

$$x^{(2\rho)}(t,\lambda_0 e^\rho) > x^{(2\rho)}(t,\lambda e^\rho), \quad 0 < t \le \tau.$$

Consequently, in view of (2.7), we easily get that

$$x^{(i)}(t,\lambda_0 e^\rho) > x^{(i)}(t,\lambda e^\rho), \quad 0 < t \le \tau,\ i = 0,1,\dots 2\rho - 1.$$

Further, the assumption (2.6) and the monotonicity of $f$ yield (by restricting, if necessary further the interval $[0,\tau]$)

$$x^{(j)}(t,\lambda_0 e^\rho) > x^{(j)}(t,\lambda e^\rho), \quad 0 < t \le \tau,\ j = 2\rho+1,\dots 2n-1.$$

Thus

$$x^{(i)}(t,\lambda_0 e^\rho) > x^{(i)}(t,\lambda e^\rho), \quad 0 < t \le \tau,\ i = 0,1,\dots 2n-1. \tag{2.12}$$

We choose $\lambda_0 \in \mathbb{R}$ (for example $\lambda_0 = 0$) such that

$$x^{(i)}(t,\lambda_0 e^\rho) > 0, \quad 0 < t \le 1 \ \ (i = 0,1,\dots 2n-1)$$

and recall the assumption (in (2.8))

$$x^{(2\rho)}(t,\lambda e^{\rho}) \geq 0, \quad 0 < t \leq 1,\ \rho = k, k+1, \dots, n-1,$$

that for every $\lambda \geq \lambda_0$. Thus noting (2.6) we clearly have

$$x^{(i)}(t,\lambda e^{\rho}) \geq 0, \quad 0 < t \leq 1,\ i = 0,1,\dots,2\rho, 2\rho+2, \dots, 2n-1 \tag{2.13}$$

Let now (in view of (2.12)) $\psi(t,\lambda) = x(t,\lambda e^{\rho}) - x(t,\lambda_0 e^{\rho})$ and suppose that there is a (minimal) $\hat{\tau} \leq 1$ and an integer $j$ with $0 \leq j \leq 2n-1$ such that for all $i = 0,1,\dots,2n-1$ :

$$\psi^{(i)}(t,\lambda) < 0,\ 0 < t < \hat{\tau} \quad \text{and} \quad \psi^{(j)}(\hat{\tau},\lambda) = 0. \tag{2.14}$$

(1) Assume that $j \leq 2\rho$. Then since $\psi^{(2n)}(t,\lambda) = x^{(2n)}(t,\lambda e^{\rho}) - x^{(2n)}(t,\lambda_0 e^{\rho})$, integrations leads (as in (2.9)) to

$$\psi^{(j)}(t,\lambda) = (\lambda_0 - \lambda)\frac{t^{2(\rho-j)}}{[2(\rho-j)]!} + \frac{t^{2n}}{(2n)!}[F(\bar{t},X(\bar{t}),X'(\bar{t})) - F(\bar{t},X(\bar{t}),X'(\bar{t}))]$$

for some $0 \leq \bar{t} \leq t$. Consequently by (2.14), for $t = \hat{\tau}$:

$$(\lambda - \lambda_0)\frac{\hat{\tau}^{2\rho+1}}{(2\rho+1)!} = \frac{\hat{\tau}^{2n}}{(2n)!}[F(\bar{t},X(\bar{t}),X'(\bar{t})) - F(\bar{t},X(\bar{t}),X'(\bar{t}))]. \tag{2.15}$$

Now since by (1.7), (2.13) and (2.14) we obtain

$$0 \leq x^{(i)}(t,\lambda e^{\rho}) \leq x^{(i)}(t,\lambda_0 e^{\rho}),\quad 0 < t \leq \hat{\tau},\ (i \neq 2\rho+1)$$

for every $\lambda \geq \lambda_0$, in view of the definition of the modification $F$, the second member of (2.15) is bounded, when $\lambda \to -\infty$ but not the first one. Thus (2.14) can not be true and so we get

$$\begin{gathered} 0 \leq x^{(j)}(t,\lambda e^{\rho}) < x^{(j)}(t,\lambda_0 e^{\rho}), \\ x^{(2\rho+1)}(t,\lambda e^{\rho}) < x^{(2\rho+1)}(t,\lambda_0 e^{\rho}) \leq 0, \quad 0 < t \leq 1,\ \lambda > \lambda_0. \end{gathered} \tag{2.16}$$

