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ASYMPTOTIC THEORY FOR WEAKLY NON-LINEAR WAVE EQUATIONS IN SEMI-INFINITE DOMAINS

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Abstract. We prove the existence and uniqueness of solutions of a class of weakly non-linear wave equations in a semi-infinite region $0 \leq x,\ t < L/\sqrt{|\epsilon|}$ under arbitrary initial and boundary conditions. We also establish the asymptotic validity of formal perturbation approximations of the solutions in this region.

1. Introduction

Many physical phenomena involve the action of weak non-linear perturbations acting over long periods of space and time. Some of these phenomena can be mathematically modeled by partial differential equations containing small non-linear terms. These non-linear effects, characterized by a small parameter ϵ, could accumulate over time and space to significantly impact the space and time evolution of the systems. Some examples of physical models involving such weakly non-linear equations are:

(i) The wave equation with a cubic non-linearity governing the slow oscillations of overhead power lines [7].
(ii) The non-linear Schroedinger equation with slowly varying coefficients with applications in water waves and non-linear optics [1, 12].
(iii) The shallow water wave equations with small initial displacement, and the weakly non-linear acoustics equations [5, 8].
(iv) The equations describing the motion of a slightly viscoelastic column with viscous damping [6].

The traditional tool to study the effect of small non-linearities is perturbation expansion in terms of the small parameter ϵ. Straight forward perturbation expansion of solutions usually become unbounded at large times (or lengths) because of unbounded growth in the wave amplitude. Averaging, matched asymptotic expansions and multiple-scale techniques are the standard techniques used to develop approximations of solutions of these non-linear perturbation problems that remain bounded at large times and distances. A review of these tools can be found in [5], and a number of examples and related approaches can be found in [8].

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Generally, multiple scale expansions of solutions of wave equations anticipate in advance the dependence of solutions on different time scales in order to construct solutions which are explicit functions of time. Prior to 1992, most studies of weakly nonlinear wave equations dealt with initial value problems on the infinite line $-\infty < x < \infty$ or initial-boundary value problems on a finite space interval $-1 \leq x \leq 1$ with fixed end conditions. The latter problem could be transformed into an initial-value problem on the infinite space interval. For such problems, a fast time scale t and a slow time scale $T = \epsilon t$ are introduced to develop asymptotic solutions. In [11], the existence, uniqueness and continuous dependence of solutions on initial data as well as the asymptotic validity of formal expansions of solutions were proved for these types of problems. This was accomplished using Green's functions and transformation of the initial-boundary value problem to an initial value problem on the infinite interval $-\infty < x < \infty$, using an odd 2π-periodic extension. For signaling problems in which the initial conditions are zero and boundary data at $x = 0$ propagate in the region $x > 0$, a perturbation scheme based on a slow spatial scale $X = \epsilon x$ can be used [4].

In [2], a multi-scale method was developed for weakly nonlinear wave equations in the region $x > 0$ with arbitrary initial and boundary data. These problems can not be transformed into initial value problems on the infinite interval $-\infty < x < \infty$, since such a transformation leads to coupled, second-order, non-linear, non-homogeneous partial differential equations that are difficult to solve. In [2], we introduced a long time scale $T = \epsilon t$ and a long space scale $X = \epsilon x$, in addition to the fast variables x and t, to develop asymptotic solutions. The necessity of both scaled time and space variables is an essential characteristic of such problems.

The main purpose of this paper is to establish the existence, uniqueness and continuous dependence of classical solutions on initial data for a class of initial-boundary value problems for the weakly non-linear wave equations in a rectangular region $0 \leq x, t < L/\sqrt{|\epsilon|}$, where ϵ is a small parameter and L is an arbitrary positive number. In addition, we establish the asymptotic validity of formal expansions of solutions for such problems. We use an integral representation of the solution to accomplish this.

