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STABILITY FOR A COUPLED SYSTEM OF WAVE EQUATIONS OF KIRCHHOFF TYPE WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. We consider a coupled system of two nonlinear wave equations of Kirchhoff type with nonlocal boundary condition and we study the asymptotic behavior of the corresponding solutions. We prove that the energy decay at the same rate of decay of the relaxation functions, that is, the energy decays exponentially when the relaxation functions decay exponentially and polynomially when the relaxation functions decay polynomially.

1. Introduction

The main purpose of this article is to study the existence of global solutions and the asymptotic behavior of the energy related to a coupled system of two nonlinear wave equations of Kirchhoff type with nonlocal boundary condition. Consider the system of equations

$$u_{tt} - M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\Delta u - \Delta u_t + f_1(u) = 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$v_{tt} - M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\Delta v - \Delta v_t + f_2(v) = 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

$$u = v = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{1.3}$$

$$\begin{gathered} u + \int_0^t g_1(t-s)((M(\|\nabla u(s)\|_2^2 + \|\nabla v(s)\|_2^2)\frac{\partial u}{\partial \nu}(s) + \frac{\partial u_t}{\partial \nu}(s))ds = 0 \\ \text{on } \Gamma_1 \times (0,\infty), \end{gathered} \tag{1.4}$$

$$\begin{gathered} v + \int_0^t g_2(t-s)((M(\|\nabla u(s)\|_2^2 + \|\nabla v(s)\|_2^2)\frac{\partial v}{\partial \nu}(s) + \frac{\partial v_t}{\partial \nu}(s))ds = 0 \\ \text{on } \Gamma_1 \times (0,\infty), \end{gathered} \tag{1.5}$$

$$(u(0,x), v(0,x)) = (u_0(x), v_0(x)), \quad (u_t(0,x), v_t(0,x)) = (u_1(x), v_1(x)) \quad \text{in } \Omega, \tag{1.6}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed, disjoint, $\Gamma_0 \neq \emptyset$ and ν is the unit normal vector pointing towards the exterior of Ω. The equations (1.4)-(1.5) are nonlocal boundary

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conditions responsible for the memory effect. Concerning the history condition, we must add the condition

$$u = v = 0 \quad \text{on } \Gamma_0 \times]-\infty, 0].$$

We observe that, u and v represent transverse displacements. The relaxation functions g_i are positive and non decreasing; while the functions $f_i \in C^1(\mathbb{R})$, $i = 1, 2$, satisfy

$$f_i(s)s \geq 0 \quad \forall s \in \mathbb{R}$$

Additionally, we suppose that f_i is superlinear, that is

$$f_i(s)s \geq (2+\delta)F_i(s), \quad F_i(z) = \int_0^z f_i(s)ds \quad \forall s \in \mathbb{R}, \quad i = 1, 2,$$

for some $\delta > 0$. Also the following growth conditions are satisifed:

$$|f_i(x) - f_i(y)| \leq c(1 + |x|^{\rho-1} + |y|^{\rho-1})|x - y|, \quad \forall x, y \in \mathbb{R}, \quad i = 1, 2,$$

for some $c > 0$ and $\rho \geq 1$ such that $(n-2)\rho \leq n$. We shall assume that the function $M \in C^1([0, \infty[)$ satisfies

$$M(\lambda) \geq m_0 > 0, \quad M(\lambda)\lambda \geq \widehat{M}(\lambda), \quad \forall \lambda \geq 0, \tag{1.7}$$

where $\widehat{M}(\lambda) = \int_0^\lambda M(s)ds$. Also, we shall assume that there exists $x_0 \in \mathbb{R}^n$ such that

$$\Gamma_0 = \{x \in \Gamma : \nu(x) \cdot (x - x_0) \leq 0\},$$
$$\Gamma_1 = \{x \in \Gamma : \nu(x) \cdot (x - x_0) > 0\}.$$

Let us denote by $m(x) = x - x_0$. Note that by the compactness of Γ_1, there exist a small positive constant δ_0 such that

