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HARMONIC SOLUTIONS TO PERTURBATIONS OF PERIODIC SEPARATED VARIABLES ODES ON MANIFOLDS

MARCO SPADINI

ABSTRACT. We study the set of harmonic solutions to perturbed periodic separated variables ordinary differential equations on manifolds. As an application, a multiplicity result is deduced.

1. Introduction

In this paper we shall investigate the structure of the set of harmonic solutions to perturbed periodic separated variable ordinary differential equations on manifolds. More precisely, let M be an m-dimensional boundaryless differentiable manifold embedded in $\mathbb{R}^k$. We consider equations of the form

$$\dot{x} = a(t)g(x), \tag{1.1}$$

where $g : M \to \mathbb{R}^k$ is a continuous tangent vector field and $a : \mathbb{R} \to \mathbb{R}$ is a continuous T-periodic function, $T > 0$ given, with nonzero average

$$\bar{a} := \frac{1}{T}\int_0^T a(t)\,\mathrm{d}t,$$

and investigate via topological methods the structure of the set of harmonic (i.e., T-periodic) solutions to perturbed equations of the form:

$$\dot{x} = a(t)g(x) + \lambda\phi(x), \quad \lambda \geq 0,$$

with $\phi : M \to \mathbb{R}^k$ a given tangent vector field. Speaking loosely, we shall prove, under appropriate conditions, the existence of a connected "branch" of T-periodic *solution pairs* (λ, x) of this equation, with the property that its closure is not contained in any compact set and meets $g^{-1}(0)$ for $\lambda = 0$.

Actually, the methods discussed in this paper shall let us treat withouth any additional effort the more general case when the perturbation is allowed to be time-dependent and periodic with the same period of a. In other words, we shall consider, for $T > 0$ given, the set of T-periodic solutions to the following parametrized differential equation

$$\dot{x} = a(t)g(x) + \lambda f(t, x), \quad \lambda \geq 0, \tag{1.2}$$

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where $f : \mathbb{R} \times M \to \mathbb{R}^k$ and $g : M \to \mathbb{R}^k$ are tangent vector fields on M, $a : \mathbb{R} \to \mathbb{R}$ and f are T-periodic in t. Therefore, our discussion will be applicable to the particular case of periodic perturbations of autonomous ODEs. This corresponds, in our notation, to $a(\cdot)$ constant (and nonzero). The more general situation considered in this paper yields a generalization of the results of [3, 4] in which $a(t) \equiv 1$.

As an application we provide a multiplicity result for equation (1.2) on compact boundaryless manifolds. Roughly speaking, we shall prove that when g has $n-1$ zeros at which the linearized unperturbed equation satisfies an appropriate "non-T-resonance" condition, and the sum of the indices of g at these zeros differs from the Euler-Poincaré characteristic of M, then (1.2) has at least n solutions of period T for $\lambda > 0$ sufficiently small. This fact will be proved via a combination of local and global results about the set of T-periodic solutions of (1.2). The multiplicity results so obtained are of topological nature: they could not, in general, be deduced via implicit function or variational methods.

2. Notation and preliminary results

We begin by recalling some facts about the function spaces used in the sequel. Let $M \subset \mathbb{R}^k$ be a differentiable manifold and $T > 0$ a given real number. The metric subspace $C_T(M)$ of $C_T(\mathbb{R}^k)$ consisting of all the T-periodic continuous functions $x : \mathbb{R} \to M$ is not complete unless M is closed in $\mathbb{R}^k$. However, $C_T(M)$ is always locally complete. This fact is a consequence of the following remark: since M is locally compact, given $x \in C_T(M)$, there exists a relatively compact open subset of M containing the image $x([0,T])$ of x.

Note that, since $a(t)$ is not identically zero, a point $p \in M$ corresponds to a constant solution to (1.1) if and only if $g(p) = 0$. This motivates the ensuing definitions.

Let f and g be as in (1.2). A pair $(\lambda, p) \in [0,\infty) \times M$ is a *starting point* (of T-periodic solutions) if the Cauchy problem

$$\dot{x} = a(t)g(x) + \lambda f(t,x)\,, \quad x(0) = p \tag{2.1}$$

has a T-periodic solution. A starting point (λ, p) is *trivial* if $\lambda = 0$ and $p \in g^{-1}(0)$.

Although the concept of starting point is essentially finite-dimensional, there is an infinite-dimensional notion strictly correlated to it: that of *T-pair*. We say that a pair $(\lambda, x) \in [0,\infty) \times C_T(M)$ is a T-pair if x satisfies (2.1). If $\lambda = 0$ and x is constant, then (λ, x) is said *trivial*.

