

## WEAK ASYMPTOTIC METHOD FOR THE STUDY OF PROPAGATION AND INTERACTION OF INFINITELY NARROW $\delta$ -SOLITONS

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ABSTRACT. We present a new method for studying the interaction of solitons for non-integrable Korteweg-de Vries (KdV) type equations with small dispersion and test this method for the KdV equation.

### INTRODUCTION

We present a method for studying the interaction of solitons for Korteweg-de Vries (KdV) type equations with small dispersion. It is well known that the KdV equation

$$u_t + (u^2)_x + \epsilon^2 u_{xxx} = 0,$$

where  $\epsilon \rightarrow +0$  is a small parameter, has a soliton solution in the form of a traveling wave

$$u = \frac{A}{2} \cosh^{-2} \left( \beta \frac{x - Vt}{\epsilon} \right), \quad \beta = \left( \frac{A}{12} \right)^{1/2}, \quad V = 4\beta^2. \quad (0.1)$$

If we consider this solution as a generalized function and calculate its asymptotic behavior in the weak sense, then we obtain

$$u \simeq \epsilon \frac{A}{\beta} \delta(x - Vt) \frac{1}{2} \int_{\mathbb{R}^1} \cosh^{-2}(\eta) d\eta. \quad (0.2)$$

Due to this formula we call the solution (0.1)  $\epsilon$ - $\delta$  solution.

Similarly, if we consider a KdV type equation in the form

$$u_t + (u^m)_x + \epsilon^2 u_{xxx} = 0, \quad (0.3)$$

where  $m$  is integer,  $m > 2$ , then we obtain a soliton solution in the form

$$u = A\omega \left( \beta \frac{x - Vt}{\epsilon} \right), \quad (0.4)$$

where  $A > 0$  is a constant,

$$\beta^2 = \gamma^{m-1} A^{m-1}, \quad V = a_m A^{m-1}, \quad \gamma = \left( \frac{m-1}{m+3} \frac{a_m a_2}{a_2'} \right)^{1/(m-1)}.$$

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Here and in the following, we denote  $a_k = \int_{\mathbb{R}^1} \omega^k(\eta) d\eta$  for any  $k \geq 1$ ,  $a'_2 = \int_{\mathbb{R}^1} (d\omega(\eta)/d\eta)^2 d\eta$ , and  $\omega(\eta)$  is the exact solution of the model equation corresponding to (0.3):

$$\frac{d\omega}{d\eta} + \frac{d\omega^m}{d\eta} + \frac{d^3\omega}{d\eta^3} = 0, \quad \omega \rightarrow 0 \quad \text{as } \eta \rightarrow \pm\infty. \quad (0.5)$$

It is clear that this solution belongs to the Schwartz space  $S(\mathbb{R}^1)$ . Again, in the weak sense, we have

$$u \simeq \epsilon \frac{A}{\beta} \delta(x - Vt) \int_{\mathbb{R}^1} \omega(\eta) d\eta.$$

It is well known that the interaction of soliton travelling waves (0.1) can be described by the famous inverse scattering transform method [1], which cannot be used for describing the interaction of waves for  $m > 3$ . However, one can easily see that, in a weak sense, the waves (0.1) and (0.4) are similar. So one can put forward the hypothesis that there must be a procedure for describing the wave interaction in both the integrable ( $m = 2, 3$ ) and non-integrable ( $m > 3$ ) cases.

The goal of the present paper is to confirm this hypothesis and to propose some general procedure (the *weak asymptotic method*) for describing the interaction of nonlinear waves.

Note that, since (0.5) has a solution rapidly decreasing as  $|\eta| \rightarrow \infty$ , (0.3) has a special class of solutions [12, 13], namely, distorted infinitely narrow solitons of the form

$$u = u_0(x, t, \epsilon) + W\left(\beta \frac{x - v(t)}{\epsilon}, t, x, \epsilon\right) + O(\epsilon^N),$$

where  $N \gg 1$ ,  $u_0(x, t, \epsilon)$  and  $W(\eta, t, x, \epsilon)$  are smooth functions, and

$$W(\eta, t, x, 0) = A(t)\omega(\eta).$$

Such solutions correspond to special initial data (of the same form as the solution) and are very sensitive to any general small perturbations [12, 13, 8]. These perturbations need not necessarily lead to instability, but they take the solutions out of the class of infinitely narrow solitons. In this paper we restrict ourselves to considering propagation and interaction of nonlinear waves corresponding to solutions from the class of infinitely narrow solitons, and we do not consider the problem of their stability [8].

We briefly consider the weak asymptotic method. The main notation of the method is the following.

1) By  $f_\epsilon(x) = O_{\mathcal{D}'}(\epsilon^\alpha)$  we denote all generalized functions (depending on  $\epsilon$  as on a parameter) that for any test function satisfy the relation

$$\langle f_\epsilon(x), \psi(x) \rangle = O(\epsilon^\alpha),$$

where  $\langle, \rangle$  denotes the action of a generalized function  $f_\epsilon(x)$  on a test function  $\psi(x)$  and the estimation on the right-hand side is treated in the usual sense.

2) Let a function  $u_\epsilon(x, t)$  belong to  $C^\infty([0, T] \times \mathbb{R}_x^1)$ ,  $T > 0$ , for  $\epsilon > 0$  and belong to  $C(0, T; \mathcal{D}'(\mathbb{R}_x^1))$  uniformly in  $\epsilon \geq 0$ . We say that  $u_\epsilon(x, t)$  is a *weak asymptotic solution* (mod  $O_{\mathcal{D}'}(\epsilon^\alpha)$ ) of the equation

$$Lu = 0$$

if  $Lu_\epsilon = O_{\mathcal{D}'}(\epsilon^\alpha)$ , where, uniformly with respect to  $\epsilon$ , the right-hand side is a weakly continuous and weakly piecewise smooth function of  $t$  (for more detail, see below).

In fact, this means that the difference between the method of weak asymptotic and the method of ordinary asymptotic expansions is that the smallness of the remainder is understood in a different way. Usually, the remainder is assumed to be small in some uniform sense with sufficient accuracy. Here we assume exactly the same but in the sense of  $O_{\mathcal{D}'}$ .

These two remarks allow us to construct asymptotic formulas that describe the propagation and interaction of nonlinear waves. Some results obtained by this method (for instance, results concerning the collision, formation, and destruction of shock waves) were published recently in the papers [3, 4, 5, 6].

In the present paper we use this method for solving the problem of solitary waves (solitons) interaction for KdV type equations with small dispersion, including the integrable case. Here we encounter a problem of constructing a suitable definition of the weak asymptotic solution in the case where the dispersion tends to zero, which is well known as the zero limit dispersion problem [9, 10]. However, in contrast to [9, 10], we deal not with oscillating solutions of the KdV equation or even with more general solutions, but we consider a special class of solutions, i.e., solitons. Therefore, in our case the zero dispersion limit leads to a system of differential equations, instead of integro-differential equations, obtained in [9, 10].

Consider in detail problems related to the definition of the weak asymptotic (mod  $O_{\mathcal{D}'}(\epsilon^2)$ ) solution  $u_\epsilon^*$  for the KdV equation. According to the notation in item 2), we define it so that

$$L_{\text{KdV}}u_\epsilon^* = O_{\mathcal{D}'}(\epsilon^2). \quad (0.6)$$

However, we note that (0.6) implies the loss of distinction between the weak asymptotic solutions of the KdV and Hopf equations. Indeed, since for any generalized functions  $f$ , we obviously have the estimate

$$\epsilon^2 f_{xxx} = O_{\mathcal{D}'}(\epsilon^2),$$

we have

$$L_{\text{KdV}}u_\epsilon^* = \frac{\partial u_\epsilon^*}{\partial t} + \frac{\partial(u_\epsilon^*)^2}{\partial x} + O_{\mathcal{D}'}(\epsilon^2) =: L_H u_\epsilon^* + O_{\mathcal{D}'}(\epsilon^2),$$

where

$$L_H u_\epsilon^* = \frac{\partial u_\epsilon^*}{\partial t} + \frac{\partial(u_\epsilon^*)^2}{\partial x}.$$

Next, let us try to construct a weak asymptotic solution in the sense of (0.6) in the simplest one-soliton situation. We write a smooth ansatz in the form

$$u_\epsilon^* = u_0(x, t) + g(t)\omega\left(\beta\frac{x-\phi}{\epsilon}\right) + e(x, t)\epsilon\omega_0\left(\beta\frac{x-\phi}{\epsilon}\right), \quad \epsilon > 0, \quad (0.7)$$

where  $u_0(x, t)$ ,  $g(t)$ ,  $e(x, t)$ ,  $\phi(t)$  are the desired smooth functions, and, according to the results obtained in [12, 13],  $\beta = \sqrt{g/6}$ . The function  $\omega(z)$  is a solution of the problem (0.5), and  $\omega_0(z) \in C^\infty(\mathbb{R})$  satisfies the conditions  $\lim_{z \rightarrow +\infty} \omega_0(z) = 1$ ,  $\lim_{z \rightarrow -\infty} \omega_0(z) = 0$ , and  $|\frac{d^\alpha \omega_0}{dz^\alpha}| \leq C_\alpha(1 + |z|)^{-3}$ . Then we have in the sense of  $\mathcal{D}'(\mathbb{R})$

$$\omega\left(\beta\frac{x-\phi}{\epsilon}\right) = \frac{\epsilon}{\beta}\delta(x-\phi) + O_{\mathcal{D}'}(\epsilon^2),$$

$$\omega_0\left(\beta\frac{x-\phi}{\epsilon}\right) = \theta(x-\phi) + O_{\mathcal{D}'}(\epsilon), \quad \epsilon \rightarrow +0,$$

where  $\theta(z)$  is the Heaviside function.

The system of equations for the functions  $u_0(x, t)$ ,  $g(t)$ ,  $e(x, t)$ ,  $\phi(t)$  follows from (0.6) (this system is derived in detail in [6]) and has the following form: (a comparative analysis of systems (0.8), (0.10), and (0.11) was performed in [6])

$$\begin{aligned} u_{0t} + (u_0^2)_x &= 0, \\ \phi_t - 2u_0(\phi(t), t) - \frac{2}{3}g(t) &= 0, \\ e(\phi(t), t) - \frac{3\sqrt{6}}{2}g_t(t)/g^{3/2}(t) &= 0, \\ (e_t(x, t) + 2(u_0(x, t)e(x, t))_x) \Big|_{x > \phi(t)} &= 0. \end{aligned} \tag{0.8}$$

It is easy to verify that under the condition  $g > 0$  (which is an analog of the admissibility condition in the theory of shock waves) the solution of system (0.8) exists on any interval  $t \in [0, T]$  such that the smooth solution  $u_0$  of the Hopf equations exists on this interval.

System (0.8) can be solved in the following way: first, one finds the smooth solution of the Hopf equation, next, one finds the function  $e(x, t)$  from the last equations (which is uniquely solvable in view of the inequality  $2u_0(\phi, t) < \phi_t$ ), then one finds the (positive) function  $g(t)$  from the next to the last equation, and finally, one finds the function  $\phi(t)$ .

Note that system (0.8) contains no obstacles to setting  $e(x, t) = 0$ . If so,  $g(t) = \text{const}$  in the case of an arbitrary (nonconstant) background function  $u_0(x, t)$ . But this conclusion is contrary to well known properties of soliton solutions of the KdV equation [12, 13].

Moreover, under our notation, the weak asymptotic of the asymptotic one-soliton solution to the KdV equation, constructed by Maslov and Omelyanov [12, 13], has the form

$$\begin{aligned} u_{1,\varepsilon}^*(x, t) = & u_{01}(x, t) + g_1(t)\omega\left(\beta\frac{x - \phi_1}{\varepsilon}\right) \\ & + e_1(x, t)\varepsilon\left[1 - \omega_0\left(\beta\frac{x - \phi_1}{\varepsilon}\right)\right] + O_{\mathcal{D}'}(\varepsilon^2), \quad \varepsilon \rightarrow +0. \end{aligned} \tag{0.9}$$

In other words, in the case (0.7), the ‘‘shock wave’’ with a small amplitude  $\varepsilon e(x, t)\theta(x - \phi_1(t))$  propagates *in front of the soliton*  $\varepsilon\delta(x - \phi_1(t))$ , but in the asymptotic one-soliton solution constructed in [12, 13] the small shock wave  $\varepsilon e_1(x, t)[1 - \theta(x - \phi_1(t))]$  arises *behind the soliton*.

When we apply (0.6) to the asymptotic solution obtained in [12, 13], whose weak asymptotic yields (0.9), we obtain the following system of equations [6]

$$\begin{aligned} u_{01t} + (u_{01}^2)_x &= 0, \\ \phi_{1t} - 2u_{01}(\phi_1(t), t) - \frac{2}{3}g_1(t) &= 0, \\ e_1(\phi(t), t) + \frac{3\sqrt{6}}{2}g_{1t}(t)/g_1^{3/2}(t) &= 0, \\ (e_{1t}(x, t) + 2(u_{01}(x, t)e_1(x, t))_x) \Big|_{x < \phi_1(t)} &= 0. \end{aligned} \tag{0.10}$$

The solution of this system for  $g_{1t}(t) \neq 0$  is not uniquely determined by the initial conditions  $e_1(x, 0)$  for  $x \leq \phi_1(0)$ , since the velocity along the characteristic ( $\dot{x} =$

$2u_{01}(x(t), t)$  is less (for  $g_1(t) > 0$ ) than that of the soliton  $\phi_{1t} = 2u_{01}(\phi_1(t), t) + \frac{2}{3}g_1(t)$ .