$$0 < \delta_0 \leq m(x) \cdot \nu(x), \quad \forall x \in \Gamma_1. \tag{1.8}$$

The existence of global solutions and exponential decay to the problem (1.1), (1.3) with $\partial\Omega = \Gamma_0$ and frictional dissipative damping has been investigated by many authors (see, e.g. [1, 2, 3, 4, 8, 9, 10, 12]). There exists a large body of literature regarding viscoelastic problems with the memory term acting in the domain or in the boundary. Among the numerous works in this direction, we can cite Rivera [5] and M. L. Santos [15, 16]. Park & Bae [13] studied the existence and uniform decay of strong solutions of the coupled wave equations (1.1)-(1.2) with nonlinear boundary damping and memory source term and $M(s) = 1 + s$. In the present paper, we obtained respectively besides the exponential decay and uniform rate of polynomial decay. Moreover, the system (1.1)-(1.6) is more general than the system considered in [13], because they only consider the case in that $M(s) = 1 + s$. As we have said before we study the asymptotic behavior of the solutions of system (1.1)-(1.6). We show that the energy of the coupled system (1.1)-(1.6) decays uniformly in time with the same rate of decay of the relaxation functions. More precisely, denoting by k_1 and k_2 the resolvent kernels of $-g_1'/g_1(0)$ and $-g_2'/g_2(0)$ respectively, we show that the energy decays exponentially to zero provided k_1 and k_2 decays exponentially to zero. When the resolvent kernels k_1 and k_2 decays polynomially, we show that energy also decays polynomially to zero. This means that the memory effect produces strong dissipation capable of making a uniform rate of decay for the energy. The method used is based on the construction of a suitable Lyapunov functional $\mathcal{L}$ satisfying

$$\frac{d}{dt}\mathcal{L}(t) \le -c_1\mathcal{L}(t) + c_2 e^{-\gamma t} \quad \text{or} \quad \frac{d}{dt}\mathcal{L}(t) \le -c_1\mathcal{L}(t)^{1+\frac{1}{\alpha}} + \frac{c_2}{(1+t)^{\alpha+1}}$$

for some positive constants c_1, c_2, γ and α. Note that, because of condition (1.3) the solution of the system (1.1)-(1.6) must belong to the space

$$V := \{v \in H^1(\Omega) : v = 0 \quad \text{on} \quad \Gamma_0\}.$$

The notation used in this paper is standard and can be found in Lion's book [7]. In the sequel by c (sometime $c_1, c_2, \dots$) we denote various positive constants independent of t and on the initial data. The organization of this paper is as follows. In section 2 we establish the existence and uniqueness of strong solutions for the system (1.1)-(1.6). In section 3 we prove the uniform rate exponential decay. In section 4 we prove the uniform rate of polynomial decay.

2. Notation and Main Results

In this section we shall study the existence and regularity of solutions for the coupled system (1.1)-(1.6). First, we shall use equations (1.4)-(1.5) to estimate the terms $M(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu}$ and $M(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu}$. Denoting by

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s)ds,$$

the convolution product operator and differentiating the equations (1.4) and (1.5) we arrive at the following Volterra equations:

$$M(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} + \frac{1}{g_1(0)} g_1' * (M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu})$$
$$= -\frac{1}{g_1(0)} u_t,$$
$$M(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu} + \frac{1}{g_2(0)} g_2' * (M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu})$$
$$= -\frac{1}{g_2(0)} v_t.$$

Applying the Volterra's inverse operator, we get

$$M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} = -\frac{1}{g_1(0)}\{u_t + k_1 * u_t\},$$
$$M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu} = -\frac{1}{g_2(0)}\{v_t + k_2 * v_t\},$$

where the resolvent kernels satisfies

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i' \quad \text{for } i = 1, 2.$$

Denoting $\tau_1 = \frac{1}{g_1(0)}$ and $\tau_2 = \frac{1}{g_2(0)}$, we obtain

$$M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} = -\tau_1\{u_t + k_1(0)u - k_1(t)u_0 + k_1' * u\} \tag{2.1}$$
$$M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu} = -\tau_2\{v_t + k_2(0)v - k_2(t)v_0 + k_2' * v\}. \tag{2.2}$$

Reciprocally, taking initial data such that $u_0 = v_0 = 0$ on Γ_1, the identities (2.1)-(2.2) imply (1.4)-(1.5). Since we are interested in relaxation functions of exponential or polynomial type and the identities (2.1)-(2.2) involve the resolvent kernels k_i, we want to know if k_i has the same properties. The following Lemma answers this question. Let h be a relaxation function and k its resolvent kernel, that is

$$k(t) - k * h(t) = h(t). \tag{2.3}$$

Lemma 2.1. *If h is a positive continuous function, then k is also a positive continuous function. Moreover,*

(1) *If there exist positive constants c_0 and γ with $c_0 < \gamma$ such that $h(t) \leq c_0 e^{-\gamma t}$, then, the function k satisfies*
$$k(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0} e^{-\epsilon t},$$
for all $0 < \epsilon < \gamma - c_0$.