Denote by $X \subset [0,\infty) \times C_T(M)$ the set of the T-pairs of (2.1) and by $S \subset [0,\infty) \times M$ the set of the starting points. Note that, as a closed subset of a locally complete space, X is locally complete.

One can show that, no matter whether or not M is closed in $\mathbb{R}^k$, the subset X of $[0,\infty) \times C_T(M)$ consisting of all the T-pairs of (1.2) is always closed and locally compact. Moreover, by the Ascoli-Arzelà Theorem, when M is closed in $\mathbb{R}^k$, any bounded closed set of T-pairs is compact.

As in [4], we tacitly assume some natural identifications. That is, we will regard every space as its image in the following diagram of closed embeddings:

$$\begin{array}{ccc} [0,\infty) \times M & \longrightarrow & [0,\infty) \times C_T(M) \\ \uparrow & & \uparrow \\ M & \longrightarrow & C_T(M), \end{array} \tag{2.2}$$

where the horizontal arrows are defined by regarding any point p in M as the constant map $\hat{p}(t) \equiv p$ in $C_T(M)$, and the two vertical arrows are the natural identifications $p \mapsto (0,p)$ and $x \mapsto (0,x)$.

According to these embeddings, if Ω is an open subset of $[0,\infty) \times C_T(M)$, by $\Omega \cap M$ we mean the open subset of M given by all $p \in M$ such that the pair $(0,p)$ belongs to Ω. If U is an open subset of $[0,\infty) \times M$, then $U \cap M$ represents the open set $\{p \in M : (0,p) \in U\}$.

Observe that any $p \in g^{-1}(0)$ can be seen –in the sense specified above– as a T-periodic solution of the unperturbed equation (1.1).

Remark 2.1. The map $h : X \to S$ given by $(\lambda, x) \mapsto \big(\lambda, x(0)\big)$ is continuous and onto. Notice that, if (λ, x) is trivial, then so is $\big(\lambda, x(0)\big)$.

In case f and g are C^1, h is also one to one. Furthermore, by the continuous dependence on initial data, we get the continuity of $h^{-1} : S \to X$. Clearly trivial solution pairs correspond to trivial starting points under this homeomorphism.

We now recall some basic facts about the topological degree of tangent vector fields on manifolds and about the fixed point index.

Let $w : M \to \mathbb{R}^k$ be a continuous tangent vector field on M, and let V be an open subset of M in which we assume w admissible for the degree, that is $w^{-1}(0) \cap V$ compact. Then, one can associate to the pair (w,V) an integer, $\deg(w,V)$, called the *degree (or characteristic) of the vector field* w in V, which, roughly speaking, counts (algebraically) the number of zeros of w in V (see e.g. [5, 6] and references therein). When $M = \mathbb{R}^k$, $\deg(w,W)$ is just the classical Brouwer degree, $\deg(w,W,0)$, of w at 0 in any bounded open neighborhood W of $w^{-1}(0) \cap V$ whose closure is in V. Moreover, when M is a compact manifold, the celebrated Poincaré-Hopf Theorem states that $\deg(w,M)$ coincides with the Euler-Poincaré characteristic of M and, therefore, is independent of w.

We recall that when p is an isolated zero of w, the index $\mathrm{i}(w,p)$ of w at p is defined as $\deg(w,V)$, where V is any isolating open neighborhood of p. If w is C^1 and p is a non-degenerate zero of w (i.e. the Fréchet derivative $w'(p) : T_pM \to \mathbb{R}^k$ is injective), then p is an isolated zero of w, $w'(p)$ maps T_pM onto itself, and $\mathrm{i}(w,p) = \operatorname{sign}\det w'(p)$ (see e.g. [6]).

Let V be an open subset of M, and let $\Psi : V \to M$ be continuous. The map Ψ is said to be admissible (for the fixed point index) on V if its set of fixed points is compact. In these conditions it is defined an integer, called the *fixed point index* of Ψ in V and denoted by $\operatorname{ind}(\Psi,V)$, which satisfies all the classical properties of the Brouwer degree: solution, excision, additivity, homotopy invariance, normalization etc. A detailed exposition of this matter can be found, for example, in [7] and references therein. The following fact deserves to be mentioned: if M is an open subset of $\mathbb{R}^m$, then $\operatorname{ind}(\Psi,V)$ is just the Brouwer degree of $I - \Psi$ in V at 0, where $I - \Psi$ is defined by $(I - \Psi)(x) = x - \Psi(x)$.