Thus, the assumption that the structure of the solution to the KdV equation is specified by (0.9) due to (0.6) leads to an ill-posed Cauchy problem (with a non-unique solution) for the functions  $u_{01}(x, t)$ ,  $g_1(t)$ ,  $e_1(x, t)$ ,  $\phi_1(t)$ .

On the other hand, the complete system of equations obtained in [12, 13] for these functions has the form

$$\begin{aligned} u_{01t} + (u_{01}^2)_x &= 0, \\ \phi_{1t} - 2u_{01}(\phi_1(t), t) - \frac{2}{3}g_1(t) &= 0, \\ e_1(\phi(t), t) + \frac{3\sqrt{6}}{2}g_{1t}(t)/g_1^{3/2}(t) &= 0, \\ (e_{1t}(x, t) + 2(u_{01}(x, t)e_1(x, t))_x) \Big|_{x < \phi_1(t)} &= 0, \\ g_1(t) + 2u_{01}(\phi_1(t), t) &= \text{const.} \end{aligned} \quad (0.11)$$

It is evident that this system differs from system (0.10) obtained from (0.6) by the additional equation  $g_1(t) + 2u_{01}(\phi_1(t), t) = g_1(0) + 2u_{01}(\phi_1(0), 0)$ . The presence of this equation implies that system (0.11) splits into the two systems

$$\begin{aligned} u_{01t} + (u_{01}^2)_x &= 0, \\ \phi_{1t} - 2u_{01}(\phi_1(t), t) - \frac{2}{3}g_1(t) &= 0, \\ g_1(t) + 2u_{01}(\phi_1(t), t) &= \text{const,} \end{aligned} \quad (0.12)$$

and

$$(e_{1t}(x, t) + 2(u_{01}(x, t)e_1(x, t))_x) \Big|_{x < \phi_1(t)} = 0, \quad (0.13)$$

$$e_1(\phi(t), t) + \frac{3\sqrt{6}}{2}g_{1t}(t)/g_1^{3/2}(t) = 0, \quad (0.14)$$

and equality (0.14) is the boundary condition for equation (0.13), which turns the Cauchy problem for equation (0.13) into the well-posed one (the Cauchy condition, in view of (0.9), has the form  $e_1(x, 0) = e_1^0(x)[1 - \theta(x - \phi_1(0))]$ ).

Moreover, if equation (0.13) is considered formally in the domain  $x > \phi_1(t)$ , which corresponds to the solution structure given by formula (0.7), then the “redundant” condition

$$e(\phi(t), t) - \frac{3\sqrt{6}}{2}g_{1t}(t)/g_1^{3/2}(t) = 0,$$

analogous to (0.14), overdetermines the problem.

Thus, the weak asymptotic corresponding to the asymptotic solution of the Cauchy problem for the KdV equation constructed in [12, 13] cannot be derived from the solution to the KdV equation with the help of (0.6), and vice versa.

Why is it so? The essence of the matter lies in the definition of weak (generalized) solution to nonlinear equation. It turns out that the definition of the weak (generalized) solution to nonlinear equation depends on the structure of the kernel of the operator adjoint to the linearized operator of the initial differential equation which arises when constructing the smooth asymptotic. This construction of the definition of weak solutions was previously discussed in [6, 7].

In the present paper we do not come into details of construction of the definition of the weak solution to our problem. We just point out that, in terms of this construction, the KdV equation is analogous to the phase field system discussed in [7]. The difference is that for the KdV equation the kernel of the adjoint operator mentioned above is *two-dimensional*. Therefore, relation (0.6) is not sufficient for the correctness and it has to be supplied with another condition.

At least for special soliton type initial data for the KdV equation, we can give a definition of a weak solution admitting the zero dispersion limit.

This definition was first presented in [6].

**Definition 0.1.** A function  $u_\epsilon(x, t)$  belonging to  $C^\infty([0, T] \times \mathbb{R}_x^1)$  for  $\epsilon > 0$  and to  $C(0, T; \mathcal{D}'(\mathbb{R}_x^1))$  uniformly in  $\epsilon \geq 0$  is called a *weak asymptotic solution* of the KdV type equation  $Lu = 0$  if the following two relations are satisfied:

$$\begin{aligned} Lu_\epsilon &= O_{\mathcal{D}'}(\epsilon^2), \\ u_\epsilon Lu_\epsilon &= O_{\mathcal{D}'}(\epsilon^2), \end{aligned}$$

where, uniformly with respect to  $\epsilon$ , the right-hand sides are weakly continuous and weakly piecewise smooth functions of  $t$ .

**Remark 0.2.** In fact, this definition means that we do not refuse to use relation (0.6), but we impose additional requirements on the right-hand side of (0.6).

Of course, these relations can be written in the form of usual integral identities (see below). The condition that the remainders  $O_{\mathcal{D}'}(\epsilon^2)$  are weakly continuous in  $t$  allows one to pass from Definition 0.1 to integral identities in the usual sense. This theme was discussed in [6] in detail. In particular, in [6] it was shown that the passage to the limit within the framework of this definition for single exact and distorted solitons leads to the same system of ordinary differential equations, which was derived by using the classical asymptotic method [12, 13]. The consequences of the use of this definition in the case of two solitons (the case of soliton interaction) are the main point of the present paper.

It should be noted that we do not study some weak solution of (0.3) in the Banach space but construct a solution of a specific structure and describe some fine properties of its dynamics. This is impossible by using methods traditional to the theory of nonlinear PDE (e.g., see [2, 11]). Earlier, such results were obtained only for integrable problems by the inverse scattering transform method or for problems similar in a sense to integrable ones and by using the inverse scattering transform method again [1, 12, 13, 14].

We also note that, even in the case of a solitary soliton, our approach allows obtaining results that are new as compared to the well-known results. Namely, the asymptotic methods known today [1, 12, 13, 14, 15] allows calculating the dynamics of a distorted soliton if there is a perturbed right-hand side and/or a variable background. However, in this case there arise very rigid restrictions on the Cauchy data. In the integrable case, in particular, for the KdV equation, it is possible to get rid of the restrictions on lower-order terms (of order  $O(\epsilon)$ ) in the initial data only by using very complicated constructions [8] based on the inverse scattering transform method. In our approach, only the leading term of the asymptotic expansion is fixed and the construction is reduced to simple algebraic calculations.

The text is organized as follows. First, we present some auxiliary results, which are useful for calculations in the weak asymptotic method. In the next section, we study the two-soliton solution of the KdV equation. In the final section, we consider non-integrable KdV type equations.

### 1. AUXILIARY FORMULAS OF THE WEAK ASYMPTOTIC METHOD

**Weak asymptotic expansions.** Let  $\omega(\eta)$  be a continuous function decreasing sufficiently fast as  $|\eta| \rightarrow \infty$  (so that the integrals below were meaningful), and let  $\beta$  be independent of  $x$ . Let us consider the expression  $\omega(\beta x/\epsilon)$  as an element of  $\mathcal{D}'$ . For any test function  $\psi(x)$  we have

$$\begin{aligned} \langle \omega(\beta \frac{x}{\epsilon}), \psi(x) \rangle &= \int_{\mathbb{R}^1} \omega(\beta \frac{x}{\epsilon}) \psi(x) dx = \frac{\epsilon}{\beta} \int_{\mathbb{R}^1} \omega(\eta) \psi(\epsilon \frac{\eta}{\beta}) d\eta \\ &= \sum_{k=0}^n \frac{\epsilon^{k+1}}{\beta^{k+1} k!} \psi^{(k)}(0) \int_{\mathbb{R}^1} \omega(\eta) \eta^k d\eta + O(\epsilon^{n+2}). \end{aligned}$$

This formula is well known in the theory of algebras of Colombeau generalized functions as the momentum decomposition.

Using the notation  $O_{\mathcal{D}'}(\epsilon^\alpha)$  introduced above, we can rewrite this relation as follows:

$$\omega(\beta \frac{x}{\epsilon}) = \sum_{k=0}^n \frac{\epsilon^{k+1}}{\beta^{k+1} k!} (-1)^k \Omega_k \delta^{(k)}(x) + O_{\mathcal{D}'}(\epsilon^{n+2}), \quad (1.1)$$

where  $\Omega_k = \int \eta^k \omega(\eta) d\eta$  and  $\delta(x)$  is the Dirac delta function. In what follows, we restrict ourselves to studying the leading term in (1.1). Then we have

$$\omega(\beta \frac{x}{\epsilon}) = \epsilon \frac{\Omega_0}{\beta} \delta(x) + O_{\mathcal{D}'}(\epsilon^2)$$

in the general case and the last relation can be made more precise by specifying the properties of the function  $\omega(\eta)$ . Namely, the term  $O_{\mathcal{D}'}(\epsilon^2)$  has the form

$$O_{\mathcal{D}'}(\epsilon^2) = -\epsilon^2 \delta'(x) \beta^{-2} \int_{\mathbb{R}^1} \eta \omega(\eta) d\eta + O_{\mathcal{D}'}(\epsilon^3).$$

Thus if  $\omega(\eta)$  is an even function, then

$$\omega(\beta \frac{x}{\epsilon}) = \epsilon \frac{\Omega_0}{\beta} \delta(x) + O_{\mathcal{D}'}(\epsilon^3).$$

Now let us consider a more complicated expression

$$f\left(\omega_1\left(\beta_1 \frac{x - \varphi_1}{\epsilon}\right) + \omega_2\left(\beta_2 \frac{x - \varphi_2}{\epsilon}\right)\right),$$

where  $f(\eta)$  is a smooth function and  $\omega_i(\eta)$ ,  $i = 1, 2$ , possess the same properties as  $\omega(\eta)$ ,  $\beta_1, \beta_2 = \text{const} > 0$ , and  $\varphi_1, \varphi_2$  are constant. We have

$$\begin{aligned} &\langle f\left(\omega_1\left(\beta_1 \frac{x - \varphi_1}{\epsilon}\right) + \omega_2\left(\beta_2 \frac{x - \varphi_2}{\epsilon}\right)\right), \psi \rangle \\ &= \left\{ \beta_1(x - \varphi_1) = \epsilon \eta \right\} \\ &= \left\langle \frac{\epsilon}{\beta_1} \int f_1\left(\omega_1(\eta) + \omega_2\left(\frac{\beta_2}{\beta_1} \eta + \frac{\beta_2}{\epsilon} \Delta \varphi\right)\right) d\eta \delta(x - \varphi_1) + f(0), \psi \right\rangle + O(\epsilon^2), \end{aligned}$$

where  $f_1(\eta) = f(\eta) - f(0)$  and  $\Delta \varphi = \varphi_1 - \varphi_2$ .

On the other hand, introducing the variable  $\eta$  by the formula  $\beta_2(x - \varphi_2) = \epsilon\eta$ , we obtain

$$\begin{aligned} & f\left(\omega_1\left(\beta_1\frac{x-\varphi_1}{\epsilon}\right) + \omega_2\left(\beta_2\frac{x-\varphi_2}{\epsilon}\right)\right) \\ &= f(0) + \frac{\epsilon}{\beta_2} \int_{\mathbb{R}^1} f_1\left(\omega_1\left(\frac{\beta_1}{\beta_2}\eta - \frac{\beta_1}{\epsilon}\Delta\varphi\right) + \omega_2(\eta)\right) d\eta \delta(x - \varphi_2) + O_{\mathcal{D}'}(\epsilon^2). \end{aligned}$$

It is easy to see that the coefficients of  $\delta(x - \varphi_1)$  and  $\delta(x - \varphi_2)$  in the last formulas are equal to each other,

$$\begin{aligned} & \beta_1^{-1} \int_{\mathbb{R}^1} f_1\left(\omega_1(\eta) + \omega_2\left(\frac{\beta_2}{\beta_1}\eta + \frac{\beta_2}{\epsilon}\Delta\varphi\right)\right) d\eta \\ &= \beta_2^{-1} \int_{\mathbb{R}^1} f_1\left(\omega_1\left(\frac{\beta_1}{\beta_2}\eta - \frac{\beta_1}{\epsilon}\Delta\varphi\right) + \omega_2(\eta)\right) d\eta := \Omega_{\Delta\varphi}. \end{aligned}$$

This implies the relation

$$\begin{aligned} & f\left(\omega_1\left(\beta_1\frac{x-\varphi_1}{\epsilon}\right) + \omega_2\left(\beta_2\frac{x-\varphi_2}{\epsilon}\right)\right) \\ &= f(0) + \epsilon\Omega_{\Delta\varphi}(\lambda\delta(x - \varphi_1) + \nu\delta(x - \varphi_2)) + O_{\mathcal{D}'}(\epsilon^2), \end{aligned}$$

where  $\lambda$  and  $\nu$  are arbitrary constants,  $\lambda + \nu = 1$ .