(2) *Given $p > 1$, let us denote by $c_p := \sup_{t \in \mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p}\, ds$. If there exists a positive constant c_0 with $c_0 c_p < 1$ such that $h(t) \leq c_0(1+t)^{-p}$, then, the function k satisfies*
$$k(t) \leq \frac{c_0}{1 - c_0 c_p} (1+t)^{-p}.$$

Proof. Note that $k(0) = h(0) > 0$. Now, we take $t_0 = \inf\{t \in \mathbb{R}^+ : k(t) = 0\}$, so $k(t) > 0$ for all $t \in [0, t_0[$. If $t_0 \in \mathbb{R}^+$, from equation (2.3) we get that $-k * h(t_0) = h(t_0)$ but this is contradictory. Therefore $k(t) > 0$ for all $t \in \mathbb{R}_0^+$. Now, let us fix ϵ, such that $0 < \epsilon < \gamma - c_0$ and denote by

$$k_\epsilon(t) := e^{\epsilon t} k(t), \quad h_\epsilon(t) := e^{\epsilon t} h(t).$$

Multiplying equation (2.3) by $e^{\epsilon t}$ we get $k_\epsilon(t) = h_\epsilon(t) + k_\epsilon * h_\epsilon(t)$, hence

$$\sup_{s \in [0,t]} k_\epsilon(s) \leq \sup_{s \in [0,t]} h_\epsilon(s) + \Big(\int_0^\infty c_0 e^{(\epsilon - \gamma)s}\, ds \Big) \sup_{s \in [0,t]} k_\epsilon(s) \leq c_0 + \frac{c_0}{(\gamma - \epsilon)} \sup_{s \in [0,t]} k_\epsilon(s).$$

Therefore,

$$k_\epsilon(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0},$$

which implies our first assertion. To show the second part we use the notation

$$k_p(t) := (1+t)^p k(t), \quad h_p(t) := (1+t)^p h(t).$$

Multiplying equation (2.3) by $(1+t)^p$ we get

$$k_p(t) = h_p(t) + \int_0^t k_p(t-s)(1+t-s)^{-p}(1+t)^p h(s)\, ds\,,$$

hence

$$\sup_{s \in [0,t]} k_p(s) \leq \sup_{s \in [0,t]} h_p(s) + c_0 c_p \sup_{s \in [0,t]} k_p(s) \ \leq \ c_0 + c_0 c_p \sup_{s \in [0,t]} k_p(s).$$

Therefore,

$$k_p(t) \leq \frac{c_0}{1 - c_0 c_p},$$

which proves our second assertion. ∎

Remark: The fact that the constant c_p is finite can be found in [14, Lemma 7.4].

Due to this Lemma, in the remainder of this paper, we shall use (2.1)-(2.2) instead of (1.4)-(1.5). Let us denote

$$(g\Box\varphi)(t) := \int_0^t g(t-s)|\varphi(t)-\varphi(s)|^2 ds.$$

The following lemma states an important property of the convolution operator.

Lemma 2.2. *For $g, \varphi \in C^1([0,\infty[: \mathbb{R})$ we have*

$$(g*\varphi)\varphi_t = -\frac{1}{2}g(t)|\varphi(t)|^2 + \frac{1}{2}g'\Box\varphi - \frac{1}{2}\frac{d}{dt}\Big[g\Box\varphi - (\int_0^t g(s)ds)|\varphi|^2\Big].$$

The proof of this lemma follows by differentiating the expression $g\Box\varphi$.

The first order energy of coupled system (1.1)-(1.6) is defined as

$$\begin{aligned}
E(t) &:= E(t,u,v)\\
&= \frac{1}{2}\int_\Omega |u_t|^2dx + \frac{1}{2}\int_\Omega |v_t|^2dx + \frac{1}{2}\widehat{M}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \int_\Omega F_1(u)dx\\
&\quad + \int_\Omega F_2(v)dx + \frac{\tau_1}{2}k_1(t)\int_{\Gamma_1}|u|^2d\Gamma_1 - \frac{\tau_1}{2}\int_{\Gamma_1} k_1'\Box u d\Gamma_1\\
&\quad + \frac{\tau_2}{2}k_2(t)\int_{\Gamma_1}|v|^2d\Gamma_1 - \frac{\tau_2}{2}\int_{\Gamma_1} k_2'\Box v d\Gamma_1.
\end{aligned}$$

The main goal of this work is given by the following Theorem.

Theorem 2.3. *Let $k_i \in C^2(\mathbb{R}^+)$ be such that $k_i, -k_i', k_i'' \geq 0$ for $i = 1,2$. If $(u_0, v_0) \in (H^2(\Omega)\cap V)^2$ and $(u_1, v_1) \in (H^2(\Omega)\cap V)^2$ satisfy the compatibility conditions*

$$M(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)\frac{\partial u_0}{\partial\nu} + \frac{\partial u_1}{\partial\nu} + \tau_1 u_1 = 0 \quad on \Gamma_1, \tag{2.4}$$

$$M(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)\frac{\partial v_0}{\partial\nu} + \frac{\partial v_1}{\partial\nu} + \tau_2 v_1 = 0 \quad on \Gamma_1. \tag{2.5}$$

Then there exists only one solution (u,v) of the system (1.1)-(1.6) satisfying

$$\begin{gathered}
u, v \in L^\infty(0,T:V), \quad u_t, v_t \in L^\infty(0,T:V),\\
u_{tt}, v_{tt} \in L^\infty(0,T:L^2(\Omega)), \quad \Delta u, \Delta v \in L^\infty(0,T:L^2(\Omega)),\\
\Delta u_t, \Delta v_t \in L^2(0,T:L^2(\Omega)).
\end{gathered}$$