Let $\gamma : \mathbb{R} \times M \to \mathbb{R}^k$ be a time-dependent tangent vector field. We will denote by P^γ_τ, $\tau \in \mathbb{R}$, the local (Poincaré) τ-translation operator associated to the equation

$$\dot{x} = \gamma(t,x). \tag{2.3}$$

One has $P^\gamma_\tau(p) = P^\gamma(\tau,p)$ where the map $P^\gamma : W \to M$ is defined on an open set $W \subset \mathbb{R} \times M$ containing $\{0\} \times M$, with the property that, for any $p \in M$, the curve $t \mapsto P^\gamma(t,p)$ is the maximal solution of (2.3) such that $P^\gamma(0,p) = p$. Therefore, given $\tau \in \mathbb{R}$, the domain of P_τ^γ is the open set consisting of those points $p \in M$ for which the maximal solution of (2.3), starting from p at $t = 0$ is defined up to τ.

Let V be an open subset of M, and let $T > 0$ be given. Assume that the solutions of (1.1) are defined in $[0, T]$ for any initial point $p \in V$, and that $\operatorname{ind}\big(P_T^{ag}, V\big)$ is well defined. This clearly implies that $g^{-1}(0)$ is compact, thus $\deg(g, V)$ is defined as well.

Notation. For the sake of simplicity, we shall often denote by $P_t(\lambda, \cdot)$ (instead of by $P_t^{ag+\lambda f}$) the t-translation operator associated to (1.2).

We shall make use of the following result of [3]:

Theorem 2.2. *Let $\gamma : M \to \mathbb{R}^k$ be a tangent vector field on a boundaryless differentiable manifold $M \subset \mathbb{R}^k$ and V a relatively compact open subset of M. Let $T > 0$ be given and assume that, for any $p \in \overline{V}$, the solution of the Cauchy problem*

$$\dot{x} = \gamma(x), \quad x(0) = p,$$

is defined on $[0, T]$. If the translation operator P_T^γ associated to $\dot{x} = \gamma(x)$ is fixed point free on ∂V, then

$$\operatorname{ind}(P_T^\gamma, V) = \deg(-\gamma, V).$$

Remark 2.3. Let $g : M \to \mathbb{R}^k$ be a C^1 tangent vector field, and let $a : \mathbb{R} \to \mathbb{R}$ be continuous, T-periodic with $1/T \int_0^T a(t)\,\mathrm{d}t = 1$. Take any $p \in M$ and consider the Cauchy problems

$$\dot{x} = g(x), \quad x(0) = p; \tag{2.4a}$$

$$\dot{x} = a(t)g(x), \quad x(0) = p\,. \tag{2.4b}$$

Denote by $x : I \to M$ and $\xi : J \to M$, $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ intervals, the (unique) maximal solution of (2.4a) and of (2.4b) respectively. Clearly, if $\int_0^\tau a(s)\,\mathrm{d}s \in I$ for all $\tau \in [0, t]$, then

$$\xi(t) = x\Big(\int_0^t a(s)\,\mathrm{d}s\Big);$$

hence, $t \in J$. Moreover, by a standard maximality argument, one can prove that $t \in J$ implies $\int_0^t a(s)\,\mathrm{d}s \in I$. In particular, if $T \in J$, then $\int_0^T a(s)\,\mathrm{d}s = T \in I$. When this happens, one has $\xi(T) = x(T)$. In other words, if $P_T^{ag}(p)$ is defined, then so is $P_T^g(p)$ and, in this case, $P_T^g(p) = P_T^{ag}(p)$.

Note also that when $a(t) > 0$ for any $t \in [0, T]$ (or, equivalently, $a(t) < 0$ for any $t \in [0, T]$) the function $t \mapsto \int_0^t a(s)\,\mathrm{d}s$ is monotone, hence invertible. In particular, $T \in J$ is equivalent to $T \in I$.

Using Remark 2.3 we obtain easily the following consequence of Theorem 2.2.

Corollary 2.4. *Let $g : M \to \mathbb{R}^k$ be a C^1 tangent vector field, and let $a : \mathbb{R} \to \mathbb{R}$ be continuous, T-periodic with $1/T \int_0^T a(t)\,\mathrm{d}t = 1$. Given an open subset U of M, if $\operatorname{ind}(P_T^{ag}, U)$ is well defined, then so is $\operatorname{ind}(P_T^g, U)$ and*

$$\operatorname{ind}(P_T^{ag}, U) = \operatorname{ind}(P_T^g, U) = \deg(-g, U).$$

Remark 2.5. Observe that if $p \in M$ is such that $p = P_T^{ag}(p)$, then any q in the image of the map $t \mapsto P_t^{ag}(p)$ is in the image of $t \mapsto P_T^g(p)$. This means that it is an initial point of a T-periodic orbit of $\dot{x} = g(x)$. Therefore q has the property that $q = P_T^g(q) = P_T^{ag}(q)$.