(2) Assume that $j > 2\rho$. Then, we also get the contradiction

$$0 = \psi^{(2j)}(\hat{\tau},\lambda) = \frac{t^{2n}}{(2n)!}[F(\bar{t},X(\bar{t};\lambda e^{\rho}),X'(\bar{t};\lambda e^{\rho})) - F(\bar{t},X(\bar{t};\lambda_0 e^{\rho}),X'(\bar{t};\lambda e^{\rho}))],$$

by noting (2.16) and the (strictly) monotonicity of $F$. Thus, (2.14) can not also be true and so we get (2.16) once again. We set now

$$\widehat{K} = \max\{x^{(j)}(t,\lambda_0 e^{\rho}) : j = 0,1,\dots,2n-1,\ 0 \leq t \leq 1,\ j \neq 2\rho+1\} > 0$$

and consider the rectangle $R = [0,1] \times [0,\widehat{K}]^{2\rho} \times (-\infty,0] \times [0,\widehat{K}]^{2(n-\rho)}$. By (1.7), (2.16) and the continuity of $F$ we get

$$\max\{F(t,X,X') : (t,X,X') \in R\} = M < +\infty$$

and thus

$$\left|F(t,x(t,\lambda e^{\rho}),x'(t,\lambda e^{\rho}),\dots,x^{(2n-1)}(t,\lambda e^{\rho}))\right| \leq M, \quad 0 < t \leq 1,\ \lambda \geq \lambda_0.$$

Hence the estimation (2.10) is established and this concludes the proof. □

We are now ready to formulate and prove our first main result.

**Theorem 2.3.** *The $(2k, 2(n-k))$ multipoint focal value problem (2.1))-(2.2) has a positive solution, provided that assumptions of the previous Lemma 2.2 are fulfilled.*

*Proof.* Let $S$ be the $(n-k)$ simplex spanning by the vertices $e^0 = 0$ and $e^\rho = -\lambda e^\rho,\ k \le \rho \le n-1$. As usual $[e^{i_0}e^{i_1}\dots e^{i_r}]$ denote the closed face of $S$ spanning by the vertices $\{e^{i_0}, e^{i_1}, \dots, e^{i_r}\}$. We choose here $\lambda$ large enough, so that (see (2.11)):

- $\lambda \ge \max\{\lambda_\rho : \rho = k, k+1, \dots, n-1\}$
- If an initial vector $v$ starts from $e^0 = 0$ and ends on the face $[e^{i_0}e^{i_1}\dots e^{i_r}]$ (which does not contains $e^0$), then at least one of its projection is greater than the corresponding $\lambda_\rho$ (we notice that in view of previous Lemma 2.2, such a trajectory $X(t,v)$ of (2.1) egresses from $\mathbb{K}$).

In the sequel, the statement "$X(t,v)$ remains asymptotic in $\mathbb{K}$" will mean that $X(t,v)$ does not egress from $\mathbb{K}$ through $H_\rho$ for some $t_1 \le 1$, i.e.

$$z_\rho(t;v) = x^{(2\rho)}(t,v) > 0, \quad 0 < t \le 1$$

for all $\rho = k, k+1, \dots, n-1$. Define now the sets

$$E_0 := cl\{v \in S : X(t,v) \text{ remains asymptotic in } \mathbb{K}\},$$
$$E_\rho = cl\{v \in S : X(t,v) \text{ egresses from } \mathbb{K} \text{ through } H_\rho \text{ for } t_1 \le 1\}$$

To apply Sperner's lemma, it is necessary to verify that

(i) The sets $\{E_\rho\}$ form a closed covering of $S$ and
(ii) The face $[e^{i_0}e^{i_1}\dots e^{i_r}] \subseteq \cup_{j=0}^r E_{i_j}$.

The closedness of $E_\rho$ is obvious due to the continuous dependence of solutions on their initial data. Further, their union covers $S$, since any trajectory either remains asymptotic in $\mathbb{K}$ or egresses from it through some plane $E_\rho$, $k \le \rho \le n-1$.