**Asymptotic linear independence.** If we consider linear combinations of generalized functions with accuracy up to  $O_{\mathcal{D}'}(\epsilon^\alpha)$ , then we need to introduce the notion of linear independence. Consider the relation

$$g_1\delta(x - \varphi_1) + g_2\delta(x - \varphi_2) = O_{\mathcal{D}'}(\epsilon^\alpha), \quad \alpha > 0, \quad \varphi_1 \neq \varphi_2,$$

where  $g_i$  are independent of  $\epsilon$ . Obviously, we obtain the relations

$$g_i = O_{\mathcal{D}'}(\epsilon^\alpha), \quad i = 1, 2,$$

which, by virtue of our assumption, imply

$$g_i = 0, \quad i = 1, 2.$$

Everything is different if we assume that the coefficients  $g_i$  can depend on  $\epsilon$ . Here we consider only a special case of such dependence, which we will use later. Namely, let

$$g_i = A_i + S_i\left(\frac{\Delta\varphi}{\epsilon}\right), \quad i = 1, 2,$$

where  $A_i$  are independent of  $\epsilon$  and continuous in  $\varphi_1$  and  $\varphi_2$  and  $S_i(\tau)$  decrease sufficiently fast as  $|\tau| \rightarrow \infty$ .

Let us find out what properties of the coefficients  $g_i$  follow from the relation

$$g_1\delta(x - \varphi_1) + g_2\delta(x - \varphi_2) = O_{\mathcal{D}'}(\epsilon).$$

Applying both sides of the equality to a test function  $\psi$ , we obtain

$$g_1\psi(\varphi_1) + g_2\psi(\varphi_2) = O(\epsilon)$$

or, which is the same,

$$[A_1\psi(\varphi_1) + A_2\psi(\varphi_2)] + [S_1\psi(\varphi_1) + S_2\psi(\varphi_2)] = O(\epsilon). \quad (1.2)$$

Let us consider the expression in the second brackets. Using Taylor's formula, we obtain

$$[S_1\psi(\varphi_1) + S_2\psi(\varphi_2)] = S_1\psi(\varphi_1) + S_2\psi(\varphi_1) + S_2(\varphi_2 - \varphi_1)\psi'(\varphi_1 + \theta\varphi_2).$$



Now we see that

$$S_2\left(\frac{\Delta\varphi}{\epsilon}\right)(\varphi_2 - \varphi_1) = \{-\rho S_2(\rho)\}\big|_{\rho=\Delta\varphi/\epsilon} \cdot \epsilon = O(\epsilon)$$

since the function  $\rho S_2(\rho)$  is assumed to be bounded uniformly in  $\rho \in \mathbb{R}^1$ . So we can rewrite relation (1.2) as

$$A_1\psi(\varphi_1) + A_2\psi(\varphi_2) + (S_1 + S_2)\psi(\varphi_1) = O(\epsilon).$$

Our next goal is to obtain from this relation the condition for the solution to be *uniform* under the choice of  $\varphi_1$  and  $\varphi_2$ . Let  $|\Delta\varphi| \geq \text{const}$ . Then  $|S_1| = O(\epsilon)$  and  $|S_2| = O(\epsilon)$ , and in the usual way we obtain

$$A_1 = 0, \quad A_2 = 0.$$

Further we obtain the relation  $S_1 + S_2 = 0$ . It follows from the continuity that  $A_1 = 0$  and  $A_2 = 0$  also for  $\Delta\varphi = 0$ . Hence we finally obtain

$$A_1 = 0, \quad A_2 = 0, \quad S_1 + S_2 = 0. \quad (1.3)$$

Another method for analyzing relation (1.2) is the following. We choose a point  $x^*$ . Then for any test function  $\psi(x)$  we have

$$\begin{aligned} \langle S_1\delta(x - \varphi_1), \psi \rangle + \langle S_2\delta(x - \varphi_2), \psi \rangle &= S_1\psi(x^*) + S_2\psi(x^*) + O(\epsilon) \\ &= \langle (S_1 + S_2)\delta(x - x^*), \psi \rangle + O(\epsilon), \end{aligned}$$

which implies

$$\begin{aligned} (A_1 + S_1)\delta(x - \varphi_1) + (A_2 + S_2)\delta(x - \varphi_2) \\ = A_1\delta(x - \varphi_1) + A_2\delta(x - \varphi_1) + (S_1 + S_2)\delta(x - x^*) + O_{\mathcal{D}'}(\epsilon). \end{aligned}$$

Following the above argument, we again obtain relations (1.3).

## 2. $\epsilon$ - $\delta$ SOLITON INTERACTION IN THE KdV MODEL

In this and in the following sections, by using the ideas briefly considered above, we qualitatively describe the interaction of  $\epsilon$ - $\delta$ -solitons (of narrow solitary waves) in models governed by KdV type equations. From methodological considerations, we first study the simplest case, i.e., the KdV equation with small dispersion

$$u_t + (u^2)_x + \epsilon^2 u_{xxx} = 0 \quad (2.1)$$

supplemented with the initial data, which are the superposition of two solitary waves

$$u|_{t=0} = u^0(x, \epsilon), \quad u^0 = A_1\omega\left(\beta_1\frac{x - x_1^0}{\epsilon}\right) + A_2\omega\left(\beta_2\frac{x - x_2^0}{\epsilon}\right). \quad (2.2)$$

We set  $\beta_i = \sqrt{A_i/12}$ ,  $\omega(\eta) = 1/2 \cosh^2(\eta)$ , and assume that

$$x_2^0 < x_1^0, \quad A_2 > A_1 > 0$$

so that there exist a time moment  $t^*$  at which the trajectories of the solitary waves corresponding to (2.2) intersect:  $V_1 t^* + x_1^0 = V_2 t^* + x_2^0$ ,  $V_i = 4\beta_i^2$ .

Needless to say, for the KdV equation, as well as for the modified KdV (MKdV) equation, the behavior of the solution is well known: the solitons interact passing through each other without changing their shapes and, after the interaction, they are shifted by some distance. This assertion can be obtained directly, for instance,

from the formula for the exact two-soliton solution of problem (2.1), (2.2) derived by the inverse scattering transform method (e.g., see [1])

$$\begin{aligned} u_{\text{sol}} &= 6\beta_1^2(\beta_2^2 - \beta_1^2)v_1(\eta_1, \rho) + 6\beta_2^2(\beta_2^2 - \beta_1^2)v_2(\eta_2, \rho), \\ v_1 &= \left\{ \beta_1 \sinh(\beta_1\eta_1 - \mu) - \beta_2 \coth(\beta_2(\eta_1 - \rho) + \mu) \cosh(\beta_1\eta - \mu) \right\}^{-2}, \\ v_2 &= \left\{ \beta_2 \cosh(\beta_2\eta_2 + \mu) - \beta_1 \tanh(\beta_1(\eta_2 + \rho) - \mu) \sinh(\beta_2\eta_2 + \mu) \right\}^{-2}, \end{aligned} \quad (2.3)$$

where  $\eta_i = (x - \varphi_{i0}(t))/\epsilon$ ,  $\varphi_{i0} = V_i t + x_i^0$ ,  $\rho = (\varphi_{20}(t) - \varphi_{10}(t))/\epsilon$ , and

$$\mu = \frac{1}{2} \ln \frac{\beta_2 - \beta_1}{\beta_2 + \beta_1} \quad (2.4)$$

is the displacement of the soliton trajectories after the interaction.

It is well known that as  $\rho \rightarrow -\infty$  (that is, for  $\varphi_{20} - \varphi_{10} < 0$ ,  $\epsilon \rightarrow 0$ , i.e., prior to the interaction) and as  $\rho \rightarrow \infty$  (that is, for  $\varphi_{20} - \varphi_{10} > 0$ ,  $\epsilon \rightarrow 0$ , i.e., after the interaction), formula (2.3) is the sum of isolated solitons with accuracy up to terms admitting the estimate  $O(\epsilon^N)$  for any  $N > 0$ . However, after the interaction, some constants are added to the arguments of the solutions.

In fact, the problem of describing the soliton interaction (as soon as the qualitative mechanism of interaction becomes clear) is just the calculation of these constants.

As mentioned above, the weak asymptotic method permits us to obtain a description of the single soliton dynamics in an extremely simple way. There is a natural question of whether we can describe the interaction of solitons passing to the limit in the weak sense as  $\epsilon \rightarrow 0$ .

In what follows, we present some results obtained in this way. According to our approach, for problem (2.1), (2.2) we present the two-soliton weak asymptotic solution  $\text{mod } \mathcal{O}_{\mathcal{D}'}(\epsilon^2)$  in the form

$$\begin{aligned} u &= g_1(\tau)\omega\left(\beta_1 \frac{x - \phi_1(t, \tau, \epsilon)}{\epsilon}\right) + g_2(\tau)\omega\left(\beta_2 \frac{x - \phi_2(t, \tau, \epsilon)}{\epsilon}\right), \\ g_i &= A_i + S_i(\tau), \quad \phi_i = \varphi_{i0}(t) + \epsilon\varphi_{i1}(\tau), \quad \tau = \beta_1(\varphi_{20}(t) - \varphi_{10}(t))/\epsilon, \end{aligned} \quad (2.5)$$

where  $\varphi_{i0}(t) = V_i t + x_i^0$  so that  $x = \varphi_{i0}(t)$  is the trajectory of motion of the solitary wave with constant amplitude  $A_i$ , the function  $\tau = \tau(t, \epsilon)$  has the meaning of "fast" time,  $\tau(t^*, \epsilon) = 0$ ,  $S_i$  and  $\varphi_{i1}$  are corrections to the amplitudes and the phases rapidly varying during the time of interaction with exponential velocities:

$$\begin{aligned} S_i(\tau) &\rightarrow 0 \quad \text{as } \tau \rightarrow \pm\infty, \\ \varphi_{i1}(\tau) &\rightarrow 0 \quad \text{as } \tau \rightarrow -\infty, \quad \varphi_{i1}(\tau) \rightarrow \varphi_{i1}^\infty = \text{const} \quad \text{as } \tau \rightarrow +\infty. \end{aligned} \quad (2.6)$$

It turned out that (in contrast to the inverse problem method) the construction of the solution significantly depends on the ratio of the parameters  $\beta_1$  and  $\beta_2$ , i.e., on  $\theta = \beta_1/\beta_2$ , and this ratio can be easily recalculated for the ratio of soliton amplitudes. Namely, it turned out that the solution in the form of (2.5) can be constructed for  $0 < \theta < 1/2$ . For  $\theta = 1/2$  the formulas degenerate, and this case must be studied separately. For  $\theta \in (\frac{1}{2}, 1)$ , the functions contained in the expression on the right-hand side of (2.5) become complex-valued, and, in order to deal with real functions, we must seek the solution in the form

$$u = g_1(\tau)\omega\left(\beta_1 \frac{x - \phi_1(t, \tau, \epsilon)}{\epsilon}\right) + g_2(\tau)\omega\left(\beta_2 \frac{x - \phi_2(t, \tau, \epsilon)}{\epsilon}\right) + \text{c.c.}$$

It is clear that these additional terms make our calculations much more complicate. So, in what follows, we restrict our study of interaction to the case  $0 < \theta < 1/2$ . Strictly speaking, we can prove that there exists a  $\theta^* < 1/2$  such that our statements are true for  $0 < \theta < \theta^*$ . But numerical calculations show that  $\theta^* = 1/2$ .

Under this restriction on the initial amplitudes of solitons, we construct an asymptotic solution of the KdV equation and, in the next section, for the KdV type equations.

Our aim is to reconstruct all the qualitative properties of soliton interaction obtained by the inverse problem method:

- 1) solitons pass through each other without changing their structure,
- 2) the result of interaction is a phase shift.

In what follows, such a mechanism of interaction of solitary waves will be called the *soliton scenario of interaction*. The description of interaction is reduced to the study of autonomous first-order differential equations. Moreover, these equations are of similar form for both the KdV equation and the non-integrable KdV type equations.

One cannot solve these equations explicitly, however, for small values of the parameter  $\theta$ , it is possible to obtain enough information about the solution and to calculate the phase shifts. The formulas thus obtained differ from (2.4). The matter is that, as will be shown below, the KdV type equations with small dispersion are not well defined in the asymptotic sense. Namely, an arbitrary small perturbation of the solution can lead to a contribution to the solution of order  $O_{\mathcal{D}'}(\epsilon^2)$  and thus to change the phase shift.

The main result of this section, which we obtain without using the inverse scattering transform method, is the following theorem.

**Theorem 2.1.** *Let  $\theta = \beta_1/\beta_2 \in (0, 1/2)$ , then the interaction of  $\epsilon$ - $\delta$ -solitons of the KdV equation follows the soliton scenario. The weak asymptotic of the solution of problem (2.1), (2.2) is asymptotically not unique in the terms of order  $O_{\mathcal{D}'}(\epsilon^2)$ .*

We prove this assertion in two steps. First, we construct the weak asymptotic modulo  $O_{\mathcal{D}'}(\epsilon^2)$  of the solution of problem (2.1), (2.2) in the form (2.5), (2.6) At the second stage of the proof, we show that a perturbation (arbitrary small in the sense of  $\mathcal{D}'$ ) of the leading term of the asymptotic of (2.5) changes the constants  $\varphi_{i1}^\infty$ .