3. Main result

Let $f : \mathbb{R} \times M \to \mathbb{R}^k$, $g : M \to \mathbb{R}^k$ and $a : \mathbb{R} \to \mathbb{R}$ be as in (1.2). In the sequel, given $X \subset \mathbb{R} \times M$ and $\lambda \in \mathbb{R}$, we will denote the slice $\{x \in M : (\lambda, x) \in X\}$ by the symbol X_λ.

By known properties of differential equations, the set $V \subset [0, \infty) \times M$, given by

$$V = \Big\{(\lambda, p) : \text{the solution } x(\cdot) \text{ of (1.2) satisfying } x(0) = p \text{ is defined in } [0, T]\Big\},$$

is open. Thus it is locally compact. Clearly V contains the set S of all starting points of (1.2). Observe that S is closed in V, even if it could be not so in $[0, +\infty) \times M$. Therefore S is locally compact. Let U be an open subset of V. Since $S \cap U$ is open in S, it is locally compact as well.

We will also use the following global connectivity result (see [1]).

Lemma 3.1. *Let Y be a locally compact metric space and let Y_0 be a compact subset of Y. Assume that any compact subset of Y containing Y_0 has nonempty boundary. Then $Y \setminus Y_0$ contains a not relatively compact component whose closure (in Y) intersects Y_0.*

We now prove a result that, when a is a nonzero constant, reduces to Theorem 3.1 in [3].

Theorem 3.2. *Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $f : \mathbb{R} \times M \to \mathbb{R}^k$ and $g : M \to \mathbb{R}^k$ be two C^1 tangent vector fields on the boundaryless manifold $M \subset \mathbb{R}^k$. Assume also that f and a are T-periodic, and the average $\bar{a}$ of a is nonzero. Denote by S the set of the starting points for* (1.2) *and let U be an open subset of* $[0, \infty) \times M$. *Assume that* $\deg(g, U \cap M)$ *is well defined and nonzero. Then the set $(S \cap U) \setminus \big(\{0\} \times g^{-1}(0)\big)$ of the nontrivial starting points (in U) of* (1.2) *admits a connected subset whose closure in $S \cap U$ meets $\{0\} \times g^{-1}(0)$ and is not compact.*

Proof. Since $\bar{a} \neq 0$, one has that

$$\deg\Big(\frac{1}{\bar{a}} g(\cdot), U \cap M\Big) = (\operatorname{sign} \bar{a})^m \deg(g, U \cap M) \neq 0,$$

where m is the dimension of M. Hence, replacing if necessary g with $\bar{a}g$ and a with $a/\bar{a}$, we shall assume $1/T \int_0^T a(s)\, \mathrm{d}s = 1$.

Since $\deg(g, U \cap M) \neq 0$, $\big(\{0\} \times g^{-1}(0)\big) \cap U$ is nonempty. Thus $S \cap U$ is nonempty as well. The assertion follows applying Lemma 3.1 to the pair

$$(Y, Y_0) = \Big(S \cap U, \big(\{0\} \times g^{-1}(0)\big) \cap U\Big).$$

In fact, if Σ is a component as in the assertion of Lemma 3.1 its closure (in $S \cap U$) meets $\{0\} \times g^{-1}(0)$ and is not compact. Assume by contradiction that there exists a compact subset C of $S \cap U$, containing $\big(\{0\} \times g^{-1}(0)\big) \cap U$ and with empty boundary in $S \cap U$. Thus, C is a relatively open subset of $S \cap U$. As a consequence, $S \cap U \setminus C$ is closed in $S \cap U$, so the distance, $\delta = \operatorname{dist}(C, S \cap U \setminus C)$, between C and $S \cap U \setminus C$ is nonzero (recall that C is compact). Consider the set

$$W = \Big\{(\lambda, p) \in U : \operatorname{dist}\big((\lambda, p), C\big) < \frac{\delta}{2}\Big\},$$

that, clearly, does not meet $S \cap U \setminus C$.

For simplicity, given $s \in [0, +\infty)$, we put

$$W_s = \{p \in M : (s, p) \in W\}.$$

Because of the compactness of $S \cap W = C$, there exists $\lambda_0 > 0$ such that $W_{\lambda_0} = \emptyset$. Moreover, the set

$$\{(\lambda, p) \in W : P_T(\lambda, p) = p\}$$

is compact. Then, from the generalized homotopy property of the fixed point index (see e.g. [7]),

$$0 = \operatorname{ind}\big(P_T(\lambda_0, \cdot), W_{\lambda_0}\big) = \operatorname{ind}\big(P_T(\lambda, \cdot), W_\lambda\big),$$

for all $\lambda \in [0, \lambda_0]$. Observe that our contradictory assumption implies that P_T^{ag} is fixed point free on the boundary of W_0, therefore $\operatorname{ind}\big(P_T^{ag}, W_0\big)$ is well defined. Applying the excision property of the degree and Corollary 2.4, we get

$$\begin{aligned}\operatorname{ind}\big(P_T^{ag}, W_0\big) &= \operatorname{ind}\big(P_T^{g}, W_0\big)\\ &= (-1)^m \deg(g, W_0) = (-1)^m \deg(g, U \cap M) \neq 0,\end{aligned}$$

contradicting the previous formula. □

We are now in a position to state and prove our main result. It is, basically, an infinite-dimensional version of Theorem 3.2 that, when a is a nonzero constant, reduces to Theorem 3.3 in [4].