2. Existence, uniqueness and continuous dependence of solutions on initial data

In this section we prove the existence and uniqueness in the classical sense of the solution to the hyperbolic system

$$\begin{gathered} u_{tt} - u_{xx} + \epsilon h(u, u_t, u_x) = 0, \quad t, x > 0, \ 0 < \epsilon \ll 1 \\ u(x,0) = a(x); \quad u_t(x,0) = b(x); \quad u(0,t) = \rho(t), \quad t, x > 0 \end{gathered} \tag{2.1}$$

Let us assume that the initial and boundary data as well as the non-linear function h satisfy the following conditions:

(A1) $a(x)$, $\rho(t)$ are twice continuously differentiable for $x \geq 0$, $t \geq 0$.
$b(x)$is continuously differentiable for $x \geq 0$, the function h and its derivatives are analytic and uniformly bounded in its arguments.
$a(0) = \rho(0)$, $b(0) = \rho'(0)$, $\rho(0) = 0$, $a'(0) = 0$, $\rho''(0) = 0$,
$-a''(0) + \epsilon h(a(0), b(0), a'(0)) = 0$

Theorem 2.1. *Suppose h, $a(x)$, $b(x)$, $\rho(t)$ satisfy the conditions (A1). Then for any ϵ with $0 < |\epsilon| \leq \epsilon_0 \ll 1$, the non-linear initial-boundary value problem* (2.1) *has* *a unique, twice continuously differentiable solution in a region of the $x-t$ plane, $0 \leq x, t \leq L/\sqrt{|\epsilon|}$, where L is a sufficiently small positive constant independent of ϵ. Moreover, this solution depends continuously on the initial-boundary data.*

The conditions on the initial values of various functions in (A1) ensure that the solution u and its first and second partial derivatives are continuous across $x = t$. The proof of this theorem depends crucially on the integral representation of the solution to the hyperbolic system (2.1)- (see [10]):

$$u(x,t) = \begin{cases} \frac{1}{2}[a(x+t) + a(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} b(\lambda)\,d\lambda \\ +\frac{\epsilon}{2}\int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} h(u, u_\tau(\lambda,\tau), u_\lambda(\lambda,\tau))d\lambda, & \text{if } 0 \leq t \leq x, \\ \rho(t-x) + \frac{1}{2}[a(t+x) + a(t-x)] + \frac{1}{2}\int_{t-x}^{t+x} b(\lambda)d\lambda \\ +\frac{\epsilon}{2}\int_0^t d\tau \int_{|x-(t-\tau)|}^{x+(t-\tau)} h(u, u_\tau(\lambda,\tau), u_\lambda(\lambda,\tau))d\lambda, & \text{if } t \geq x \geq 0, \end{cases} \tag{2.2}$$

By direct differentiation, one can show the equivalence of the system (2.2) to (2.1) whenever C^2 solutions of (2.1) exist.

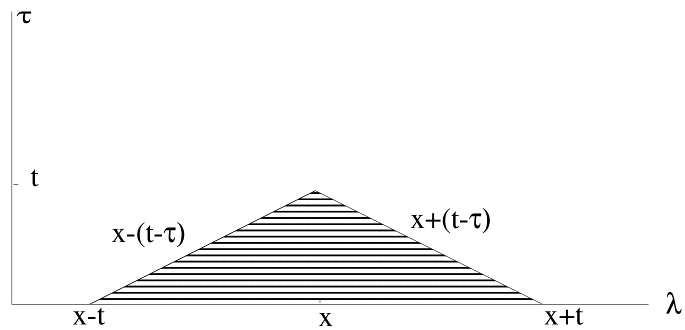

FIGURE 1. Region of integration for the first case in (2.2). This region has area t^2

Using the above representation of solutions, and the assumptions on the regularity of initial and boundary conditions, one can show that a twice continuously differentiable solution $u(x,t;\epsilon)$ of the hyperbolic system exists in a rectangle $0 \leq t, x \leq O(1/\sqrt{\epsilon})$. This solution depends continuously on the initial-boundary data, and formal perturbation series expansions asymptotically converge to the solution in this rectangle.

Let $T : S \to S$ denote the integral operator defined by (2.2), where

$$S = \{(x,t) | 0 \leq x, t \leq \frac{L}{\sqrt{|\epsilon|}}\}$$

We represent the integral equations (2.2) in the form $u = Tu$. The proof consists of showing that T is a contraction on a space of twice continuously differentiable functions defined on S, and therefore by Banach's Fixed Point Theorem, a unique solution u exists.