Prior to proving the theorem, let us discuss the leading term of the weak asymptotic of (2.5) and compare it with the exact solution (2.3). First, recall that we understand the solution of problem (2.1), (2.2) in the weak sense (see Introduction), that is, by a weak asymptotic modulo  $O_{\mathcal{D}'}(\epsilon^2)$  of the solution we mean a function  $u = u(x, t, \epsilon)$  such that for any test function  $\psi = \psi(x)$  the following relations are satisfied:

$$\begin{aligned} \frac{d}{dt} \int u\psi \, dx - \int u^2 \frac{\partial \psi}{\partial x} \, dx &= O(\epsilon^2), \\ \frac{d}{dt} \int u^2 \psi \, dx - \frac{4}{3} \int u^3 \frac{\partial \psi}{\partial x} \, dx + 3 \int \left( \epsilon \frac{\partial u}{\partial x} \right)^2 \frac{\partial \psi}{\partial x} \, dx &= O(\epsilon^2), \\ \int (u|_{t=0} - u^0(x, \epsilon)) \psi \, dx &= O(\epsilon^2). \end{aligned} \quad (2.7)$$

Here and in the following, the notation  $\int \dots dx$  means the integration over  $\mathbb{R}^1$ .

The most unexpected result obtained while proving the theorem is that the following fact.

**Theorem 2.2.** *The condition  $\theta \in (0, 1/2)$ , the conservation laws*

$$\frac{d}{dt} \int u \, dx = 0, \quad (2.8)$$

$$\frac{d}{dt} \int u^2 \, dx = 0, \quad (2.9)$$

and the energy relations

$$\frac{d}{dt} \int xu \, dx - \int u^2 \, dx = 0, \quad (2.10)$$

$$\frac{d}{dt} \int xu^2 \, dx - \frac{4}{3} \int u^3 \, dx + 3 \int \left( \epsilon \frac{\partial u}{\partial x} \right)^2 \, dx = 0 \quad (2.11)$$

are necessary and sufficient conditions for the function (2.5) to be a weak asymptotic of the solution of problem (2.1), (2.2) in the sense of (2.7).

Let us discuss the structure of the asymptotic solution of (2.5). Obviously, the above formulas imply that the function (2.5) is the sum of solitary waves until they interact

$$u = \sum_{i=1}^2 A_i \omega \left( \beta_i \frac{x - \varphi_{i0}(t)}{\epsilon} \right) + O(\epsilon^\infty), \quad t < t^*,$$

and the sum of solitary waves with phase shift

$$u = \sum_{i=1}^2 A_i \omega \left( \beta_i \frac{x - \varphi_{i0}(t)}{\epsilon} - \beta_i \varphi_{i1}^\infty \right) + O(\epsilon^\infty), \quad t > t^*,$$

after the interaction. We stress that the resulting phase shifts  $\beta_i \varphi_{i1}^\infty$  satisfy the same relation

$$\beta_1 \varphi_{11}^\infty + \beta_2 \varphi_{21}^\infty = 0 \quad (2.12)$$

as those for the exact two-soliton solution (2.3). In the sense of  $\mathcal{D}'$ , for the function (2.5) uniformly in  $t$  we have

$$u = \epsilon \sum_{i=1}^2 \frac{g_i(\tau)}{\beta_i} \delta(x - \phi_i) + O_{\mathcal{D}'}(\epsilon^3). \quad (2.13)$$

However,  $\epsilon \delta(x - \phi_i) = \epsilon \delta(x - \varphi_{i0}) + O_{\mathcal{D}'}(\epsilon^2)$  and we see that only soliton components of the amplitude  $S_i$  are essential, while the role of phase shifts  $\epsilon \varphi_{i1}$  seems to be unessential. Moreover, a similar conclusion can readily be obtained by calculating the weak asymptotic of the exact solution (2.3):

$$u_{\text{sol}} = \epsilon \sum_{i=1}^2 \frac{P_i(\tau)}{\beta_i} \delta(x - \varphi_{i0}(t)) + O_{\mathcal{D}'}(\epsilon^2), \quad (2.14)$$

where

$$P_i = 6\beta_i^2(\beta_2^2 - \beta_1^2) \int v_i(\eta, \tau/\beta_1) \, d\eta \rightarrow A_i \quad \text{as } \tau \rightarrow \pm\infty.$$

Nevertheless, the phase shifts play an important role. To verify this, we calculate the weak asymptotic for the derivatives of the function (2.5). Taking into account that fact that  $S_i$  and  $\varphi_{i1}$  vary rapidly with time, we obtain

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sum_{i=1}^2 \left\{ \frac{c}{\beta_i} \frac{\partial S_i}{\partial \tau} \delta(x - x^*) - \epsilon \frac{A_i}{\beta_i} V_i \delta'(x - \varphi_{i0}) \right. \\ &\quad \left. - c\epsilon \frac{\partial}{\partial \tau} \left[ V_i \frac{\tau}{c} \frac{S_i}{\beta_i} + \frac{g_i}{\beta_i} \varphi_{i1} \right] \delta'(x - x^*) \right\} + O_{\mathcal{D}'}(\epsilon^2), \end{aligned}$$

where  $c = \beta_1(V_2 - V_1)$  and  $x^* = \varphi_{i0}(t^*)$  is the point of intersection of the soliton trajectories. Similar calculations for the exact solution (2.3) lead to the expansion

$$\begin{aligned} \frac{\partial u_{\text{sol}}}{\partial t} &= \sum_{i=1}^2 \left\{ \frac{c}{\beta_i} \frac{\partial P_i^s}{\partial \tau} \delta(x - x^*) - \epsilon \frac{A_i}{\beta_i} V_i \delta'(x - \varphi_{i0}) \right. \\ &\quad \left. - c\epsilon \frac{\partial}{\partial \tau} \left[ V_i \frac{\tau}{c} \frac{P_i^s}{\beta_i} + \frac{D_i}{\beta_i} \right] \delta'(x - x^*) \right\} + O_{\mathcal{D}'}(\epsilon^2), \end{aligned}$$

where  $D_i = 6\beta_i^2(\beta_2^2 - \beta_1^2) \int \eta v_i(\eta, \tau/\beta_1) d\eta$  and we set  $P_i = A_i + P_i^s$  so that  $P_i^s \rightarrow 0$  as  $\tau \rightarrow \pm\infty$ . Comparing these two formulas, as well as (2.13) and (2.14), we readily see that the varying phase shifts of the function (2.5) play the same role as the odd (with respect to  $\eta$ ) parts of the functions  $v_i(\eta, \tau/\beta_i)$  in the exact solution.

If we discuss the interaction process and compare the functions (2.3) and (2.5) in the pointwise sense, then we see that these functions are, of course, different. Nevertheless, they possess an important common property: the soliton type (with respect to time) components of the amplitudes compensate one another. We mean that uniformly in  $\tau \in \mathbb{R}^1$ , the relation

$$S_1(\tau)/\beta_1 + S_2(\tau)/\beta_2 = 0 \tag{2.15}$$

holds for the weak asymptotic (2.5) (we prove this later). This coincides with the identity

$$P_1(\tau)/\beta_1 + P_2(\tau)/\beta_2 = A_1/\beta_1 + A_2/\beta_2,$$

which can be derived directly for the exact solution by rewriting formula (2.3) in the form standard for the inverse problem method or by using the conservation law (2.8).

Prior to proving Theorem 2.1, we introduce some auxiliary functions we need to describe the weak asymptotic (2.5) exactly. By  $\lambda_j = \lambda_j(\sigma)$  and  $L_i = L_i(\sigma)$  we denote the following convolutions depending on the parameter  $\theta = \beta_1/\beta_2$ :

$$\begin{aligned} \lambda_0 &= \frac{1}{a_2} \int \omega(\eta)\omega(\theta\eta + \sigma) d\eta, & \lambda_1 &= \frac{1}{a_2} \int \eta\omega(\eta)\omega(\theta\eta + \sigma) d\eta, \\ \lambda_2 &= \frac{1}{a_2} \int \omega'(\eta)\omega'(\theta\eta + \sigma) d\eta, & \omega'(z) &= \partial\omega(z)/\partial z, \end{aligned} \tag{2.16}$$

$$\begin{aligned} L_1 &= \frac{1}{a_3} \int \left\{ \theta(\theta + k)\omega(\theta\eta + \sigma) + (1 - k)\omega(\eta) \right\}^3 d\eta - (1 + \theta^5), \\ L_2 &= \theta^3 k(2\theta + k) + k^2 - 2k + 2\theta^2(\theta + k)(1 - k)\lambda_2, \end{aligned} \tag{2.17}$$

where  $a_2, a_3, a'_2$  are the numbers

$$\begin{aligned} a_2 &= \int \omega^2(\eta) d\eta = \frac{1}{3}, & a_3 &= \int \omega^3(\eta) d\eta = \frac{2}{15}, \\ a'_2 &= \int (\omega'(\eta))^2 d\eta = \frac{4}{15}, \end{aligned} \quad (2.18)$$

and  $k = k(\sigma)$  is the function determined for  $\theta \in (0, 1/2)$  by the expression

$$\begin{aligned} k &= \{(1 - \theta)(1 + \theta - \lambda_0\theta) - \sqrt{I}\}(1 + \theta - 2\lambda_0\theta)^{-1}, \\ I &= (1 + \theta)^2(1 - \theta - \lambda_0\theta)^2 + 2\lambda_0\theta^2(1 + \theta)(1 - 2\theta). \end{aligned} \quad (2.19)$$

For  $\theta = 1/2$  we set

$$k = \lambda_0 / (3 - 2\lambda_0). \quad (2.20)$$

Next, we introduce a function  $\mathcal{F} = \mathcal{F}(\sigma)$  by the formula

$$\mathcal{F} = F / (1 - \theta + \mathcal{J}'), \quad (2.21)$$

where

$$\begin{aligned} F &= \left( \frac{4}{3} \frac{a_3}{a_2^2} (L_1 - L_2) + L_2 \right) (\theta(1 - \theta)^2)^{-1} - \mathcal{J}', \\ \mathcal{J}' &= k \left( \frac{1 - \theta - \theta^2}{\theta} - k \right) + \left\{ \left( \frac{1 - \theta - \theta^2}{\theta} - 2k \right) \sigma + 2\theta(1 - \theta - 2k)\lambda_1 \right\} \frac{\partial k}{\partial \sigma} \\ &\quad + 2\theta(\theta + k)(1 - k) \frac{\partial \lambda_1}{\partial \sigma}. \end{aligned}$$

Let us define a function  $\sigma_0 = \sigma_0(\tau)$  depending also on the parameter  $\theta$  as a solution of the problem

$$\frac{d\sigma_0}{d\tau} = \mathcal{F}(\tau + \sigma_0), \quad \sigma_0 \rightarrow 0 \quad \text{as } \tau \rightarrow -\infty. \quad (2.22)$$

Next, we show that problem (2.22) has solution for all  $\tau \in \mathbb{R}^1$ . Moreover,

$$\sigma_0 \rightarrow \sigma_0^\infty(\theta) \quad \text{as } \tau \rightarrow +\infty \quad (2.23)$$

as a function of exponential type.

**Theorem 2.3.** *Let  $\theta \in (0, 1/2)$ , then the weak asymptotic solution of problem (2.1), (2.2) has the form (2.5), where*

$$S_1 = \sqrt{A_1 A_2} k(\tau + \sigma_0), \quad S_2 = -A_2 k(\tau + \sigma_0), \quad (2.24)$$

$$\begin{aligned} \varphi_{11} &= \frac{1}{\beta_1(1 + \theta)} \{ \tau k(\tau + \sigma_0) - \sigma_0(1 - k(\tau + \sigma_0)) \}, \\ \varphi_{21} &= \frac{1}{\beta_2\theta(1 + \theta)} \{ \theta\sigma_0 + (\tau + \sigma_0)k(\tau + \sigma_0) \}. \end{aligned} \quad (2.25)$$

**Remark 2.4.** The above formulas readily imply that

$$\begin{aligned} k(\sigma) &\rightarrow 0 \quad \text{as } \sigma \rightarrow \pm\infty, & \varphi_{i1}(\tau) &\rightarrow 0 \quad \text{as } \tau \rightarrow -\infty, \quad i = 1, 2, \\ \beta_1\varphi_{11}(\tau) &\rightarrow -\sigma_0^\infty / (1 + \theta), & \beta_2\varphi_{21}(\tau) &\rightarrow \sigma_0^\infty / (1 + \theta) \quad \text{as } \tau \rightarrow +\infty \end{aligned}$$

as functions of exponential type.