Theorem 3.3. *Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $f : \mathbb{R} \times M \to \mathbb{R}^k$ and $g : M \to \mathbb{R}^k$ be two continuous tangent vector fields on the boundaryless manifold $M \subset \mathbb{R}^k$. Assume that f and a are T-periodic, with average $\bar{a} \neq 0$. Let Ω be an open subset of $[0, \infty) \times C_T(M)$, and assume that the degree $\deg(g, \Omega \cap M)$ is well-defined and nonzero. Then there exists a connected set Γ of nontrivial T-pairs in Ω whose closure in $[0, \infty) \times C_T(M)$ meets $g^{-1}(0) \cap \Omega$ and is not contained in any compact subset of Ω. In particular, if M is closed in $\mathbb{R}^k$ and $\Omega = [0, \infty) \times C_T(M)$, then Γ is unbounded.*

Proof. As in the proof of Theorem 3.2 we shall assume, without loss of generality, that $1/T \int_0^T a(s)\,\mathrm{d}s = 1$.

Let X denote the set of T-pairs of (1.2). Since X is closed, it is enough to show that there exists a connected set Γ of nontrivial T-pairs in Ω whose closure in $X \cap \Omega$ meets $g^{-1}(0)$ and is not compact.

Assume first that f and g are smooth. Denote by S the set of all starting points of (1.2), and take

$$\tilde{S} = \{(\lambda, p) \in S : \text{the solution of (2.1) is contained in } \Omega\}.$$

Obviously $\tilde{S}$ is an open subset of S, thus we can find an open subset U of V such that $S \cap U = \tilde{S}$, where V is the set of all the pairs (λ, p) such that the solution of (2.1) is defined in $[0, T]$. We have that

$$g^{-1}(0) \cap \Omega = g^{-1}(0) \cap \tilde{S} = g^{-1}(0) \cap U,$$

thus $\deg(g, U \cap M) = \deg(g, \Omega \cap M) \neq 0$. Applying Theorem 3.2, we get the existence of a connected set $\Sigma \subset \big(S \cap U\big) \setminus g^{-1}(0)$ such that its closure in $S \cap U$ is not compact and meets $g^{-1}(0)$. Let $h : X \to S$ be the map which assigns to any T-pair (λ, x) the starting point $\big(\lambda, x(0)\big)$. By Remark 2.1, h is a homeomorphism and trivial T-pairs correspond to trivial starting points under h. This implies that $\Gamma = h^{-1}(\Sigma)$ satisfies the requirements.

Let us remove the smoothness assumption on g and f. Take $Y_0 = g^{-1}(0) \cap \Omega$ and $Y = X \cap \Omega$. We have only to prove that the pair (Y, Y_0) satisfies the hypothesis of Lemma 3.1. Assume the contrary. We can find a relatively open compact subset C of Y containing Y_0. Thus there exists an open subset W of Ω such that the closure $\overline{W}$ of W in $[0,\infty) \times C_T(M)$ is contained in Ω, $W \cap Y = C$ and $\partial W \cap Y = \emptyset$. Since C is compact and $[0,\infty) \times M$ is locally compact, we can choose W in such a way that the set

$$\Big\{\big(\lambda, x(t)\big) \in [0,\infty) \times M : (\lambda, x) \in W,\ t \in [0,T]\Big\}$$

is contained in a compact subset K of $[0,\infty) \times M$. This implies that W is bounded with complete closure in Ω and $W \cap M$ is a relatively compact subset of $\Omega \cap M$. In particular g is nonzero on the boundary of $W \cap M$ (relative to M). By known approximation results, there exist sequences $\{g_i\}$ of smooth tangent vector fields uniformly approximating g on M. For $i \in \mathbb{N}$ large enough, we get

$$\deg(g_i, W \cap M) = \deg(g, W \cap M).$$

Furthermore, by excision,

$$\deg(g, W \cap M) = \deg(g, \Omega \cap M) \neq 0.$$

Therefore, given i large enough, the first part of the proof can be applied to the equation

$$\dot{x} = a(t) g_i(x) + \lambda f_i(t, x), \tag{3.1}$$

where $\{f_i\}$ is a sequence of smooth T-periodic tangent vector fields uniformly approximating f on K.