Consider now any vector $v \in [e^{i_0}e^{i_1}\dots e^{i_r}]$ (more precisely $v$ starts from $e^0 = 0$ and ends on the face $[e^{i_0}e^{i_1}\dots e^{i_r}]$) and assume that $X(t,v) \in \mathbb{K}$, $0 < t < t_1$, where $t_1$ is a egress point, as it has been established in Lemma 2.2. We examine the next two cases:

(a) If $0 \notin \{i_0, i_1, \dots, i_r\}$, then by the choice of $\lambda$ and the previous analysis, obviously the solution $X(t,v)$ egresses (strictly) from $\mathbb{K}$. If $\rho \notin \{i_0, i_1, \dots, i_r\}$, then $z_\rho'(0,v) = x^{(2\rho+1)}(0,v) = 0$ and, as we pointed out above by the nature of the vector field, $z_\rho(t,v)$ is a positive increasing map, that is $z_\rho(t,v) > 0,\quad 0 < t < t_1$. Thus $X(t,v)$ egresses from $\mathbb{K}$ on some hyperplane $E_i$ with $i \ne \rho$. But this means that $\rho \in \{i_0, i_1, \dots, i_r\}$ i.e. $v \in E_\rho \subseteq \cup_{l=1}^r E_{i_l}$.

(b) If $0 \in \{i_0, i_1, \dots, i_r\}$, then if $X(t,v)$ remains asymptotic in $\mathbb{K}$, then $v \in E_0 \subseteq \cup_{j=0}^r E_{i_j}$. Otherwise as we showed at the previous case, $X(t,v)$ egresses from $\mathbb{K}$ on some hyperplane $H_{i_j}$ with $0 \le j \le r$, that is we obtain once again $v \in \cup_{j=0}^r E_{i_j}$.

Finally, by applying Sperner's Lemma 1.1, we get a trajectory $X(t,v_0)$ with initial value $v_0 \in E_0 \cap \{\cap_{i=k}^{n-1} E_i\}$. Now for all $k \le j \le n-1$ and in view of definition of $E_0$ and $E_\rho$, such a trajectory must satisfy

$$z_j(t;v_0) > 0, \quad z_j'(t;v_0) < 0, \quad 0 < t < 1$$
$$z_j(1;v_0) = 0.$$

Moreover $X(t,v_0)$ remains asymptotic in $\mathbb{K}$. Thus we must also have

$$z_{n-1}'(t;v_0) = x^{(2n-1)}(t;v_0) < 0,\ 0 < t < 1 \quad \text{and} \quad x^{(2n-1)}(1;v_0) = 0.$$

For if we suppose that $x^{(2n-1)}(1) < 0$, then the map $z_{n-1}(t;v_0) = x^{(2n-2)}(t;v_0)$ is strictly decreasing and thus the trajectory must egresses strictly from the cone at $t_1 = 1$ through the plane $H_{n-1}$, a contradiction, since $v_0 \in E_0$.

From the above procedure, it is clear that the obtaining solution $(t, v_0)$ of the differential equation (2.5), satisfies the focal condition

$$\begin{gathered} x^{(i)}(0)=0, \quad 0 \leq j \leq 2k-1 \\ x^{(j)}(1)=0, \quad 2k \leq j \leq 2n-1, \end{gathered}$$

and further the initial conditions

$$z_j(0, v_0)=\alpha, \quad z_j'(0, v_0)=\lambda_{0j}, \quad k \leq j \leq n-1,$$

where $v_0=(\lambda_{01}, \lambda_{02}, \dots, \lambda_{0n-k})$ and the only restriction on the parameter $a$ is that $a>0$. In this way, we get a whole $(n-k)-$parametric family of solutions of the modified differential equation (2.5) satisfying the above focal boundary conditions.

We assert that there is an $a>0$ such that the obtaining trajectory $X(t, v_0)$ satisfies further the additional condition

$$\sum_{i=1}^{m} \alpha_i x^{(p)}(\xi_i)=0, \quad p=2\rho,\ (\rho=k, k+1, \dots, n-1). \tag{2.17}$$

Recalling that $\sum_{i=1}^{m} \alpha_i=0$, we set

$$\alpha_i= \begin{cases} a_i^{+}, & \text{if } \alpha_i \geq 0 \\ a_i^{-}, & \text{if } \alpha_i<0, \end{cases}$$