*Proof of Theorem 2.3.* We present the leading term of the weak asymptotic of the solution of problem (2.1), (2.2) in the form (2.5), where  $S_i = S_i(\tau)$  and  $\phi_i = \varphi_{i0}(t) + \epsilon\varphi_{i1}(\tau)$  are the desired smooth functions. We assume that

$$\begin{aligned} S_i(\tau) &\rightarrow 0 \quad \text{as } \tau \rightarrow \pm\infty, \quad i = 1, 2, \\ \varphi_{i1}(\tau) &\rightarrow 0 \quad \text{as } \tau \rightarrow -\infty, \quad \varphi_{i1}(\tau) \rightarrow \varphi_{i1}^\infty \quad \text{as } \tau \rightarrow +\infty \end{aligned}$$

as functions of exponential type. Now we apply formulas from Sec. 1 and calculate the weak asymptotic

$$\begin{aligned} u^n &= \epsilon a_n \sum_{i=1}^2 K_{i0}^{(n)} \delta(x - \phi_i) + \epsilon a_n \left\{ \sum_{i=1}^2 K_{i1}^{(n)} + R_n \right\} \delta(x - x^*) + O_{\mathcal{D}'}(\epsilon^2), \\ (\epsilon u_x)^2 &= \epsilon a_2' \sum_{i=1}^2 \beta_i^2 K_{i0}^{(2)} \delta(x - \phi_i) + \epsilon a_2' \left\{ \sum_{i=1}^2 \beta_i^2 K_{i1}^{(2)} + Q_2 \right\} \delta(x - x^*) + O_{\mathcal{D}'}(\epsilon^2), \quad (2.26) \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial t} &= c \frac{\partial}{\partial \tau} \sum_{i=1}^2 K_{i1}^{(1)} \delta(x - x^*) - \epsilon \sum_{i=1}^2 V_i K_{i0}^{(1)} \delta'(x - \phi_i) \\ &\quad - \epsilon c \frac{\partial}{\partial \tau} \left\{ \sum_{i=1}^2 K_i^{(1)} \varphi_{i1} + \frac{V_i}{c} \tau K_{i1}^{(1)} \right\} \delta'(x - x^*) + O_{\mathcal{D}'}(\epsilon^2), \end{aligned}$$

$$\begin{aligned} \frac{\partial u^2}{\partial t} &= \epsilon a_2 \frac{\partial}{\partial \tau} \left\{ \sum_{i=1}^2 K_{i1}^{(2)} + R_2 \right\} \delta(x - x^*) - \epsilon a_2 \sum_{i=1}^2 V_i K_{i0}^{(2)} \delta'(x - \phi_i) \quad (2.27) \\ &\quad - \epsilon c a_2 \frac{\partial}{\partial \tau} \left\{ \sum_{i=1}^2 \left( K_i^{(2)} \varphi_{i1} + \frac{V_i}{c} \tau K_{i1}^{(2)} \right) + R_2^{(1)} \right\} \delta'(x - x^*) + O_{\mathcal{D}'}(\epsilon^2). \end{aligned}$$

Here  $K_i^{(n)} = g_i^n / \beta_i$ ,  $K_{i0}^{(n)} = A_i^n / \beta_i$ ,  $K_{i1}^{(n)} = K_i^{(n)} - K_{i0}^{(n)}$ , and the numbers  $a_n$  for  $n = 2, 3$  and  $a_2'$  are calculated by formulas (2.18), while  $a_1 = 1$ . We also set  $V_i = \varphi'_{i0t}(t^*)$ ,  $c = \beta_1(V_2 - V_1)$ , and

$$R_n = \frac{1}{\beta_2 a_n} \int (g_1 \omega(\theta \eta + \sigma) + g_2 \omega(\eta))^n d\eta - \sum_{i=1}^2 K_i^{(n)}.$$

Moreover, using the notation (2.16), we obtain

$$R_2 = 2 \frac{g_1 g_2}{\beta_2} \lambda_0, \quad R_2^{(1)} = 2 \frac{g_1 g_2}{\beta_2} \left\{ \left( \frac{V_2}{c} \tau + \varphi_{21} \right) \lambda_0 + \frac{1}{\beta_2} \lambda_1 \right\}, \quad Q_2 = 2 g_1 g_2 \beta_1 \lambda_2. \quad (2.28)$$

By definition, the weak asymptotic of the solution must satisfy the relations

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = O_{\mathcal{D}'}(\epsilon^2), \quad \frac{\partial u^2}{\partial t} + \frac{4}{3} \frac{\partial u^3}{\partial x} - 3 \frac{\partial}{\partial x} (\epsilon u_x)^2 = O_{\mathcal{D}'}(\epsilon^2). \quad (2.29)$$

We substitute expansions (2.26), (2.27) into (2.29) and collect the term containing  $\delta(x - x^*)$ ,  $\epsilon \delta'(x - \phi_i)$ , and  $\epsilon \delta'(x - x^*)$ . By equating the obtained relations with

zero, we arrive at the relations

$$\frac{d}{d\tau} \sum_{i=1}^2 K_{i1}^{(1)} = 0, \quad (2.30)$$

$$\frac{d}{d\tau} \left\{ \sum_{i=1}^2 K_{i1}^{(2)} + R_2 \right\} = 0, \quad (2.31)$$

$$\frac{d}{d\tau} \sum_{i=1}^2 \left\{ K_i^{(1)} \varphi_{i1} + \tau \frac{V_i}{c} K_{i1}^{(1)} \right\} = 0, \quad (2.32)$$

$$\begin{aligned} c \frac{d}{d\tau} \left\{ \sum_{i=1}^2 \left( K_i^{(2)} \varphi_{i1} + \tau \frac{V_i}{c} K_{i1}^{(2)} \right) + R_2^{(1)} \right\} \\ = \frac{4}{3} \frac{a_3}{a_2} \left\{ \sum_{i=1}^2 K_{i1}^{(3)} + R_3 \right\} - 3 \frac{a'_2}{a_2} \left\{ \sum_{i=1}^2 \beta_i^2 K_{i1}^{(2)} + Q_2 \right\}, \end{aligned} \quad (2.33)$$

as well as the relations

$$-V_i K_{i0}^{(1)} + a_2 K_{i0}^{(2)} = 0, \quad -a_2 V_i K_{i0}^{(2)} + \frac{4}{3} a_3 K_{i0}^{(3)} - 3a'_2 \beta_i^2 K_{i0}^{(2)} = 0, \quad i = 1, 2, \quad (2.34)$$

describing the motion of solitary waves without taking into account their interaction. Relations (2.34) readily imply

$$V_i = a_2 A_i, \quad \beta_i^2 = \gamma A_i, \quad \gamma = \left( \frac{4}{3} a_3 - a_2^2 \right) / 3a'_2. \quad (2.35)$$

Let us transform system (2.30)–(2.33). Integrating the first three relations and performing simple transformations, we obtain

$$K_{11}^{(1)} + K_{21}^{(1)} = 0, \quad (2.36)$$

$$\sum_{i=1}^2 K_{i1}^{(2)} + R_2 = 0, \quad (2.37)$$

$$\frac{\sigma}{\beta_1} K_{11}^{(1)} - \sum_{i=1}^2 K_{i0}^{(1)} \varphi_{i1} = 0. \quad (2.38)$$

Next, we use (2.28), (2.36) and rewrite (2.37) as follows:

$$-2(A_2 - A_1) K_{11}^{(1)} + (\beta_1 + \beta_2) (K_{11}^{(1)})^2 + \frac{2}{\beta_2} (A_1 + \beta_1 K_{11}^{(1)}) (A_2 - \beta_2 K_{11}^{(1)}) \lambda_0 = 0. \quad (2.39)$$

Solving (2.39) for  $K_{11}^{(1)}$  and choosing the root from the condition that  $K_{11}^{(1)} \rightarrow 0$  as  $\tau \rightarrow -\infty$ , we obtain

$$K_{11}^{(1)} = \frac{\beta_2}{\gamma} k(\sigma), \quad (2.40)$$

where the function  $k(\sigma)$  is defined in (2.19) for  $\theta \neq 1/2$  and in (2.20) for  $\theta = 1/2$ . Let us analyze formulas (2.19), (2.20). The obvious inequality  $\lambda_0(\sigma, \theta) \leq 1/\sqrt{\theta}$  for  $\theta < 1$  implies

$$1 + \theta - 2\lambda_0(\sigma, \theta)\theta \geq (1 - \sqrt{\theta})^2 \geq \text{const} > 0.$$

Next, let us consider the expression  $I$  defined in (2.19). It is easy to see that the maximal value (with respect to  $\sigma$ ) of  $\lambda_0$ , which we denote by  $\lambda_0(0, \theta)$ , is a function of  $\theta \in [0, 1]$  monotonically decreasing from  $3/2$  to  $1$ . Therefore, for  $\theta \geq 2/5$  there exists  $\sigma^* = \sigma^*(\theta)$  for which the first term in the formula for  $I$  is zero. Nevertheless,



the second term in this formula is positive for  $\theta < 1/2$  and negative for  $\theta > 1/2$ . Hence for  $\theta < 1/2$  we have the following estimate uniform with respect to  $\sigma$ :

$$I \geq \text{const} > 0, \tag{2.41}$$

and we see that for  $\theta > 1/2$  this inequality does not hold.

For  $\theta = 1/2$  we define  $k$  by formula (2.20). Since  $3 - 2\lambda_0|_{\theta=1/2} \geq \text{const} > 0$ , this formula determines the smooth function  $k(\sigma)$ .

Finally, for  $\theta \in (c_1, 1/2]$ ,  $c_1 > 0$ , the function  $k(\sigma)$  is smooth. It is also easy to see that this function is positive, even with respect to  $\sigma$ , and tends to zero as  $\sigma \rightarrow \pm\infty$  as an exponential function.

Now let us consider relation (2.38) together with the relation

$$\sigma = \tau + \beta_1(\varphi_{21} - \varphi_{11}). \tag{2.42}$$

By solving (2.38), (2.42) for  $\varphi_{i1}$ , we obtain

$$\begin{aligned} \varphi_{11} &= (\beta_1 \sum_{i=1}^2 K_{i0}^{(1)})^{-1} (\tau K_{20}^{(1)} - \sigma K_2^{(1)}), \\ \varphi_{21} &= (\beta_1 \sum_{i=1}^2 K_{i0}^{(1)})^{-1} (-\tau K_{10}^{(1)} + \sigma K_1^{(1)}). \end{aligned} \tag{2.43}$$

Let us transform (2.33). After simple calculations, taking into account (2.28), (2.36), (2.40), and (2.43), we obtain

$$\sum_{i=1}^2 \left( K_i^{(2)} \varphi_{i1} + \tau \frac{V_i}{c} K_{i1}^{(2)} \right) + R_2^{(1)} = \frac{\beta_2^2}{\gamma^2} \{ -\tau(1 - \theta) + \sigma(1 - \theta) + \mathcal{J}(\sigma) \},$$

where

$$\mathcal{J}(\sigma) = \sigma(\theta^{-1}(1 - \theta - \theta^2)k - k^2) + 2\theta(\theta + k)(1 - k)\lambda_1. \tag{2.44}$$

Now we note that the following relations hold:

$$\begin{aligned} \sum_{i=1}^2 K_{i1}^{(3)} + R_3 &= \frac{\beta_2^5}{\gamma^3} L_1(\sigma), \quad \sum_{i=1}^2 \beta_i^2 K_{i1}^{(2)} + Q_2 = \frac{\beta_2^5}{\gamma^2} L_2(\sigma), \\ c &= a_2 \frac{\beta_2^3}{\gamma} \theta(1 - \theta^2), \quad 3\gamma a'_2 = \frac{4}{3} a_3 - a_2^2, \end{aligned}$$

where  $L_1, L_2$  are defined in (2.17). Hence we can rewrite Eq. (2.33) in the autonomous form

$$(1 - \theta + \mathcal{J}') \frac{d\sigma}{d\tau} = 1 - \theta + \frac{F_1(\sigma)}{\theta(1 - \theta^2)}, \tag{2.45}$$

where

$$F_1 = \frac{4}{3} \frac{a_3}{a_2^2} (L_1 - L_2) + L_2, \quad \mathcal{J}' = \frac{\partial \mathcal{J}}{\partial \sigma}.$$

Formulas (2.16), (2.44) and the properties of  $k(\sigma)$  readily imply that  $\mathcal{J}(\sigma) \in C^\infty(\mathbb{R}^1)$  for  $0 < \theta \leq 1/2$ . Moreover,  $\mathcal{J}(\sigma)$  is odd and tends to zero as  $\sigma \rightarrow \pm\infty$  as an exponential function.

Consider  $\mathcal{J}(\sigma)$  for small  $\theta$ . Expanding  $\omega(\theta\eta + \sigma)$  in the Taylor series around the point  $\theta = 0$ , we obtain

$$\lambda_0 = \frac{1}{a_2} \omega(\sigma) + O(\theta^2), \quad \lambda_1 = O(\theta).$$

Thus we have

$$k(\sigma) = \frac{\theta^2}{a_2} \omega(\sigma) + O(\theta^3), \quad \mathcal{J}(\sigma) = \frac{\theta(1-\theta)}{a_2} \sigma \omega(\sigma) + O(\theta^3), \quad (2.46)$$

and there exists  $\theta_1^* \in (0, 1/2]$  such that uniformly in  $\sigma$

$$1 - \theta + \mathcal{J}'(\sigma) \geq \text{const} > 0 \quad \text{for } \theta < \theta_1^*. \quad (2.47)$$

It follows from this relation and the fact that  $F_1(\sigma)$  is smooth that Eq. (2.45) has a solution for all  $\tau \in \mathbb{R}^1$ . Next, for small  $\theta$  we find

$$\lambda_2 = O(\theta), \quad L_1 = -\frac{3}{2} \theta^2 \omega(\sigma) + O(\theta^3), \quad L_2 = -6\theta^2 \omega(\sigma) + O(\theta^4).$$

Now we readily calculate that for small  $\theta$

$$1 - \theta + \frac{F_1(\sigma)}{\theta(1-\theta^2)} = 1 - \theta + \frac{6}{5} \theta \omega(\sigma) + O(\theta^2) > 0. \quad (2.48)$$

Hence there exists  $\theta_2^* \in (0, 1/2]$  such that uniformly in  $\sigma$

$$1 - \theta + \frac{F_1(\sigma)}{\theta(1-\theta^2)} \geq \text{const} > 0 \quad \text{for } \theta < \theta_2^*. \quad (2.49)$$

Inequalities (2.47) and (2.49) and the fact that the function  $F_1(\sigma)$  is even in  $\sigma$  imply that  $\sigma(\tau)$  tends to  $\pm\infty$  as  $\tau \rightarrow \pm\infty$  and is odd in  $\tau$  up to an additive constant.