Let X_i denote the set of T-pairs of (3.1). There exists a connected subset Γ_i of $\Omega \cap X_i$ whose closure in Ω meets $g_i^{-1}(0) \cap W$ and is not contained in any compact subset of Ω. Let us prove that, for i large enough, $\Gamma_i \cap \partial W \neq \emptyset$. It is sufficient to show that $X_i \cap \overline{W}$ is compact. In fact, if $(\lambda, x) \in X_i \cap \overline{W}$ we have, for any $t \in [0,T]$,

$$\|\dot{x}(t)\| \leq \max\big\{\|a(\tau) g(p) + \mu f(\tau, p)\| : (\mu, p) \in K\ ,\ \tau \in [0,T]\big\}.$$

Hence, by Ascoli's theorem, $X_i \cap \overline{W}$ is totally bounded and, consequently, compact, since X_i is closed and $\overline{W}$ is complete. Thus, for i large enough, there exists a T-pair $(\lambda_i, x_i) \in \Gamma_i \cap \partial W$ of (3.1). Again by Ascoli's theorem, we may assume that $x_i \to x_0$ in $C_T(M)$ and $\lambda_i \to \lambda_0$ with $(\lambda_0, x_0) \in \partial W$. Therefore

$$\dot{x}_0(t) = a(t) g\big(x_0(t)\big) + \lambda_0 f\big(t, x_0(t)\big), \quad t \in \mathbb{R}.$$

Hence (λ_0, x_0) is a T-pair in ∂W. This contradicts the assumption $\partial W \cap Y = \emptyset$.

It remains to prove the last assertion. Let M be closed. There exists a connected set Γ of T-pairs of (1.2) whose closure is not compact and meets $g^{-1}(0)$. We need to show that Γ is unbounded. Assume the contrary. As we already observed, when M is closed any bounded closed set of T-pairs is compact. Thus the closure of Γ in $[0,\infty) \times C_T(M)$ is compact. This yields a contradiction. $\square$

Note that the connected set of T-pairs of Theorem 3.3 can be completely contained in the slice $\{0\} \times C_T(M)$, as in the following simple example where $M = \mathbb{R}^2$, $T = 2\pi$, $a(t) \equiv 1$ and $\Omega = [0,\infty) \times C_{2\pi}(\mathbb{R}^2)$:

$$\dot{x} = y, \quad \dot{y} = -x + \lambda \sin t$$

Corollary 3.4. *Let $M \subset \mathbb{R}^k$ be a compact boundaryless manifold with $\chi(M) \neq 0$. Take a, g and f as in Theorem 3.3. Then there exists an unbounded connected set Γ of T-pairs whose closure meets $g^{-1}(0)$ and is such that*

$$\pi_1(\Gamma) = [0, \infty), \tag{3.2}$$

where π_1 denotes the projetion onto the first factor of $[0, \infty) \times C_T(M)$.

Proof. Take $\Omega = [0, \infty) \times C_T(M)$, so that $\Omega \cap M = M$. By the Poincaré-Hopf theorem

$$\deg(g, \Omega \cap M) = \deg(g, M) = \chi(M) \neq 0.$$

Theorem 3.3 yields the existence of an unbounded connected set Γ of T-pairs whose closure meets $g^{-1}(0)$. Since $C_T(M)$ is bounded, (3.2) holds. □

4. Applications to multiplicity results

Note that in the previous section, where only "global" properties of the set of T-pairs were studied, we merely require the average of the function a to be nonzero. In this section, where we look also at "local" behaviour, we will need to require explicitly that

$$\frac{1}{T}\int_0^T a(s)\,\mathrm{d}s = 1.$$

As we have seen in the proof of Theorem 3.2, this can be assumed without any loss of generality.

Below, we shall obtain a multiplicity result. In order to do that we will need to consider also the behavior of the set of T-pairs near $g^{-1}(0)$. Loosely speaking, in order to find multiplicity results for the periodic solutions of (1.2) it is necessary to avoid the somehow degenerate situation when the "branch" of T-pairs "sticks" to the manifold. We first tackle this problem from an abstract viewpoint.

We need some notation. Let Y be a metric space and C a subset of $[0, \infty) \times Y$. Given $\lambda \geq 0$, we denote by C_λ the slice $\big\{y \in Y : (\lambda, y) \in C\big\}$. In what follows, Y will be identified with the subset $\{0\} \times Y$ of $[0, \infty) \times Y$.

Definition 4.1 ([2])**.** Let C be a subset of $[0, \infty) \times Y$. We say that a subset A of C_0 is an *ejecting set* (for C) if it is relatively open in C_0 and there exists a connected subset of C which meets A and is not included in C_0.