Let $C_M^2(S)$ be the space of twice continuously differentiable functions on S with norm

$$\|f\| = \sum_{i,j=0}^{2} \sum_{i+j \leq 2} \max_{(x,t)\in S} \Big|\frac{\partial^{i+j} f(x,t)}{\partial x^i \partial t^j}\Big| \leq M$$

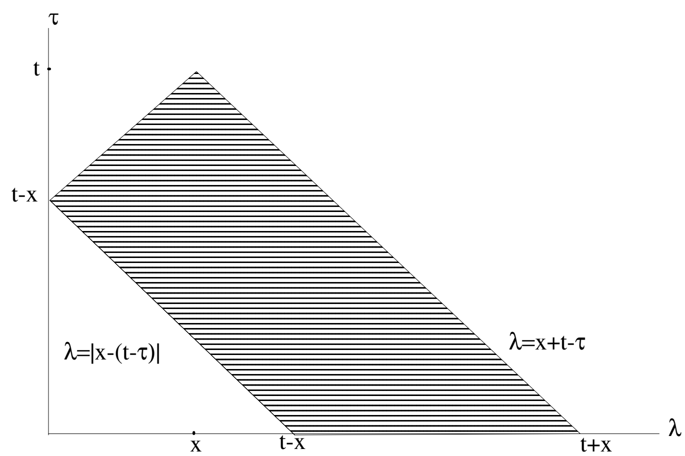

FIGURE 2. Region of integration for the second case in (2.2). This region has area $2tx - x^2$

Let us write

$$Tu = u_I + T_\epsilon u \tag{2.3}$$

where

$$u_I(x,t) = \begin{cases} \frac{1}{2}[a(x+t) + a(x-t)] + \int_{x-t}^{x+t} b(\lambda)d\lambda, & \text{if } 0 \le t \le x \\ \rho(t-x) + \frac{1}{2}[a(t+x) + a(t-x)] + \frac{1}{2}\int_{t-x}^{t+x} b(\lambda)d\lambda, & \text{if } t \ge x \ge 0 \end{cases} \tag{2.4}$$

and

$$T_\epsilon u = \begin{cases} \frac{\epsilon}{2}\int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} h(u, u_\tau(\lambda,\tau), u_\lambda(\lambda,\tau))d\lambda, & \text{if } 0 \le t \le x \\ = \frac{\epsilon}{2}\int_0^t d\tau \int_{|x-(t-\tau)|}^{x+(t-\tau)} h(u, u_\tau(\lambda,\tau), u_\lambda(\lambda,\tau))d\lambda, & \text{if } t \ge x \ge 0 \end{cases} \tag{2.5}$$

From (2.3) we get

$$\|Tu\| \le \|u_I\| + \|T_\epsilon u\| \tag{2.6}$$

Because of the boundedness conditions on ρ, a and b, there exists a nonnegative constant M_1, independent of ϵ, such that

$$\|u_I\| \le \frac{M_1}{2} \tag{2.7}$$

From Figures 2 and 3, it follows that there exist constants M_2^1 and M_2^2 such that

$$\|T_\epsilon u\| \le \begin{cases} M_2^1 \epsilon t^2, & \text{if } t \le x \\ M_2^2 \epsilon(2tx - x^2), & \text{if } t \ge x \end{cases}$$

Thus for t and x in the region S, one can find an ϵ-independent constant M_2 such that

$$\|T_\epsilon u\| \le M_2 L^2$$

Combining this inequality with (2.7) and (2.6),

$$\|Tu\| \le \frac{M_1}{2} + M_2 L^2 \le M_1$$

for sufficiently small L. This shows that $T : C^2_{M_1}(S) \to C^2_{M_1}(S)$.

We next show that T is a contraction on $C^2_{M_1}(S)$. Using the Lipschitz property of h, there exists a constant M':

$$\|h(u,u_t,u_x) - h(v,v_t,v_x)\| \le M'\|u-v\|$$

for all $(x,t) \in S$. It follows from (2.5) that one can find constants K_1 and K_2 satisfying

$$\|T_\epsilon u - T_\epsilon v\| \le \begin{cases} \epsilon t^2 K_1\|u-v\|, & \text{if } t \le x \\ \epsilon(2tx - x^2)K_2\|u-v\|, & \text{if } t \ge x \end{cases}$$