We set  $\sigma_0 = \sigma - \tau$  and  $\theta^* = \min\{\theta_1^*, \theta_2^*\}$ . As was mentioned above, we cannot rigorously prove a more precise estimate for  $\theta^*$ , but computer calculations give  $\theta^* = 1/2$ .

Obviously, it follows from the above that problem (2.22) has a solution smooth in  $\tau$  and odd up to an additive constant  $\sigma_0(0)$ . The exponential stabilization rate of  $\sigma_0$  as  $\tau \rightarrow \infty$  is a direct consequence of the fact that the function  $\mathcal{F}(\sigma)$  decreases as an exponential function as  $\tau \rightarrow \infty$ .

To complete the proof of Theorem 2.3, it remains to note that formulas (2.21), (2.22) readily follow from (2.33), (2.37), and (2.40). The third relation in (2.7) for functions of the form (2.5) is obvious.  $\square$

*Proof.* Proof of Theorem 2.2 Let us consider (2.27). Taking into account the normalization of the function  $\omega$ , we obtain

$$\begin{aligned} \sum_{i=1}^2 K_{i1}^{(1)} &= \sum_{i=1}^2 \frac{g_i - A_i}{\beta_i} \int \omega(\eta) d\eta = \sum_{i=1}^2 g_i \int \omega(\beta_i z) dz + \text{const} \\ &= \frac{1}{\epsilon} \int \sum_{i=1}^2 g_i \omega\left(\beta_i \frac{x - \phi_i}{\epsilon}\right) dx + \text{const}. \end{aligned}$$

Since  $cd/d\tau = \epsilon d/dt$ , we see that (2.27) is exactly the conservation law (2.8) calculated for functions  $u$  of the form (2.5). Consider relation (2.30). With the help of the identity

$$\frac{V_i}{c} \tau = \frac{\varphi_{i0}}{\epsilon} + q, \quad q = \frac{V_1 \varphi_{20} - V_2 \varphi_{10}}{\epsilon(V_2 - V_1)}, \quad i = 1, 2,$$

by setting  $\xi = \theta\eta + \sigma$ , we calculate

$$\begin{aligned} M &:= a_2 \sum_{i=1}^2 \left\{ K_i^{(2)} \varphi_{i1} + \frac{V_i}{c} \tau K_{i1}^{(2)} \right\} + a_2 R_2^{(1)} \\ &= \frac{g_1^2}{\beta_1} \frac{\phi_1}{\epsilon} \int \omega^2(\xi) d\xi + \frac{g_2^2}{\beta_2} \frac{\phi_2}{\epsilon} \int \omega^2(\eta) d\eta + 2 \frac{g_1 g_2}{\beta_2} \left\{ \left( \frac{\phi_2}{\epsilon} + q \right) \int \omega(\eta) \omega(\xi) d\eta \right. \\ &\quad \left. + \frac{1}{\beta_2} \int \eta \omega(\eta) \omega(\xi) d\eta \right\} + a_2 \sum_{i=1}^2 \left( q K_{i1}^{(2)} - \frac{\varphi_{i0}}{\epsilon} K_{i0}^{(2)} \right). \end{aligned}$$

Performing the change  $\eta = \beta_2(x - \phi_2)/\epsilon$ , we prove the relation

$$\begin{aligned} &\frac{\phi_2}{\epsilon} \int \omega(\eta) \omega(\xi) d\eta + \frac{1}{\beta_2} \int \eta \omega(\eta) \omega(\xi) d\eta \\ &= \frac{\beta_2}{\epsilon^2} \int x \omega\left(\beta_1 \frac{x - \phi_1}{\epsilon}\right) \omega\left(\beta_2 \frac{x - \phi_2}{\epsilon}\right) dx. \end{aligned}$$

Now it is easy to see that even functions  $\omega(\eta)$  satisfy the relation

$$\begin{aligned} M &= \frac{1}{\epsilon^2} \int x \left( g_1 \omega\left(\beta_1 \frac{x - \phi_1}{\epsilon}\right) + g_2 \omega\left(\beta_2 \frac{x - \phi_2}{\epsilon}\right) \right)^2 dx \\ &\quad + a_2 q \left( \sum_{i=1}^2 K_{i1}^{(2)} + R_2 \right) - \frac{a_2}{\epsilon} \sum_{i=1}^2 \varphi_{i0} K_{i0}^{(2)}. \end{aligned}$$

In a similar way, we obtain

$$\begin{aligned} a_3 \left\{ \sum_{i=1}^2 K_{i1}^{(3)} + R_3 \right\} &= \frac{1}{\epsilon} \int \left( \sum_{i=1}^2 g_i \omega\left(\beta_i \frac{x - \phi_i}{\epsilon}\right) \right)^3 dx - a_3 \sum_{i=1}^2 K_{i0}^{(3)}, \\ a'_2 \left\{ \sum_{i=1}^2 \beta_i K_{i1}^{(2)} + Q_2 \right\} &= \frac{1}{\epsilon} \int \left\{ \epsilon \frac{\partial}{\partial x} \left( \sum_{i=1}^2 g_i \omega\left(\beta_i \frac{x - \phi_i}{\epsilon}\right) \right) \right\}^2 dx \\ &\quad - a'_2 \sum_{i=1}^2 \beta_i^2 K_{i0}^{(2)}. \end{aligned}$$

We substitute the relations obtained into (2.33) and take into account (2.29). Then (2.33) takes the form

$$\begin{aligned} &\frac{1}{\epsilon} \left\{ \frac{d}{dt} \int x u^2 dx - \frac{4}{3} \int u^3 dx + 3 \int (\epsilon u_x)^2 dx \right\} \\ &= \sum_{i=1}^2 \left\{ a_2 \frac{d}{dt} \varphi_{i0} K_{i0}^{(2)} - \frac{4}{3} a_3 K_{i0}^{(3)} + 3 a'_2 \beta_i^2 K_{i0}^{(2)} \right\}. \end{aligned} \tag{2.50}$$

However, we have  $d\varphi_{i0}/dt = V_i$  and thus the right-hand in (2.50) is zero by virtue of the second relation in (2.34). Hence Eq. (2.33) is exactly the energy relation (2.11) calculated for the function  $u$  of the form (2.5).

In a similar way we can prove that (2.31) and (2.32) coincide with the equalities (2.9) and (2.10) calculated for the function  $u$  of the form (2.5).

To complete the proof of Theorem 2.2, it remains to recall that relations (2.30) and (2.33) are necessary and sufficient conditions for the function (2.5) to satisfy (2.7) with any test function  $\psi$ . It should be noted that relations (2.34) also coincide

exactly with the energy relations (2.10) and (2.11) whose left-hand sides, however, are calculated for the solitary wave  $A_i\omega(\beta_i(x - \varphi_{i0})/\epsilon)$ .  $\square$

*Proof of Theorem 2.1.* Let  $W_i(\eta)$  be some even functions from  $C^\infty(\mathbb{R}^1)$  decreasing faster than  $|\eta|^{-2}$  as  $\eta \rightarrow \pm\infty$ . By  $W_{i,\ell}(\eta)$  we denote their  $2\ell + 1$ -order derivatives with respect to  $\eta$  for arbitrary  $\ell \geq 0$  and consider the function

$$U = \sum_{i=1}^2 \left\{ g_i(\tau) \omega\left(\beta_i \frac{x - \phi_i}{\epsilon}\right) + A_i G_i(\tau) W_{i,\ell}\left(\frac{\beta_i}{\nu_i(\theta)} \frac{x - \phi_i}{\epsilon}\right) \right\}, \quad (2.51)$$

where  $\omega, g_i, \beta_i, \phi_i$  are the same as in (2.5),  $G_i(\tau) \in C^\infty$  are arbitrary functions exponentially decreasing as  $\tau \rightarrow \pm\infty$ , and  $\nu_i > 0$  are sufficiently small variables depending on the parameter  $\theta$ .

It is easy to prove that in the sense of  $\mathcal{D}'$  the function (2.51) differs from the function (2.5) by a value of order  $O(\epsilon^{2\ell+2})$ , while their first derivatives differ by a value of order  $O(\epsilon^{2\ell+1})$ . Nevertheless, we have

$$\begin{aligned} U^n &= (u^n)_0 + \epsilon a_n M_{(n)}(\tau, \nu, G) \delta(x - x^*) + O_{\mathcal{D}'}(\epsilon^2), \\ (\epsilon U_x)^2 &= (\epsilon u_x)_0^2 + \epsilon a_2' P(\tau, \nu, G) \delta(x - x^*) + O_{\mathcal{D}'}(\epsilon^2), \\ \frac{\partial U^2}{\partial t} &= \left(\frac{\partial u^2}{\partial t}\right)_0 + a_2 C(\tau, \nu, G) \delta(x - x^*) \\ &\quad + \epsilon a_2 C^{(1)}(\tau, \nu, G) \delta'(x - x^*) + O_{\mathcal{D}'}(\epsilon^2), \end{aligned}$$

where  $(u^n)_0$ ,  $(\epsilon u_x)_0$ , and  $(\partial u^2/\partial t)_0$  are distributions defined in formulas (2.26), (2.27), the functions  $M_{(n)}$ ,  $P$ ,  $C$ , and  $C^{(1)}$  tend to zero as  $\tau \rightarrow \pm\infty$  as exponential functions, and

$$M_{(n)} = O(\nu_1 + \nu_2), \quad P = O(1/\nu_1 + 1/\nu_2), \quad C = O(\nu_1 + \nu_2).$$

We require that the function  $U$  is the weak modulo  $O_{\mathcal{D}'}(\epsilon^2)$  asymptotic of the solution of problem (2.1), (2.2). Just as above, we obtain a system of model equations from (2.7). Four of these equations coincide with (2.30), (2.32), and (2.34), and the other two have the form

$$\begin{aligned} \frac{d}{d\tau} \left\{ \sum_{i=1}^2 K_{i1}^{(2)} + R_2 + M_{(2)} \right\} &= 0, \quad (2.52) \\ c \frac{d}{d\tau} \left\{ \sum_{i=1}^2 (K_i^{(2)} \varphi_{i1} + \tau \frac{V_i}{c} K_{i1}^{(2)}) + R_2^{(1)} + C^{(1)} \right\} \\ &= \frac{4}{3} \frac{a_3}{a_2} \left\{ \sum_{i=1}^2 K_{i1}^{(2)} + R_3 + M_{(3)} \right\} - 3 \frac{a_2'}{a_2} \left\{ \sum_{i=1}^2 \beta_i^2 K_{i1}^{(2)} + Q_2 + P \right\}. \quad (2.53) \end{aligned}$$

Repeating the above constructions and using the notation (2.40), we obtain the auxiliary function  $k$  from (2.30) and (2.52). However, in contrast to (2.19), we now have  $k = k(\sigma, \theta, \nu, G)$ . Here it is important that we can choose parameters  $\nu_i$  arbitrary small so that the corresponding quadratic equation has a real root for

$\theta < \theta_1^*$ . Further, formulas (2.52) and (2.42) yield

$$\begin{aligned} \varphi_{11} &= \left( \beta_1 \sum_{i=1}^2 K_{i0}^{(1)} \right)^{-1} (\tau K_{20}^{(1)} - \sigma K_2^{(1)} + D_1), \\ \varphi_{21} &= \left( \beta_1 \sum_{i=1}^2 K_{i0}^{(1)} \right)^{-1} (-\tau K_{10}^{(1)} + \sigma K_1^{(1)} + D_2), \end{aligned}$$

which differ from (2.43) by an additional function  $D_i = D_i(\tau, \sigma, \nu, G)$  such that  $D_i \rightarrow 0$  as  $\tau \rightarrow \pm\infty$ . Here it should be pointed out that, since  $M_{(2)}$  vanishes as  $\tau \rightarrow \pm\infty$ , the symmetry law for the resulting phase shifts of trajectories (2.12) remains valid for all perturbations of the function  $u$  contained in (2.51).