In addition, considering (1.8) and assuming that there exist positive constants b_1, b_2 such that

$$k_i(0) > 0, \quad k_i'(t) \leq -b_1 k_i(t), \quad k_i''(t) \geq -b_2 k_i'(t), \quad i = 1,2, \quad or \tag{2.6}$$

$$k_i(0) > 0, \quad k_i'(t) \leq -b_1 k_i'(t)^{1+\frac{1}{p}}, \quad k_i''(t) \geq b_2[-k_i'(t)]^{1+\frac{1}{p+1}}, \quad p > 1, \quad i = 1,2 \tag{2.7}$$

then the energy $E(t)$ associated to problem (1.1)-(1.6) decays, respectively, a the following rate

$$E(t) \leq \alpha_1 e^{-\alpha_2 t}E(0), \tag{2.8}$$

$$E(t) \leq \frac{c}{(1+t)^{p+1}}E(0), \tag{2.9}$$

where α_1, α_2 and c are positive constants.

Proof of the existence of regular solutions. The main idea is to use the Galerkin method. To do this let us take a basis $\{w_j\}_{j\in\mathbb{N}}$ to V which is orthonormal in $L^2(\Omega)$ and we represent by V_m the subspace of V generated by the first m vectors. Standard results on ordinary differential equations guarantee that there exists only one local solution

$$(u^m(t), v^m(t)) := \sum_{j=1}^{m} (g_{j,m}(t), h_{j,m}(t))w_j,$$

of the approximate systems

$$\begin{aligned}
&\int_\Omega u^m_{tt} w dx + M(\|\nabla u^m(t)\|_2^2 + \|\nabla v^m(t)\|_2^2) \int_\Omega \nabla u^m \cdot \nabla w\, dx \\
&+ \int_\Omega \nabla u^m_t \cdot \nabla w dx + \int_\Omega f_1(u^m) w\, dx \\
&= -\tau_1 \int_{\Gamma_1} \{u^m_t + k_1(0)u^m - k_1(t)u^m(0) + k_1' * u^m\} w d\Gamma_1
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\int_\Omega v^m_{tt} w dx + M(\|\nabla u^m(t)\|_2^2 + \|\nabla v^m(t)\|_2^2) \int_\Omega \nabla v^m \cdot \nabla w\, dx \\
&+ \int_\Omega \nabla v^m_t \cdot \nabla w\, dx + \int_\Omega f_2(v^m) w\, dx \\
&= -\tau_2 \int_{\Gamma_1} \{v^m_t + k_2(0)v^m - k_2(t)v^m(0) + k_2' * v^m\} w d\Gamma_1,
\end{aligned} \tag{2.11}$$

for all $w \in V_m$ with the initial data

$$(u^m(0), v^m(0)) = (u_0, v_0), \quad (u^m_t(0), v^m_t(0)) = (u_1, v_1).$$

The extension of these solutions to the whole interval $[0, T]$, $0 < T < \infty$, is a consequence of the first estimate below.
A priori estimate I. Replacing w by $u'_m(t)$ in (2.10) and $v'_m(t)$ in (2.11), respectively, and then adding the results and using Lemma 2.2 we conclude that

$$\frac{d}{dt} E(t, u^m, v^m) \le cE(0, u^m, v^m).$$

Integrating over $[0, t]$ and taking into account the definition of the initial data of (u^m, v^m) we conclude that

$$E(t, u^m, v^m) \le c, \quad \forall t \in [0, T], \quad \forall m \in \mathbb{N}. \tag{2.12}$$

A priori estimate II. First, we estimate the initial data $u^m_{tt}(0)$ and $v^m_{tt}(0)$ in the L^2-norm. Letting $t \to 0^+$ in the equations (2.10)-(2.11), replacing w by $u''_m(0)$ and $v''_m(0)$, respectively, and using the compatibility conditions (2.4)-(2.5) we get

$$\begin{aligned}
&\|u^m_{tt}(0)\|_2^2 \\
&= M(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2) \int_\Omega \Delta u_0 u^m_{tt}(0) dx + \int_\Omega \Delta u_1 u^m_{tt}(0) dx - \int_\Omega f_1(u_0) u^m_{tt}(0) dx, \\
&\|v^m_{tt}(0)\|_2^2 \\
&= M(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2) \int_\Omega \Delta v_0 v^m_{tt}(0) dx + \int_\Omega \Delta v_1 v^m_{tt}(0) dx - \int_\Omega f_2(v_0) v^m_{tt}(0) dx.
\end{aligned}$$