$I_{+}=\{i: \alpha_i \geq 0\}$, $I_{-}=\{i: \alpha_i<0\}$ and

$$A=\sum_{i \in I_{+}} \alpha_i^{+}=\sum_{i \in I_{-}} \alpha_i^{-}.$$

Since the solution $x=x^{(2\rho)}(t)$, $0 \leq t \leq 1$ is decreasing, we get

$$\begin{aligned} \sum_{i=1}^{m} \alpha_i x^{(2\rho)}(\xi_i) &= \sum_{i \in I_{+}} \alpha_i^{+} x^{(2\rho)}(\xi_i)-\sum_{i \in I_{-}}(-\alpha_i^{-}) x^{(2\rho)}(\xi_i) \\ &\leq \sum_{i \in I_{+}} \alpha_i^{+} x^{(2\rho)}(\xi_1)-\sum_{i \in I_{-}}(-\alpha_i^{-}) x^{(2\rho)}(\xi_m) \\ &= A\big[x^{(2\rho)}(\xi_1)-x^{(2\rho)}(\xi_m)\big]. \end{aligned} \tag{2.18}$$

Suppose that (2.17) is not fulfilled and so there exists an $\varepsilon_0>0$ such that for every $a>0$,

$$\sum_{i=1}^{m} \alpha_i x^{(2\rho)}(\xi_i) \geq \varepsilon_0.$$

If we choose $a \leq \varepsilon_0/(2A)$, then by positivity and monotonicity of $x=x^{(2\rho)}(t)$, $0 \leq t \leq 1$ and noticing (2.18), we get the contradiction

$$\sum_{i=1}^{m} \alpha_i x^{(2\rho)}(\xi_i) \leq A x^{(2\rho)}(\xi_1) \leq A \frac{\varepsilon_0}{2A}=\frac{\varepsilon_0}{2}.$$

It is worth noticing (see the following Remark) that the solution, $x=x(t; v_0)$, of the modified differential equation (2.5) is actually a solution of the original equation (2.1). $\square$

**Remark 2.4.** The solution $x(t; v_0)$ of (2.5) fulfilling the boundary conditions (2.2), satisfies further the inequalities:

$$x^{(2\rho)}(t, v_0) > 0, \quad x^{(2\rho+1)}(t, v_0) < 0, \quad 0 < t < 1,\ k \le \rho < n - 1.$$

Especially, since $x^{(2k)}(t, v_0) \ge 0$, it follows that

$$x^{(i)}(t) > 0, \quad 0 < t \le 1,\ 0 \le i \le 2k - 1.$$

As a result, the solution of (2.5)-(2.2) is positive. Consequently, in view of the definition of $F$, the function $x = x(t; v_0)$ is also a solution of the original equation (2.1)

In the previous approach we have assumed that $n$ and $p$ were even integer. We consider now the $(2n+1)$-order differential equation

$$x^{(2n+1)}(t) = f(t, x(t), x'(t), \dots x^{(2n)}(t)), \quad 0 \le t \le 1, \tag{2.19}$$

along with the associated $(2k+1, 2(n-k))$ focal boundary multi-value problem

$$\begin{gathered} x^{(j)}(0) = 0, \quad 0 \le j \le 2k, \\ x^{(j)}(1) = 0, \quad 2k+1 \le j \le 2n-1, \sum_{i=1}^{m} \alpha_i x^{(p)}(\xi_i) = 0, \end{gathered} \tag{2.20}$$

for an even integer $p \in \{2(k+1), 2(k+2), \dots, 2n\}$.

**Theorem 2.5.** *Under the assumptions of Theorem 2.3, the* $(2k+1, 2(n-k))$ *multipoint focal value problem* (2.19)-(2.20) *has a solution* $x = x(t)$, $t \in [0, 1]$ *such that*

$$x^{(i)}(t) > 0, \quad 0 < t < 1,\ 0 \le i \le 2k$$

*and further*

$$x^{(2\rho+1)}(t) > 0, \quad x^{(2\rho+2)}(t) < 0,\ 0 < t < 1,\ k \le \rho < n - 1.$$