Obviously, we now can transform (2.53) to the equation similar to (2.45):

$$\frac{d}{d\tau} \{ (\sigma - \tau)(1 - \theta) + \mathcal{J}(\sigma) + \mathcal{J}_1(\tau, \sigma, \nu, G) \} = \frac{F_2}{\theta(1 - \theta^2)}, \tag{2.54}$$

where

$$F_2 = F_1(\sigma) + \tilde{F}_1(\tau, \sigma, \nu, G), \quad \tilde{F}_1 = \frac{4}{3} \frac{a_3}{a_2^2} (M_{(3)} - P) + P,$$

$\mathcal{J}$  and  $F_1$  are the same functions as in (2.45), and  $\mathcal{J}_1$  is a smooth function tending to zero as  $\tau \rightarrow \pm\infty$ . Let us integrate (2.54) with respect to  $\tau$  and let  $\tau \rightarrow \infty$ . Since  $\sigma = \tau + \sigma_0$ , we obtain the relation

$$\sigma_0^\infty(1 - \theta) = \left( \int_{-\infty}^\infty F_1 d\tau + \int_{-\infty}^\infty \tilde{F}_1 d\tau \right) / \theta(1 - \theta^2). \tag{2.55}$$

The first integral on the right-hand side of (2.55) corresponds to the fixed value of  $\sigma_0^\infty(\theta)$  obtained in the proof of Theorem 2.3. We see that the perturbations of  $u$  contained in (2.51) yield a function of soliton type (with respect to  $\tau$ ), namely the function  $\tilde{F}_1 = (1 - 4a_3/3a_2^2)P + O(\nu_1 + \nu_2)$ . Hence, by changing  $G_i$  and  $\nu_i$ , we can make the limit  $\sigma_0^\infty$  to be an arbitrary finite number. This fact means the asymptotic non-uniqueness. The proof of Theorem 2.1 is complete.  $\square$

In conclusion, we analyze (2.45) for small  $\theta$ . As shown in (2.48),  $F_1(\sigma) = \frac{6}{5}\theta^2\omega(\sigma) + O(\theta^3)$ . Hence we have  $(1 - \theta)\sigma_0^\infty = \frac{6}{5}\theta + O(\theta^2)$ . Thus the trajectory of the first soliton is shifted by  $-\epsilon\beta_1\varphi_{11}^\infty = \frac{6}{5}\epsilon\theta + O(\epsilon\theta^2)$ , while according to the exact formula this shift is equal to  $-2\epsilon\mu = \epsilon\theta + O(\epsilon\theta^2)$ . In view of the fact that the weak asymptotic (2.5) is asymptotically non-unique, this coincidence is astonishingly good.

### 3. $\epsilon$ - $\delta$ -INTERACTION IN KdV TYPE MODELS

In this section we discuss a natural generalization of the KdV equation

$$u_t + (u^m)_x + \epsilon u_{xxx} = 0. \tag{3.1}$$

Under the assumption that  $m \geq 2$  is an integer, we study the problem of interaction of two solitary waves. Suppose that

$$u|_{t=0} = u^0(x, \epsilon), \quad u^0 = A_1\omega\left(\beta_1 \frac{x - x_1^0}{\epsilon}\right) + A_2\omega\left(\beta_2 \frac{x - x_2^0}{\epsilon}\right), \tag{3.2}$$

where  $\omega(\eta)$  is the exact solution of the model equation (0.5) corresponding to (3.1) (see Introduction) and

$$\beta_i^2 = \gamma^{m-1} A_i^{m-1}, \quad \gamma = \left( \frac{m-1}{m+3} \frac{a_m a_2}{a_2'} \right)^{1/(m-1)}. \quad (3.3)$$

We assume that  $A_i > 0$  for even  $m$  and that the sign of  $A_i$  can be arbitrary for odd  $m$ . In both cases, we assume that

$$|A_2| > |A_1|, \quad x_1^0 < x_1^0,$$

which ensures the interaction of solitons at time  $t^*$ .

Precisely as in Sec. 2, a weak modulo  $O_{\mathcal{D}'}(\epsilon^2)$  asymptotic of the solution of problem (3.1) (3.2) is a function  $u$  satisfying for all  $\psi = \psi(x) \in \mathcal{D}(\mathbb{R}^1)$  the relations

$$\begin{aligned} \frac{d}{dt} \int u \psi \, dx - \int u^m \frac{\partial \psi}{\partial x} \, dx &= O(\epsilon^2), \\ \frac{d}{dt} \int u^2 \psi \, dx - \frac{2m}{m+1} \int u^{m+1} \frac{\partial \psi}{\partial x} \, dx + 3 \int \left( \epsilon \frac{\partial u}{\partial x} \right)^2 \frac{\partial \psi}{\partial x} \, dx &= O(\epsilon^2), \\ \int (u|_{t=0} - u^0) \psi \, dx &= O(\epsilon^2). \end{aligned} \quad (3.4)$$

The main result of this section is the following assertion.

**Theorem 3.1.** *There exists  $\theta^*(m) \in (0, 1)$  such that for  $\theta := \beta_1/\beta_2 \in (0, \theta^*(m))$  the interaction of solitary waves in the KdV type equation (3.1) follows the soliton scenario. The weak asymptotic of the solution of problem (3.1), (3.2) is asymptotically non-unique in the terms of order  $O_{\mathcal{D}'}(\epsilon^2)$ .*

The proof of this theorem differs from the proof of a similar statement in Sec. 2 only in small details. Namely, we represent the leading term of the weak asymptotic in the form

$$u = \sum_{i=1}^2 (A_i + S_i(\tau)) \omega \left( \beta_i \frac{x - \varphi_{i0}(t) - \epsilon \varphi_{i1}(\tau)}{\epsilon} \right), \quad (3.5)$$

where  $S_i$  and  $\varphi_{i1}$  are assumed to satisfy the same properties as in Sec. 2, while the function  $\omega(\eta)$ , precisely as in (3.2), depends on the parameter  $m$ . As in Sec. 2, we establish the following assertion.

**Theorem 3.2.** *The condition  $\theta \in (0, \theta^*(m))$ , the conservation laws*

$$\frac{d}{dt} \int u \, dx = 0, \quad \frac{d}{dt} \int u^2 \, dx = 0,$$

and the energy relations

$$\begin{aligned} \frac{d}{dt} \int x u \, dx - \int u^m \, dx &= 0, \\ \frac{d}{dt} \int x u^2 \, dx - \frac{2m}{m+1} \int u^{m+1} \, dx + 3 \int \left( \epsilon \frac{\partial u}{\partial x} \right)^2 \, dx &= 0 \end{aligned}$$

are necessary and sufficient for the function (3.5) to be a weak modulo  $O_{\mathcal{D}'}(\epsilon^2)$  asymptotic of the solution of problem (3.1), (3.2).

The key point in the proof of these statements is the derivation of model equations for  $S_i$ ,  $\phi_i$  and their analysis. To state the result, we introduce some auxiliary notation and functions. We write

$$s_i = \text{sign } A_i, \quad s = s_1 s_2, \quad \kappa = \frac{2}{m-1},$$

and the convolutions  $\lambda_l = \lambda_l(\sigma)$ ,  $l = 0, 1, 2$ , are calculated by formulas of the form (2.16). By  $\theta_0^* \in (0, 1)$  we denote the number determined by the relation

$$1 - \theta_0^* = 2(\theta_0^*)^\kappa,$$

and introduce a function  $k = k(\sigma)$  determined in the case  $s = 1$  and  $\theta = \theta_0^*$  by the formula

$$k = \frac{\lambda_0(1 - \theta)}{1 + \theta - 2\lambda_0\theta} \Big|_{\theta=\theta_0^*}, \quad (3.6a)$$

and in all other cases by the formula

$$k = \frac{1 - \theta\lambda_0 + s(\lambda_0 - 1)\theta^\kappa - \sqrt{I}}{1 + \theta - 2\theta\lambda_0}, \quad (3.6)$$

where

$$I = (1 - s\theta^\kappa - \lambda_0(\theta^\kappa + \theta))^2 + 2\lambda_0\theta^\kappa(1 - s\theta - \theta^\kappa(1 + s)). \quad (3.7)$$

Further, we introduce the convolutions  $\lambda_{n,j} = \lambda_{n,j}(\sigma)$ :

$$\lambda_{n,j} = \frac{1}{a_n} \int \omega^{n-j}(\theta\eta + \sigma)\omega^j(\eta) d\eta \quad (3.8)$$

and the functions  $L_{i,n} = L_{i,n}(\sigma)$

$$L_{1,n} = \theta^{n-1} [(k + s\theta^{\kappa-1})^n - (s\theta^{\kappa-1})^n] + (1 - k)^n - 1 + \sum_{j=1}^{n-1} c_n^j \theta^{n-j} (k + s\theta^{\kappa-1})^{n-j} (1 - k)^j \lambda_{n,j}, \quad (3.9)$$

$$L_{2,n} = (1 + \theta^3)k^2 - 2k(1 - s\theta^{\kappa+2}) + 2\theta^2(k + s\theta^{\kappa-1})(1 - k)\lambda_2, \quad (3.10)$$

where  $c_n^j$  are binomial coefficients. Now for all  $m \geq 2$  and any sign of  $s$  except the case

$$s = -1, \quad m = 3, \quad (3.11)$$

we defined a function  $\sigma_0 = \sigma_0(\tau)$  as a solution of the problem

$$\frac{d\sigma_0}{d\tau} = \frac{\mathcal{F}_m(\tau + \sigma_0) - \mathcal{J}'_m(\tau + \sigma_0)}{r_m + \mathcal{J}'_m(\tau + \sigma_0)}, \quad \sigma_0 \rightarrow 0 \quad \text{as } \tau \rightarrow -\infty. \quad (3.12)$$

Here  $\mathcal{J}'_m(\sigma) = \partial \mathcal{J}_m(\sigma) / \partial \sigma$ ,

$$\mathcal{J}_m(\sigma) = \sigma k \left[ \frac{1 - 2s\theta^\kappa - \theta^{2\kappa-1}}{1 + s\theta^{\kappa-1}} - \theta k \right] + 2\theta(\theta k + s\theta^\kappa)(1 - k)\lambda_1, \quad (3.13)$$

$$r_m = \theta^{\kappa-1} \frac{s - \theta^\kappa}{1 + s\theta^{\kappa-1}}, \quad (3.14)$$

$$\mathcal{F}_m(\sigma) = (1 - \theta^2)^{-1} \left\{ \frac{2m}{m+1} \cdot \frac{a_{m+1}}{a_2 a_m} (L_{1,m+1} - L_{2,m}) + L_{2,m} \right\} - (1 - r_m)L_{1,m}. \quad (3.15)$$

In the exceptional case (3.11) we find the function  $\sigma_0 = \sigma_0(\tau)$  as the solution of the problem

$$\frac{d\sigma_0}{d\tau} = \frac{L_{1,3}^-(\tau + \sigma_0) - \mathcal{J}'_3(\tau + \sigma_0)}{1 + \mathcal{J}'_3(\tau + \sigma_0)}, \quad \sigma_0 \rightarrow 0 \quad \text{as } \tau \rightarrow -\infty, \quad (3.16)$$

where  $\mathcal{J}'_3(\sigma) = \partial \mathcal{J}_3(\sigma) / \partial \sigma$ ,  $\mathcal{J}_3(\sigma) = -\sigma k(\sigma)$ , and  $L_{1,3}^- = L_{1,3}|_{s=-1}$ .