Since $(u_0, v_0) \in [H^2(\Omega)]^2$, the growth hypothesis for the functions f_1 and f_2 together with the Sobolev's imbedding imply $f_1(u_0), f_2(v_0) \in L^2(\Omega)$. Hence

$$\|u_{tt}^m(0)\|_2 + \|v_{tt}^m(0)\|_2 \le c_1, \quad \forall m \in \mathbb{N}. \tag{2.13}$$

Differentiating the equations (2.10)-(2.11) with respect to the time, replacing w by $u_m''(t)$ and $v_m''(t)$, respectively, and summing the results we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |u_{tt}^m|^2dx + \int_\Omega |v_{tt}^m|^2dx\Big\} + \int_\Omega |\nabla u_{tt}^m|^2dx + \int_\Omega |\nabla v_{tt}^m|^2dx \\
&= -M(\|\nabla u^m\|_2^2 + \|\nabla v^m\|_2^2)\Big\{\int_\Omega \nabla u_t^m \cdot \nabla u_{tt}^m dx + \int_\Omega \nabla v_t^m \cdot \nabla v_{tt}^m dx\Big\} \\
&\quad + 2M'(\|\nabla u^m\|_2^2 + \|\nabla v^m\|_2^2)\Big\{\int_\Omega \nabla u^m \nabla u_t^m dx + \int_\Omega \nabla v^m \nabla v_t^m dx\Big\} \\
&\quad \times \Big\{\int_\Omega \nabla u^m \cdot \nabla u_{tt}^m dx + \int_\Omega \nabla v^m \cdot \nabla v_{tt}^m dx\Big\} - \int_\Omega f_1'(u^m)u_t^m u_{tt}^m dx \\
&\quad - \int_\Omega f_2'(v^m)v_t^m v_{tt}^m dx - \tau_1 \int_{\Gamma_1} |u_{tt}^m|^2dx - \tau_1 \int_{\Gamma_1} k_1(0)u_t^m u_{tt}^m d\Gamma_1 \\
&\quad + \tau_1 k_1'(t)\int_{\Gamma_1} u^m(0)u_{tt}^m d\Gamma_1 - \tau_1 \int_{\Gamma_1} (k_1' * u^m)_t u_{tt}^m d\Gamma_1 - \tau_2 \int_{\Gamma_1} |v_{tt}^m|^2dx \\
&\quad - \tau_2 \int_{\Gamma_1} k_2(0)v_t^m v_{tt}^m d\Gamma_1 + \tau_2 k_2'(t)\int_{\Gamma_1} v^m(0)v_{tt}^m d\Gamma_1 - \tau_2 \int_{\Gamma_1} (k_2' * v^m)_t v_{tt}^m d\Gamma_1.
\end{aligned}$$

Noting that

$$\begin{aligned}
(k_1' * u^m)_t &= k_1'(t)u_0^m + \int_0^t k_1'(t-s)u^m(\cdot,s)ds, \\
(k_2' * v^m)_t &= k_2'(t)v_0^m + \int_0^t k_2'(t-s)v^m(\cdot,s)ds
\end{aligned}$$

and using Lemma 2.2, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |u_{tt}^m|^2dx + \int_\Omega |v_{tt}^m|^2dx + \tau_1 k_1(t)\int_{\Gamma_1} |u_t^m|^2d\Gamma_1 - \tau_1 \int_{\Gamma_1} k_1'\Box u_t^m d\Gamma_1 \\
&- \tau_2 k_2(t)\int_{\Gamma_1} |v_t^m|^2d\Gamma_1 - \tau_2 \int_{\Gamma_1} k_2'\Box v_t^m d\Gamma_1\Big\} + \int_\Omega |\nabla u_{tt}^m|^2dx + \int_\Omega |\nabla v_{tt}^m|^2dx \\
&= -M(\|\nabla u^m\|_2^2 + \|\nabla v^m\|_2^2)\Big\{\int_\Omega \nabla u_t^m \cdot \nabla u_{tt}^m dx + \int_\Omega \nabla v_t^m \cdot \nabla v_{tt}^m dx\Big\} \\
&\quad + 2M'(\|\nabla u^m\|_2^2 + \|\nabla v^m\|_2^2)\Big\{\int_\Omega u^m u_t^m dx + \int_\Omega v^m v_t^m dx\Big\} \\
&\quad \times \Big\{\int_\Omega \nabla u^m \cdot \nabla u_{tt}^m dx + \int_\Omega \nabla v^m \cdot \nabla v_{tt}^m dx\Big\} - \int_\Omega f_1'(u^m)u_t^m u_{tt}^m dx \\
&\quad - \int_\Omega f_2'(v^m)v_t^m v_{tt}^m dx - \tau_1 \int_{\Gamma_1} |u_{tt}^m|^2dx + \tau_1 k_1'(t)\int_{\Gamma_1} u^m(0)u_{tt}^m d\Gamma_1 \\
&\quad + \frac{\tau_1}{2}k_1'(t)\int_{\Gamma_1} |u_t^m|^2d\Gamma_1 - \frac{\tau_1}{2}\int_{\Gamma_1} k_1''\Box u_t^m d\Gamma_1 - \tau_2 \int_{\Gamma_1} |v_{tt}^m|^2dx \\
&\quad + \tau_1 k_2'(t)\int_{\Gamma_1} v^m(0)v_{tt}^m d\Gamma_1 + \frac{\tau_1}{2}k_2'(t)\int_{\Gamma_1} |u_t^m|^2d\Gamma_1 - \frac{\tau_1}{2}\int_{\Gamma_1} k_1''\Box u_t^m d\Gamma_1.
\end{aligned}$$