We shall simply say that $q \in C_0$ is an *ejecting point* if $\{q\}$ is an ejecting set. In this case, $\{q\}$ being open in C_0, it is clearly isolated in C_0.

In [2] the following theorem which relates ejecting sets and multiplicity results was proved.

Theorem 4.2. *Let Y be a metric space and let C be a locally compact subset of $[0, \infty) \times Y$. Assume that C_0 contains n pairwise disjoint ejecting sets, $n-1$ of which are compact. Then, there exists $\delta > 0$ such that the cardinality of C_λ is greater than or equal to n for any $\lambda \in [0, \delta)$.*

Let p be a zero of g. We give a condition which ensures that p (regarded as the trivial T-pair $(0, \hat{p})$, where $\hat{p}$ denotes the function $\hat{p}(t) \equiv p$) is an ejecting point for the set X of the T-pairs of (1.2).

Definition 4.3. A point $p \in g^{-1}(0)$ is said *T-resonant* provided that

(1) g is C^1 in a neighborhood of p;

(2) the only T-periodic solution of the linearized equation at p (on T_pM)

$$\dot{\xi} = a(t)g'(p)\xi \tag{4.1}$$

is trivial (i.e., $\xi(t) \equiv 0$).

Remark 4.4. If g is C^1 in a neighborhood of $p \in g^{-1}(0)$, the T-resonancy condition at p can be read on the spectrum $\sigma\big(g'(p)\big)$ of the endomorphism $g'(p) : T_pM \to T_pM$.

In fact, as one can easily check,

$$\xi(t) = e^{\int_0^t a(s)\,\mathrm{d}s\, g'(p)} u$$

is the solution of (4.1) with initial condition $\xi(0) = u$, $u \in T_pM$. Therefore, $u \in T_pM$ is a starting point for a periodic solution of (4.1) if and only if

$$u \in \ker\left(I - e^{Tg'(p)}\right),$$

where $I : T_pM \to T_pM$ denotes the identity (we are assuming $1/T \int_0^T a(t)\,\mathrm{d}t = 1$). Thus p is T-resonant if and only if, for some $n \in \mathbb{Z}$

$$\frac{2n\pi i}{T} \in \sigma\big(g'(p)\big).$$

Observe also that if $p \in g^{-1}(0)$ is non-T-resonant then the fixed point index of the Poincaré T-translation operators associated to the two following linearized equations at p: $\dot{y} = a(t)g'(p)y$ and $\dot{y} = g'(p)y$, coincide with $\mathrm{i}(-g,p)$.

Lemma 4.5. *Assume that g is C^1 in a neighborhood of a non-T-resonant $p \in g^{-1}(0)$. Then, p (regarded as a trivial T-pair) is an ejecting point for the set X of the T-pairs of* (1.2)*.*

Proof. Observe first that, since p is non-T-resonant, it is an isolated zero of g, and there exists a neighborhood V of p such that $g^{-1}(0) \cap \overline{V} = \{p\}$ and

$$\deg(g,V) = \mathrm{i}(g,p) = \operatorname{sign}\det g'(p) \neq 0.$$

Therefore, taking

$$\Omega = [0,\infty) \times C_T(V) \subset [0,\infty) \times C_T(M),$$

one has $\deg(g, \Omega \cap M) = \deg(g,V) \neq 0$. Thus, Theorem 3.3 yields the existence of a connected set Γ of T-pairs for (1.2) whose closure in Ω contains p and is not compact.

We now prove that, for V small enough and with compact closure $\overline{V}$, no T-periodic solution to (1.1) touches the boundary ∂V of V. Assume by contradiction that this is not the case. Take a sequence $\{V_n\}$ of open neighborhoods of p such that $\bigcap_{n\in\mathbb{N}} V_n = \{p\}$ and $\overline{V_{n+1}} \subset V_n$ for all $n \in \mathbb{N}$. Then, there exists a sequence $\{x_n\}$ of T-periodic solutions to (1.1) with the property that $x_n([0,T]) \cap \partial V_n \neq \emptyset$. By Remark 2.5 we can assume $x_n(0) \in \partial V_n$. Clearly, due to Remark 2.3, it is also not restrictive to assume $x_n(0) \neq x_m(0)$ for $m \neq n$. Put

$$p_n = x_n(0), \quad \text{and} \quad u_n = \frac{p_n - p}{|p_n - p|}.$$

Clearly $p_n \to p$. We can assume $u_n \to u \in T_pM$.