*Proof.* we set now

$$X = (\hat{Y}, \hat{Z}), \hat{Y} = (x, x', x'', \dots, x^{(2k)}) \in \mathbb{R}^{2k+1}$$

and generally as above

$$\hat{Z} = (x^{(2k+1)}, x^{(2k+3)}, \dots, x^{(2n-1)}) \in \mathbb{R}^{n-k}.$$

Then, the boundary conditions (2.20) take the form

$$\hat{Y}(0) = 0 \quad \text{and} \quad \hat{Z}(1) = \hat{Z}'(1) = 0.$$

On the other hand, the initial conditions (2.7) take the form

$$\begin{gathered} x^{(i)}(0, \lambda e^{\rho}) = 0,\ (i = 0, 1, \dots, 2k), \\ x^{(2j+1)}(0, \lambda e^{\rho}) = \alpha, \quad z_j^{(2j+2)}(0, \lambda e^{\rho}) = 0, \quad k \le j \le n-1,\ j \ne \rho \\ x^{(2\rho+1)}(0, \lambda e^{\rho}) = \alpha, \quad x^{(2\rho+2)}(0, \lambda e^{\rho}) = -\lambda. \end{gathered}$$

Then Lemma 2.2 and Theorem 2.3 can be carried out readily, under the obvious modifications in their proofs. $\square$

**Remark 2.6.** By Theorems 2.3 and 2.5 it follows that there are not different results if the natural number $p$ in (1.3) is odd or even. The keypoint is the value (even or not) of the number $n - p$.

So we may now consider the general differential equation

$$x^{(2n+1)}(t) = f(t, x(t), x'(t), \dots x^{(2n-1)}(t), x^{(2n)}(t)) \tag{2.21}$$

and the associated $(2k, 2(n-k)+1)$ multipoint focal value problem

$$\begin{gathered} x^{(i)}(0) = 0, \quad 0 \le i \le 2k-1, \\ x^{(j)}(1) = 0, \quad 2k \le j \le 2n, \\ \sum_{i=1}^{m} a_i x^{(2p+1)}(\xi_j) = 0. \end{gathered} \tag{2.22}$$

Consider the cone

$$\begin{aligned} \mathbb{K}_0^* = \big\{ &(x, x', x'', \dots, x^{(2n-1)}) \in \mathbb{R}^{2n} \setminus \{0\} : x^{(i)} \ge 0,\ 0 \le i \le 2k \\ &\text{and } (-1)^j x^{(j)} \ge 0,\ 2k+1 \le j \le 2n \big\} \end{aligned}$$

and assume for the rest of this paper that the function $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ is continuous, negative: $f(t, x, x', x'', \dots, x^{(2n-1)}) < 0$, on the cone $\mathbb{K}_0^*$ and further

(1) $f$ is nondecreasing on every of its last $2(n-k)$ variables as well as (strictly) increasing in (at least) one of $n-k-1$ odd-order derivatives: $x^{(2k+1)}$, $x^{(2k+3)}$, ..., $x^{(2n-1)}$,
(2) Bounded at $+\infty$ on every of its last $n-k+1$ even-order derivatives: $x^{(2k)}, x^{(2k+2)}, \dots, x^{(2n)}$, uniformly for
$$\big(t, x, x', \dots, x^{(2k-2)}, x^{(2k-1)}, x^{(2k+1)}, x^{(2k+3)}, \dots, x^{(2n-1)}\big) \in W,$$
where $W$ is any compact subset of $[0,1] \times \mathbb{R}^{n-k}$:
$$\lim_{x^{(2\rho)} \to +\infty} f(t, x(t), x'(t), \dots x^{(2n-1)}(t)) \le K, \quad (\rho = k+1, k+2, \dots, n).$$

**Theorem 2.7.** *Under the above assumptions, the* $(2k, 2(n-k)+1)$ *multipoint focal value problem* (2.21)-(2.22) *has a solution* $x = x(t)$, $t \in [0,1]$ *such that* $x^{(i)}(t) > 0$, $0 < t < 1$, $0 \le i \le 2k$ *and further*

$$x^{(2\rho+1)}(t) < 0, \quad x^{(2\rho+2)}(t) > 0, \quad 0 < t < 1,\ k \le \rho < n-1.$$

*Proof.* We set $X = (Y^*, V^*, Z^*)$, where

$$\begin{gathered} Y^* = (x, x', \dots, x^{(2k-1)}) \in \mathbb{R}^{2k}, \quad V^* = x^{(2k)} \in \mathbb{R}, \\ Z^* = (x^{(2k+1)}, x^{(2k+3)}, \dots, x^{(2n-1)}) \in \mathbb{R}^{(n-k)}. \end{gathered}$$