Let us take $p_n = 2n/(n-2)$. From the growth condition of the functions f_i and from the Sobolev imbedding we have

$$\begin{aligned}
&\int_\Omega f_1'(u^m)u_t^m u_{tt}^m dx \\
&\le c\int_\Omega (1+2|u^m|^{\rho-1})|u_t^m||u_{tt}^m|dx \\
&\le c\Big[\int_\Omega (1+2|u^m|^{\rho-1})^n dx\Big]^{1/n}\Big[\int_\Omega |u_t^m|^{p_n}dx\Big]^{1/p_n}\Big[\int_\Omega |u_{tt}^m|^2 dx\Big]^{1/2} \\
&\le c\Big[\int_\Omega (1+|\nabla u^m|^2)dx\Big]^{(\rho-1)/2}\Big[\int_\Omega |\nabla u_t^m|^2 dx\Big]^{1/2}\Big[\int_\Omega |u_{tt}^m|^2 dx\Big]^{1/2}.
\end{aligned}$$

Taking into account the estimate (2.12) we conclude that

$$\begin{aligned}
\int_\Omega f_1'(u^m)u_t^m u_{tt}^m dx &\le c\Big[\int_\Omega |\nabla u_t^m|^2 dx\Big]^{1/2}\Big[\int_\Omega |u_{tt}^m|^2 dx\Big]^{1/2} \\
&\le c\Big\{\int_\Omega |\nabla u_t^m|^2 dx + \int_\Omega |u_{tt}^m|^2 dx\Big\}.
\end{aligned} \tag{2.14}$$

Similarly we get

$$\int_\Omega f_2'(v^m)v_t^m v_{tt}^m dx \le c\Big\{\int_\Omega |\nabla v_t^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx\Big\}. \tag{2.15}$$

Note that Young's inequality, the first estimate and hypothesis on M give us

$$\begin{aligned}
&M(\|\nabla u^m\|_2^2 + \|\nabla v^m\|_2^2)\Big\{\int_\Omega \nabla u_t^m\cdot\nabla u_{tt}^m dx + \int_\Omega \nabla v_t^m\cdot\nabla v_{tt}^m dx\Big\} \\
&\le c\Big\{\int_\Omega (|\nabla u_t^m|^2 + |\nabla v_t^m|^2)dx + \frac14\int_\Omega |\nabla u_{tt}^m|^2 dx + \frac14\int_\Omega |\nabla v_{tt}^m|^2 dx\Big\}
\end{aligned} \tag{2.16}$$

and similarly

$$\begin{aligned}
&2M'(\|\nabla u^m\|_2^2 + \|\nabla v^m\|_2^2)\Big\{\int_\Omega u^m u_t^m dx + \int_\Omega v^m v_t^m dx\Big\} \\
&\times\Big\{\int_\Omega \nabla u^m\cdot\nabla u_{tt}^m dx + \int_\Omega \nabla v^m\cdot\nabla v_{tt}^m dx\Big\} \\
&\le c\Big\{\int_\Omega (|\nabla u_t^m|^2 + |\nabla v_t^m|^2)dx + \frac14\int_\Omega |\nabla u_{tt}^m|^2 dx + \frac14\int_\Omega |\nabla v_{tt}^m|^2 dx\Big\}.
\end{aligned} \tag{2.17}$$

Substitution of inequalities (2.14)-(2.17) into (2.14) yields

$$\begin{aligned}
&\frac12\frac{d}{dt}\Big\{\int_\Omega |u_{tt}^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx + \tau_1 k_1(t)\int_{\Gamma_1} |u_t^m|^2 d\Gamma_1 - \tau_1\int_{\Gamma_1} k_1'\Box u_t^m d\Gamma_1 \\
&+ \tau_2 k_2(t)\int_{\Gamma_1} |v_t^m|^2 d\Gamma_1 - \tau_2\int_{\Gamma_1} k_2'\Box v_t^m d\Gamma_1\Big\} + \frac12\int_\Omega |\nabla u_{tt}^m|^2 dx + \frac12\int_\Omega |\nabla v_{tt}^m|^2 dx \\
&\le \frac{\tau_1 c}{2}\int_{\Gamma_1} |u_0|^2 d\Gamma_1 + \frac{\tau_2 c}{2}\int_{\Gamma_1} |v_0|^2 d\Gamma_1 + c\int_\Omega (|\nabla u_t^m|^2 + |\nabla v_t^m|^2)dx \\
&+ c\Big\{\int_\Omega |u_{tt}^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx\Big\}.
\end{aligned}$$