Using the functions  $\sigma_0(\tau)$  and setting  $\sigma = \tau + \sigma_0(\tau)$ , we define  $\psi_1 = \psi_1(\tau)$  in all the cases except the case (3.11) by the formula

$$\psi_1(\tau) = (1 + s\theta^{k-1})^{-1} \left\{ (\sigma_0 + \tau)k(\sigma_0 + \tau) - \sigma_0 + \int_{-\infty}^{\tau} L_{1,m}(\tau' + \sigma_0(\tau')) d\tau' \right\}. \quad (3.17)$$

In the exceptional case (3.11) we define

$$\psi_1(\tau) = (1 + \theta)^{-1} \left\{ \theta\sigma(k^2 - 2k) - \sigma_0 + 2\theta^2(1 - k)^2\lambda_1 + \int_{-\infty}^{\tau} (\mathcal{F}_3 + (1 - r_3)L_{1,3}^-) d\tau' \right\}. \quad (3.18)$$

**Theorem 3.3.** *There exists a number  $0 < \theta^* < 1$  depending on  $m$  and  $s$  such that for  $0 < c_1 \leq \theta \leq \theta^*$  problems (3.12) and (3.15) have unique solutions. For such  $\theta$  the weak asymptotic of the solution of problem (3.1), (3.2) has the form (3.5), where*

$$\begin{aligned} S_1 &= \theta A_2 k(\tau + \sigma_0), \quad S_2 = -A_2 k(\tau + \sigma_0), \\ \varphi_{i0} &= V_i t + x_i^0, \quad V_i = a_m A_i^{m-1}, \quad \tau = \beta_1 \frac{\varphi_{20}(t) - \varphi_{10}(t)}{\epsilon}, \\ \psi_2(\tau) &= (\psi_1(\tau) + \sigma_0(\tau)) / \theta, \quad \varphi_{i1} = \psi_i / \beta_i, \quad i = 1, 2. \end{aligned} \quad (3.19)$$

*Proof.* Proof of Theorem 3.3 We represent the weak asymptotic of the solution in the form (3.5) and apply formulas from Sec. 1. By substituting these expansions into (3.4) and equating the coefficients of  $\delta'(x - x^*)$ ,  $\epsilon\delta'(x - \phi_i)$ , and  $\epsilon\delta'(x - x^*)$  with zero, we obtain the following system of equations similar to system (2.30)–(2.34):

$$\frac{d}{d\tau} \sum_{i=1}^2 K_{i1}^{(1)} = 0, \quad (3.20)$$

$$\frac{d}{d\tau} \left\{ \sum_{i=1}^2 K_{i1}^{(2)} + R_2 \right\} = 0, \quad (3.21)$$

$$c \frac{d}{d\tau} \sum_{i=1}^2 \left\{ K_i^{(1)} \varphi_{i1} + \frac{V_i}{c} \tau K_{i1}^{(1)} \right\} = f_m, \quad (3.22)$$

$$c \frac{d}{d\tau} \left\{ \sum_{i=1}^2 \left( K_i^{(2)} \varphi_{i1} + \frac{V_i}{c} \tau K_{i1}^{(2)} \right) + R_2^{(1)} \right\} = F_m, \quad (3.23)$$

$$V_i K_{i0}^{(1)} = a_m K_{i0}^{(m)}, \quad (3.24)$$

$$V_i K_{i0}^{(2)} = \frac{2m}{m+1} \frac{a_{m+1}}{a_2} K_{i0}^{(m+1)} - 3 \frac{a'_2}{a_2} \beta_i^2 K_{i0}^{(2)}, \quad (3.25)$$



where we use the notation

$$f_m = a_m \left( \sum_{i=1}^2 K_{i1}^{(m)} + R_m \right), \tag{3.26}$$

$$F_m = \frac{2m}{m+1} \frac{1}{a_2} f_{m+1} - 3 \frac{a_2'}{a_2} \left( \sum_{i=1}^2 \beta_i^2 K_{i1}^{(2)} + Q_2 \right). \tag{3.27}$$

Clearly, expressions (3.24), (3.25) imply the same relations between the amplitude  $A_i$ , the velocity  $V_i$ , and the coefficient  $\beta_i$  as in the problem of motion of a single solitary wave:

$$V_i = a_m A_i^{m-1}, \quad \beta_i^2 = \gamma^{m-1} A_i^{m-1}, \quad \gamma^{m-1} = \frac{(2m/(m+1))a_{m+1} - a_2 a_m}{3a_2'}. \tag{3.28}$$

Since  $\omega(\eta)$  is the solution of (0.5), by using the properties of the coefficients  $a_1$  and  $a_2'$ , we can rewrite the coefficient  $\gamma$  in the form (3.3).

Let us consider (3.20) and (3.21). Integrating by  $\tau$ , we arrive at an equation of the form (2.36). We set

$$A_i = s_i \frac{\beta_i^\kappa}{\gamma}, \quad K_{11}^{(1)} = s_2 \frac{\beta_2^{\kappa-1}}{\gamma} k, \quad \kappa = \frac{2}{m-1}. \tag{3.29}$$

Then, solving this equation for  $k = k(\sigma)$ , we obtain formulas (3.6), (3.7). Since  $\lambda_0 \leq 1/\sqrt{\theta}$ , we have  $1 + \theta - 2\theta\lambda_0 \geq \text{const} > 0$  for  $\theta < 1$ . Next, relation (3.7) readily implies that for  $s = -1$  the inequality  $I \geq \text{const} > 0$  holds uniformly in  $\sigma \in \mathbb{R}^1$  and  $\theta \in (0, 1)$ . Let us consider the case  $s = 1$ . Obviously, the second term in (3.7) is positive only for  $\theta < \theta_0^*$ . For  $\theta = \theta_0^*$  the first term takes the form  $(1 + \theta)^2(1 - \lambda_0)^2/4$ . Since  $\lambda_0(0)$  is a monotonically decreasing function of  $\theta$  and  $\lambda_0(0)|_{\theta=1} = 1$ , for all  $\theta \geq \theta_0^*$  there exists a value  $\sigma = \sigma^*(m, \theta)$  such that the first term is zero at this point. Thus we obtain the condition  $\theta < \theta_0^*$  for the estimate

$$I \geq \text{const} > 0$$

to be uniform with respect to  $\tau$ .

Finally, the function  $k$  defined in (3.6a), (3.6) is real and smooth only if

$$\theta \leq \theta_1^*, \quad \theta_1^* = \theta_0^* \text{ for } s = 1 \quad \text{and} \quad \theta_1^* = 1 - \mu, \quad \mu > 0, \quad \text{for } s = -1.$$

Let us consider Eqs. (3.22) and (3.23). Using formulas (3.9), (3.10), and (3.29), we transform these equations to the form

$$\frac{d}{d\tau} \{ -\sigma k + s\theta^{\kappa-1}\psi_1 + \theta\psi_2 \} = L_{1,m}, \tag{3.30}$$

$$\begin{aligned} (1 - \theta^2) \frac{d}{d\tau} \{ -\sigma k(2s\theta^\kappa + \theta k) + \theta^{2\kappa-1}\psi_1 + \theta\psi_2 + 2\theta(s\theta^\kappa + \theta k)(1 - k)\lambda_1 \} \\ = \frac{2m}{m+1} \frac{a_{m+1}}{a_2 a_m} (L_{1,m+1} - L_{2,m}) + L_{2,m}. \end{aligned} \tag{3.31}$$

Here we have used the notation

$$\psi_i = \beta_i \varphi_{i1}, \quad i = 1, 2. \tag{3.32}$$

We supplement relations (3.30) and (3.31) by the identity

$$\sigma = \tau + \sigma_0, \quad \sigma_0 = \theta\psi_2 - \psi_1, \tag{3.33}$$

and consider this system in the exceptional case (3.11). Since  $\kappa = 1$  in this case, (3.30) takes the form

$$\{1 - (\sigma k)'\} \frac{d\sigma}{d\tau} = 1 + L_{1,3}^-(\sigma), \tag{3.34}$$

where the prime denotes the derivative with respect to  $\sigma$  and the superscript  $-$  means that we set  $s = -1$  in the formula for  $L_{1,3}(\sigma)$ . Now by using the explicit form of the function  $k$ , we can verify that

$$1 - \frac{\sqrt{1+\theta}}{1-\sqrt{\theta}} \leq k \leq 0, \quad 1 - \frac{c\theta}{(1-\sqrt{\theta})^2} \leq 1 - (\sigma k)' \leq \frac{\sqrt{1+\theta}}{1-\sqrt{\theta}},$$

where the constant  $c$  independent of  $\theta$  is the maximum value of the function  $-\sigma \lambda_0'(\sigma)$ .

The above formulas readily imply that there exists  $\theta_2^* > 0$  such that for  $0 < \theta \leq \theta_2^*$  equation (3.34) can be solved for all  $\tau \in \mathbb{R}^1$ . Obviously, problem (3.16) follows from relations (3.33) and (3.34) and the condition  $\varphi_{i1} \rightarrow 0$  as  $\tau \rightarrow -\infty$ .

Let us study the general case. We set  $\theta\psi_2 = \psi_1 + \sigma - \tau$  and, by using (3.30), define  $d\psi_1/d\tau$  as a function of  $\sigma$  and  $\tau$ . Then (3.31) can be transformed to the form

$$(r_m + \mathcal{J}'_m(\sigma)) \frac{d\sigma}{d\tau} = \mathcal{F}_m(\sigma) + r_m, \tag{3.35}$$

where the notation (3.13), (3.15) are used. Let us prove that (3.35) is globally solvable for, at least, sufficiently small  $\theta$ . For this purpose, we first note that for  $\theta \ll 1$ ,

$$\begin{aligned} \lambda_0 &= \lambda_0^0 + O(\theta^2), & \lambda_0^0 &= \frac{1}{a_2} \omega(\sigma), \\ \lambda_1 &= \theta \lambda_1^0 + O(\theta^3), & \lambda_1^0 &= \frac{1}{a_2} \omega'(\sigma) \int \eta^2 \omega(\eta) d\eta, \end{aligned} \tag{3.36}$$

Therefore,

$$k = s_1 \theta^\kappa \lambda_0^0 + O(\theta^{\kappa+1}),$$

and we calculate

$$\begin{aligned} \mathcal{J}_3 &= \frac{1}{2} \sigma \lambda_0^0 \theta + O(\theta^2), & \mathcal{J}_4 &= \sigma \lambda_0^0 \theta + O(\theta^{4/3}), \\ \mathcal{J}_5 &= -2s_1 \sigma \lambda_0^0 \theta^{3/2} + O(\theta^2), & \mathcal{J}_m &= -s_2 \sigma \lambda_0^0 \theta^{2\kappa} + O(\theta) \quad \text{for } m \geq 6. \end{aligned}$$

At the same time, we have

$$r_3 = \frac{1}{2} + O(\theta), \quad r_4 = 1 + O(\theta^{1/3}), \quad r_m = 1 + O(\theta^\kappa) \quad \text{for } m \geq 5.$$

Hence there exists  $\theta_m^*$  such that

$$r_m + \mathcal{J}'_m(\sigma) \geq \text{const} > 0 \quad \text{for } \theta < \theta_m^*.$$

It remains to note that  $\mathcal{J}'_m(\sigma)$  and  $\mathcal{F}_m(\sigma)$  are bounded by a constant and tend to zero both as  $\sigma \rightarrow \pm\infty$  and  $\theta \rightarrow 0$ . This implies that (3.35) is solvable for  $\theta \leq \theta^* = \min\{\theta_1^*, \theta_m^*\}$ , and hence problem (3.12) is solvable.

The last step in deriving formulas (3.19) is the calculation of the function  $\psi_1$ . In the exceptional case (3.11) formula (3.18) is obtained by integrating (3.33) for  $m = 3$  and  $s = -1$ . In the general case we obtain (3.17) by integrating (3.32). The proof of Theorem 3.3 is complete.  $\square$

The proof of Theorem 3.2 and of the asymptotic non-uniqueness of the weak asymptotic is carried out in the same way as the proofs in Sec. 2.

## REFERENCES

- [1] M. J. Ablowitz, M. J., & Segur, H., *Solitons and the Inverse Scattering Transform*, SIAM (Philadelphia, 1981).
- [2] Bona, J. L., & Sant, J. C., Dispersive blow-up of solutions of generalized Korteweg-de Vries equations, *J. Differential Equations* **103**, 3–57 (1993).
- [3] Danilov, V. G., Generalized solutions describing singularity interaction, *International Journal of Math. and Mathematical Sci.* **30**, 1–14 (2002).
- [4] Danilov, V. G., & Shelkovich, V. M., Propagation and interaction of shock waves of quasilinear equation, *Nonlinear Studies* **8**, 135–170 (2001).
- [5] Danilov, V. G., & Shelkovich, V. M., Propagation and interaction of nonlinear waves to quasilinear equations. In: *Proc. Eighth Inter. Conference on Hyperbolic Problems*, Birkhauser (2001).
- [6] Danilov, V. G., & Shelkovich, V. M., Propagation of infinitely narrow  $\delta$ -solitons, <http://arXiv.org/abs/math-ph/0012002>
- [7] Danilov, V. G., Omel'yanov, G. A., & Radkevich, E. V., Hugoniot-type conditions and weak solutions to the phase-field system, *European J. Appl. Math.* **10**, 55–77 (1999).
- [8] Il'in, A. M., & Kalyakin, L. A., Perturbation of finite-soliton solutions to the Korteweg-de Vries equation, *Dokl. Ross. Akad. Nauk* **336**, 595–598 (1994); English transl. in *Phys. Dokl.* **40**, 603–606 (1995).
- [9] Lax, P. D., & Levermore, C. D., The small dispersion limit of the KdV equation, I, *Comm. Pure and Appl. Math.*, 253–291 (1983).
- [10] Lax, P. D., & Levermore, C. D., The small dispersion limit of the KdV equation, II, *Comm. Pure and Appl. Math.*, 577–593, (1983).
- [11] Linares, F., & Scialom, M., On the smoothing properties of solutions to the modified Korteweg-de Vries equations, *J. Differential Equations* **106**, 141–154 (1993).
- [12] Maslov, V. P., & Omel'yanov, G. A., Asymptotic soliton-form solutions of equations with small dispersion, *Uspekhi Mat. Nauk* **36**, 63–126 (1981); English transl. in *Russian Math. Surveys* **36**, 73–149 (1981).
- [13] Maslov, V. P., & Omel'yanov, G. A., *Geometric Asymptotics for Nonlinear PDE*, I, AMS, MMONO 202 (Providence, RI, 2001).
- [14] McLaughlin, D. W., & Scott, A. C., A multisoliton perturbation theory, in: *Solitons in Action*, Academic Press, New York, 1978. pp. 201–256.
- [15] Molotkov, I. A., & Vakulenko, S. A., *Concentrated Nonlinear Waves* (in Russian), Leningrad Univ (Leningrad, 1988).
- [16] Omel'yanov, G.A., Propagation of singularities: multidimensional effects (KP and phase field type equations). In: *Proc. I. G. Petrovskii Seminar*, MGU (Moscow, 2002).

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