Thus there is a nonnegative constant K such that

$$\|Tu - Tv\| = \|T_\epsilon u - T_\epsilon v\| \le KL^2\|u-v\|$$

for all $(x,t) \in S$. Then for sufficiently small L, independent of ϵ,

$$\|Tu - Tv\| \le k\|u-v\| \tag{2.8}$$

for all $u, v \in C^2_{M_1}(S)$ and $0 \le k < 1$. This shows that T is a contraction of $C^2_{M_1}(S)$ into itself for sufficiently small L. By applying Banach fixed point theorem, it follows that T has a unique fixed point in $C^2_{M_1}(S)$. Since solutions of the integral equation (2.2) are also solutions of the hyperbolic system (2.1), we have proved that (2.1) has a unique solution in the space S.

We next prove that the solutions of the hyperbolic system (2.1) depends continuously on the initial-boundary values, in the sense that small changes in these values result in small changes in the solution within the region S in which existence-uniqueness of solutions have been proved.

Let $\tilde{u}$ be the solution of the hyperbolic system (2.1) corresponding to the initial boundary conditions

$$u(x,0) = \tilde{a}(x); \quad u_t(x,0) = \tilde{b}(x); \quad u(0,t) = \tilde{\rho}(t), \quad t,x > 0 \tag{2.9}$$

Assume that the initial-boundary data in (24) satisfy conditions similar to (A1). Following (2.3), we let

$$T\tilde{u} = \tilde{u}_I + T_\epsilon \tilde{u}.$$

Then the following estimate can be made:

$$\|u - \tilde{u}\| \le \|u_I - \tilde{u}_I\| + \|T_\epsilon u - T_\epsilon \tilde{u}\|$$

From an argument similar to (2.8), there exists a k, $0 \le k < 1$, such that

$$\|T_\epsilon u - T_\epsilon \tilde{u}\| \le k\|u - \tilde{u}\|$$

so that

$$\|u - \tilde{u}\| \le \|u_I - \tilde{u}_I\| + k\|u - \tilde{u}\|$$

and

$$\|u - \tilde{u}\| \le \frac{1}{1-k}\|u_I - \tilde{u}_I\|$$

whenever u, $\tilde{u}$ are in $C^2_{M_1}(S)$. This proves that small changes in initial data lead to small changes in the solutions.

3. Asymptotic validity of formal expansions

Next we prove that perturbation series expansions of solutions of (2.1) are asymptotically convergent to the exact solutions. Let $v(x,t)$ be defined on S satisfying

$$\begin{gathered} v_{tt} - v_{xx} + \epsilon h(v, v_t, v_x) = \epsilon^n r_1(x,t) \\ v(x,0) = a(x) + \epsilon^{n-1} r_2(x) \\ v_t(x,0) = b(x) + \epsilon^{n-1} r_3(x) \\ v(0,t) = \rho(t) + \epsilon^{n-1} r_4(t) \end{gathered} \tag{3.1}$$

The motivation to study this system is that they satisfy the original partial differential equation and initial/boundary conditions to $O(\epsilon^n)$ and $O(\epsilon^{n-1})$ respectively, where usually $n = 2$.

(A2) Assume that h, r_1, r_2, r_3, r_4 satisfy boundary conditions as in (A1), and that r_3, r_4 are in C^1

Theorem 3.1. *Let v satisfy (3.1) and the $r_{i(1\le i\le 4)}$ satisfy (A2). Then for $n > 1$, $\|u - v\| = O(\epsilon^{n-1})$ for all $t, x \in S$. Thus in the limit of small ϵ, the formal approximation v converges to the solution u.*

Proof. Note that a formal integral representation of (3.1) can be written in the form

$$v = \tilde{v}_I + T_\epsilon v + \tilde{T}_\epsilon r_1 \tag{3.2}$$

where

$$\tilde{T}_\epsilon r_1 = \begin{cases} \frac{\epsilon^n}{2} \int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} r_1(\lambda,\tau) d\lambda, & \text{if } 0 \le t \le x \\ \frac{\epsilon^n}{2} \int_0^t d\tau \int_{|x-(t-\tau)|}^{x+(t-\tau)} r_1(\lambda,\tau) d\lambda, & \text{if } t \ge x \ge 0 \end{cases}$$

and v_I is analogous to (2.4). From the geometry of Figures 1 and 2, it follows that there exists constants M_3^1 and M_3^2 such that

$$\|\tilde{T}_\epsilon r_1\| \le \begin{cases} M_3^1 \epsilon^n t^2, & \text{if } t \le x \\ M_3^2 \epsilon^n (2tx - x^2), & \text{if } t \ge x. \end{cases}$$