Then the focal boundary conditions at (2.22) take the form

$$Y^*(0) = 0, \quad V^*(1) = 0, \quad Z^*(1) = Z^{*\prime}(1) = 0.$$

and the initial conditions (2.7) $(a > 0,\ \lambda > 0,\ m \ge 0)$

$$\begin{gathered} x^{(i)}(0) = 0, \quad 0 \le i \le 2k-1, \quad x^{(2k)}(0) = m, \\ x^{(2j+1)}(0) = -\alpha, \quad x^{(2j+2)}(0) = 0, \quad k \le j \le n-1.\ j \ne \rho \\ x^{(2\rho+1)}(0) = -\alpha, \quad x^{(2\rho+2)}(0) = \lambda_{\rho+1}, \end{gathered} \tag{2.23}$$

Consider the modification (replace now by 0 only the negative coordinates of $V^*$ and/or $Z^{*\prime}$)

$$F^*(t, Y^*, V^*, Z^*, Z^{*\prime}) := \begin{cases} f(t, Y^*, V_0^*, Z^*, Z_0^{*\prime}) & \text{if } V_0^* \not> 0 \text{ or/and } Z^{*\prime} \not> 0 \\ f(t, Y^*, V^*, Z^*, Z^{*\prime}) & \text{otherwise} \end{cases}$$

and the differential equation

$$x^{(2n+1)}(t) = F^*(t, Y^*, V^*, Z^*, Z^{*\prime}). \tag{2.24}$$

For the moment, we fix the initial value $x^{(2k)}(0) = m > 0$, and then we may follow the lines of Lemma 2.2 and Theorem 2.3, under the obvious symmetrical alterations to get a solution

$$x = x_m(t; v_0), \quad 0 \le t \le 1$$

of (2.23) such that for $k \le \rho \le n-1$,

$$x_m^{(2\rho+1)}(t; v_0) < 0, \quad x_m^{(2\rho+2)}(t; v_0) > 0, \quad 0 \le t \le 1$$
$$x_m^{(2\rho+1)}(1; v_0) = 0, \quad x_m^{(2\rho+2)}(1; v_0) = 0,$$

Especially, since $x_m^{(2k+1)}(t; v_0) < 0$, $0 \le t \le 1$, the map $x_m^{(2k)}(t; v_0)$ is decreasing on $0 \le t \le 1$ and we may show that there is an $m_0 > 0$ such that

$$x_{m_0}^{(2k)}(t; v_0) > 0, \quad 0 \le t < 1 \quad \text{and} \quad x_{m_0}^{(2k)}(1; v_0) = 0. \tag{2.25}$$

Indeed, suppose that for an $m_1$ (for example $m_1 = 0$)

$$x_{m_1}^{(2k)}(t; v_0) > 0,\ 0 \le t < \hat{\tau} \quad \text{and} \quad x_{m_1}^{(2k)}(t; v_0) \le 0,\ \hat{\tau} \le t \le 1.$$

Taking now $m \to +\infty$ and noticing that the function $f$ is bounded, we may get (following a procedure similar to the given one in the proof of Lemma 2.2) an $m_2 > m_1$ such that

$$x_{m_2}^{(2k)}(t; v_0) > 0, \quad 0 \le t \le 1.$$

Hence by the continuity (Knesser's property) of solutions upon their initial values (see for details [21] and the references therein), we obtain the requesting in (2.25) $m_0 \in (m_1, m_2)$. Finally, the obtaining solution $x(t) = x_{m_0}(t; v_0)$ clearly satisfies, for $0 < t < 1$,

$$x_{m_0}^{(i)}(t; v_0) > 0, \quad i = 0, 1, \dots, 2k,$$
$$x_{m_0}^{(2\rho+1)}(t; v_0) < 0,$$
$$x_{m_0}^{(2\rho+2)}(t; v_0) > 0, \quad \rho = k, k+1, \dots, n-1,$$

and thus noticing the definition of the modification $F$, we conclude that it is actually a solution of the initial differential equation. □

**Remark 2.8.** By the construction of the simplex $S$ (see Theorem 2.3) and especially the choice of initial conditions (2.7), we clearly get a whole $(n-k-1)$-parametric family of solutions of BVP (2.19)-(2.20). Indeed, the only restriction of the constant $\alpha$ comes from the inequality $a > 0$.

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