Since g is C^1, it is known that $P_T^{ag}(\cdot)$ is differentiable. Define $\Phi : M \to \mathbb{R}^k$ by $\Phi(q) = q - P_T^{ag}(q)$. Clearly Φ is differentiable and $\Phi(p_n) = 0$, hence

$$\Phi'(p)u = \lim_{n\to\infty} \frac{\Phi(p_n) - \Phi(p)}{|p_n - p|} = 0.$$

On the other hand, $\Phi'(p)v = v - [P_T^{ag}]'(p)v$ for any $v \in T_pM$. One can easily verify that the map $\alpha : t \mapsto [P_t^{ag}]'(p)v$ satisfies the following Cauchy problem

$$\dot\alpha(t) = a(t)g'(p)\alpha(t)\,, \quad \alpha(0) = u.$$

Since p is non-T-resonant, $\Phi'(p)u = \alpha(0) - \alpha(T) \neq 0$. This is a contradiction.

We now prove that p is an ejecting point for X. Clearly, if Γ is contained in $\{0\} \times C_T(M)$, then it must be contained into $\{0\} \times C_T(\overline{V})$ since no T-periodic solution to (1.1) touches ∂V. Let us prove that this is impossible. Assume the contrary. Then, Γ, as a bounded set of T-pairs is totally bounded. Moreover, $\{0\} \times C_T(\overline{V})$ being complete, the closure of Γ is compact. This proves that Γ cannot be contained in $\{0\} \times C_T(M)$. The assertion follows. □

We are now in a position to establish a multiplicity result for forced oscillations.

Theorem 4.6. *Let M be a compact boundaryless manifold, and take continuous tangent vector fields $f : \mathbb{R} \times M \to \mathbb{R}^k$ and $g : M \to \mathbb{R}^k$, a continuous function $a : \mathbb{R} \to \mathbb{R}$, and let f and a be T-periodic with $1/T \int_0^T a(t)\,\mathrm{d}t = 1$. Then, if g has $n-1$, $n > 1$, non-T-resonant zeros $p_1,\dots,p_{n-1}$ with*

$$\sum_{k=1}^{n-1} \mathrm{i}(p_k, g) \neq \chi(M),$$

there are at least n solutions of period T of equation (1.2) *for λ sufficiently small.*

Proof. Since $p_1,\dots,p_{n-1}$ are non-T-resonant, there exist neighborhoods $V_1,\dots,V_{n-1}$ such that

$$\overline{V_i} \cap g^{-1}(0) = \{p_i\} \quad \text{for } i = 1, \dots, n-1$$

Clearly, by excision, $\deg(g, V_i) = \mathrm{i}(g, p_i)$, for $i = 1, \dots, n-1$. Define

$$V_0 = M \setminus \bigcup_{i=1}^{n-1} \overline{V_i}.$$

By the Poincaré-Hopf Theorem, $\deg(g, M) = \chi(M)$. The additivity property of the degree yields

$$\deg(g, V_0) = \chi(M) - \sum_{i=1}^{n-1} \mathrm{i}(p_i, g) \neq 0$$

Define

$$\Omega = [0, \infty) \times C_T(V_0) \subset [0, \infty) \times C_T(M).$$

Theorem 3.3 implies that $g^{-1}(0) \cap V_0$ is an ejecting set of the set of T-pairs for (1.2). The assertion now follows from Lemma 4.5 and Theorem 4.2. □

In the following example we exibit a tangent vector field g to the unit sphere S^2 centered at the origin of $\mathbb{R}^3$ with the property that, for any $T > 0$, only one of its two zeros can be T-resonant. Theorem 4.6 implies that any small enough T-periodic perturbation of equation

$$\dot x = a(t)g(x),$$

where $a : \mathbb{R} \to \mathbb{R}$ is any T-periodic continuous function with average equal to 1, has at least two T-periodic solutions.

Example 4.7. Take $M = S^2 \subset \mathbb{R}^3$ and let g be the tangent vector field given by

$$(x, y, z) \mapsto e^z\big(- xz, -yz, 1 - z^2\big).$$

That is, g is the gradient on the manifold $M = S^2$ of the functional $(x, y, z) \mapsto e^z$.

Note that g has the "poles" $\mathrm{N} = (0, 0, 1)$ and $\mathrm{S} = (0, 0, -1)$ as its only two zeros, and $\sigma\big(g'(\mathrm{N})\big) = \{-e\}$ and $\sigma\big(g'(\mathrm{S})\big) = \{e^{-1}\}$.

Then, for any $T > 0$ for which N is T-resonant, S is non-T-resonant. Consequently, for any $T > 0$, any T-periodic $a : \mathbb{R} \to [0, \infty)$ with $\bar{a} = 1$, and any T-periodic $f : \mathbb{R} \times M \to \mathbb{R}^3$, there exists $\lambda_0 > 0$ such that (1.2) admits two T-periodic solutions for $\lambda \in [0, \lambda_0)$.

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Dipartimento di Matematica Applicata, Università di Firenze, Via S. Marta 3 - 50139 Firenze, Italy

E-mail address: spadini@dma.unifi.it