Integrating with respect to time and applying Gronwall's inequality we conclude that for all $m \in \mathbb{N}$ and all $t \in [0,T]$,

$$\int_\Omega |u_{tt}^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx + \int_0^t \int_\Omega |\nabla u_{tt}^m|^2 dx + \int_0^t \int_\Omega |\nabla v_{tt}^m|^2 dx \le c\,. \qquad (2.18)$$

A priori estimate III. Replacing w by $-\Delta u_t^m$ in (2.10) and by $-\Delta v_t^m$ in (2.11), respectively, and then using the Green's formula and adding the results yields

$$\begin{aligned}
&- \int_\Omega u_{tt}^m \Delta u_t^m dx - \int_\Omega v_{tt}^m \Delta v_t^m dx + M(\|\nabla u^m\|_2^2 + \|\nabla v^m\|_2^2)\{\int_\Omega \Delta u^m \Delta u_t^m dx \\
&+ \int_\Omega \Delta v^m \Delta v_t^m dx\} + \int_\Omega |\Delta u_t^m|^2 dx + \int_\Omega |\Delta v_t^m|^2 dx \\
&= \int_\Omega f_1(u^m)\Delta u_t^m dx + \int_\Omega f_2(v^m)\Delta v_t^m dx.
\end{aligned}$$

Using similar arguments as in (2.18) we conclude that for all $m \in \mathbb{N}$ and all all $t \in [0,T]$,

$$\|\Delta u^m\|_2^2 + \int_0^T \|\Delta u_t^m(t)\|_2^2 dt + \|\Delta v^m\|_2^2 + \int_0^T \|\Delta v_t^m(t)\|_2^2 dt \le c\,. \qquad (2.19)$$

Now, from estimates (2.12), (2.18) and (2.19) and of the Lions-Aubin's compactness Theorem we can pass to the limit in (2.10)-(2.11). The rest of the proof is a matter of routine.

3. Uniform Rate of Exponential Decay

In this section we shall study the asymptotic behavior of the solutions of system (1.1)-(1.6) when the resolvent kernels k_1 and k_2 satisfy (2.6). Our point of departure will be to establish some inequalities for the strong solution of coupled system (1.1)-(1.6).

Lemma 3.1. *Any strong solution (u,v) of the system (1.1)-(1.6) satisfy*

$$\begin{aligned}
\frac{d}{dt}E(t) \le &-\frac{\tau_1}{2}\int_{\Gamma_1} |u_t|^2 d\Gamma_1 + \frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1 + \frac{\tau_1}{2}k_1'(t)\int_{\Gamma_1} |u|^2 d\Gamma_1 \\
&- \frac{\tau_1}{2}\int_{\Gamma_1} k_1'' \Box u d\Gamma_1 - \frac{\tau_2}{2}\int_{\Gamma_1} |v_t|^2 d\Gamma_1 + \frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_1} |v_0|^2 d\Gamma_1 \\
&+ \frac{\tau_2}{2}k_2'(t)\int_{\Gamma_1} |v|^2 d\Gamma_1 - \frac{\tau_2}{2}\int_{\Gamma_1} k_2'' \Box v d\Gamma_1 - \int_\Omega \{|\nabla u_t|^2 + |\nabla v_t|^2\} dx.
\end{aligned}$$

Proof. Multiplying the equation (1.1) by u_t and integrating by parts over Ω we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 dx + \frac{1}{2}M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \int_\Omega F_1(u)dx + \int_\Omega |\nabla u_t|^2 dx \\
&= \int_{\Gamma_1} \{(M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu}\} u_t d\Gamma_1.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |v_t|^2 dx + \frac{1}{2}M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{d}{dt}\int_\Omega |\nabla v|^2 dx + \int_\Omega F_2(v)dx + \int_\Omega |\nabla v_t|^2 dx \\
&= \int_{\Gamma_1} \{(M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu}\} v_t d\Gamma_1.
\end{aligned}$$

Summing the above identities, substituting the boundary terms by (2.1)-(2.2) and using Lemma 2.1 our conclusion follows. □

Let us consider the binary operator

$$(k \diamond \varphi)(t) := \int_0^t k(t-s)(\varphi(t) - \varphi(s))ds.$$

Then applying the Hölder's inequality for $0 \le \mu \le 1$ we have

$$|(k \diamond \varphi)(t)|^2 \le \Big[\int_0^t |k(s)|^{2(1-\mu)}ds\Big](|k|^{2\mu}\Box\varphi)(t). \tag{3.1}$$

Let us introduce the functionals

$$\begin{aligned}
\mathcal{N}(t) &:= \int_\Omega (|u_t|^2 + |v_t|^2 + F_1(u) + F_2(v))dx + \widehat{M}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2), \\
\psi(t) &= \int_\Omega \{m \cdot \nabla u + (\frac{n}{2} - \theta)u\}u_t dx + \int_\Omega \{m \cdot \nabla v + (\frac{n}{2} - \theta)v\}v_t dx,
\end{aligned}$$

where θ is a small positive constant. The following Lemma plays an important role for the construction of the Lyapunov functional.