Thus for t and x in the region S, one can find a constant M_3 such that

$$\|\tilde{T}_\epsilon r_1\| \le \epsilon^{n-1} M_3 L^2.$$

In addition, there exists a non-negative constant M_4 such that,

$$\|u_I - \tilde{v}_I\| \le \epsilon^{n-1} M_4.$$

From (2.3) and (3.2) it follows that

$$\|u - v\| \le \|T_\epsilon u - T_\epsilon v\| + \|u_I - \tilde{v}_I\| + \|\tilde{T}_\epsilon r_1\| \le k\|u - v\| + \epsilon^{n-1}(M_3 L^2 + M_4)$$

so that

$$\|u - v\| \le \frac{\epsilon^{n-1}}{1-k}(M_3 L^2 + M_4)$$

showing that $\|u - v\| = O(\epsilon^{n-1})$ for $x, t \in S$. This establishes Theorem 3.1. □

4. Applications of the asymptotic theory

The results obtained above give formal theoretical support to a multiple-scale solution technique for weakly non-linear wave equations developed in [2]. It is shown there that the solution of (2.1) involves two scaled (or slow) variables, $X = \epsilon x$ and $T = \epsilon t$, in addition to the regular (or fast) variables x and t. The perturbation solution then defines two regions, $t > x$ and $t \le x$. For $t \le x$, the first order solution has the form

$$u_0(x,t,T) = f(\sigma,T) + g(\xi,T)$$

where $\sigma = x - t$ and $\xi = x + t$ are the forward and backward going characteristics of the wave equation. It is shown in [2] that f and g are governed by:

$$\begin{aligned} 2f_{\sigma T} - \lim_{M\to\infty} \frac{1}{M}\int_0^M h(f+g, g_\xi - f_\sigma, g_\xi + f_\sigma)\, d\xi = 0, \\ 2g_{\xi T} + \lim_{M\to\infty} \frac{1}{M}\int_0^M h(f+g, g_\xi - f_\sigma, g_\xi + f_\sigma)\, d\sigma = 0 \end{aligned}$$

with appropriate initial-boundary conditions.

For the region $t > x$, the first order solution takes the form

$$u_0(x,t,X,T) = p(\mu,X,T) + q(\xi,X,T)$$

where $\mu = t - x$ and $\xi = t + x$. In general the PDEs governing p and q are complicated coupled nonlinear equations; but for special cases, they are considerably simplified. For example, when h involves only the first derivatives of u, $h = h(u_t, u_x)$, it can be shown that

$$\begin{aligned} 2p_{\mu T} + 2p_{\mu X} + \lim_{M\to\infty} \frac{1}{M}\int_0^M h(p_\mu + q_\xi, q_\xi - p_\mu) d\xi = 0\,, \\ 2q_{\xi T} - 2q_{\xi X} + \lim_{M\to\infty} \frac{1}{M}\int_0^M h(p_\mu + q_\xi, q_\xi - p_\mu) d\mu = 0\,. \end{aligned}$$

The important point here is that the two equations governing the interaction of backward and forward propagating waves form a pair of coupled, nonlinear first-order PDEs whose theory is well-developed. This is a considerable simplification from the original second order nonlinear equation. We refer to [2] for details of solutions of these equations as well as explicit solutions for the cases $h = 2u_t + u(u_t - u_x)$ and $h = 2u_t$.

Concluding remarks. For weakly non-linear hyperbolic partial differential equations in the region $t > 0$, $x > 0$, we established the existence, uniqueness and continuous dependence of solutions on initial data within a rectangle $0 \le x, t < L/\sqrt{|\epsilon|}$. Although it appears from numerical evidence that the existence-uniqueness theorems could hold in much longer space and time intervals, it is not clear how to establish that. We also proved the asymptotic validity of formal perturbation expansions of solutions within this rectangle.

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