Lemma 3.2. *For any strong solution of the system (1.1)-(1.6) we get*

$$\begin{aligned}
\frac{d}{dt}\psi(t) \le &-\frac{\theta}{2}\mathcal{N}(t) + c\int_{\Gamma_1}(|u_t|^2 + |k_1(t)u|^2 + |k_1' \diamond u|^2 + |k_1(t)u_0|^2)d\Gamma_1 \\
&+ c\int_{\Gamma_1}(|v_t|^2 + |k_2(t)v|^2 + |k_2' \diamond v|^2 + |k_2(t)v_0|^2)d\Gamma_1 \\
&+ c_\epsilon\int_\Omega(|\nabla u_t|^2 + |\nabla v_t|^2)dx,
\end{aligned}$$

for some positive constants c and ϵ.

Proof. From equation (1.1) we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t\{m \cdot \nabla u + (\frac{n}{2} - \theta)u\}dx \\
&= \int_\Omega u_t m \cdot \nabla u_t dx + (\frac{n}{2} - \theta)\int_\Omega |u_t|^2 dx \\
&\quad + \int_\Omega m \cdot \nabla u(M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\Delta u + \Delta u_t - f_1(u))dx \\
&\quad + (\frac{n}{2} - \theta)\int_\Omega u(M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\Delta u + \Delta u_t - f_1(u))dx.
\end{aligned}$$

Performing a integration by parts and using the Young's inequality, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t\{m\cdot\nabla u+big(\frac{n}{2}-\theta)u\}dx\\
&\le \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|u_t|^2d\Gamma_1-\theta\int_\Omega|u_t|^2dx\\
&\quad+\int_{\Gamma_1}(M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})\Big\{m\cdot\nabla u+(\frac{n}{2}-\theta)u\Big\}\,d\Gamma_1\\
&\quad-(1-\theta)M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_\Omega|\nabla u|^2dx\\
&\quad+\epsilon cM(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_\Omega|\nabla u|^2dx+c_\epsilon\int_\Omega|\nabla u_t|^2dx\\
&\quad-(\frac{n}{2}-\theta)\int_\Omega f_1(u)udx+n\int_\Omega F_1(u)dx\\
&\quad-\frac{1}{2}M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_{\Gamma_1} m\cdot\nu|\nabla u|^2d\Gamma_1,
\end{aligned}$$

where ϵ is a positive constant. Taking into account that f_i is superlinear we conclude that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t\{m\cdot\nabla u+(\frac{n}{2}-\theta)u\}dx\\
&\le \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|u_t|^2d\Gamma_1-\theta\int_\Omega|u_t|^2dx\\
&\quad+\int_{\Gamma_1}(M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})\Big\{m\cdot\nabla u+(\frac{n}{2}-\theta)u\Big\}\,d\Gamma_1\\
&\quad-(1-\theta)M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_\Omega|\nabla u|^2dx\\
&\quad+\epsilon cM(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_\Omega|\nabla u|^2dx+c_\epsilon\int_\Omega|\nabla u_t|^2dx\\
&\quad-(\frac{n}{2}-\theta)(2+\delta)\int_\Omega F_1(u)dx-\frac{1}{2}M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_{\Gamma_1} m\cdot\nu|\nabla u|^2d\Gamma_1.
\end{aligned}$$

Similarly, using equation (1.2) instead of (1.1) we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v_t\{m\cdot\nabla v+(\frac{n}{2}-\theta)v\}dx\\
&\le \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|v_t|^2d\Gamma_1-\theta\int_\Omega|v_t|^2dx\\
&\quad+\int_{\Gamma_1}(M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\frac{\partial v}{\partial\nu}+\frac{\partial v_t}{\partial\nu})\Big\{m\cdot\nabla v+(\frac{n}{2}-\theta)v\Big\}\,d\Gamma_1\\
&\quad-(1-\theta)M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_\Omega|\nabla v|^2dx\\
&\quad+\epsilon cM(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_\Omega|\nabla v|^2dx+c_\epsilon\int_\Omega|\nabla v_t|^2dx\\
&-(\frac{n}{2}-\theta)(2+\delta)\int_\Omega F_2(v)dx-\frac{1}{2}M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_{\Gamma_1} m\cdot\nu|\nabla v|^2d\Gamma_1.
\end{aligned}$$

Summing these two last inequalities we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\psi(t) \le \frac{1}{2}\int_{\Gamma_1} m\cdot\nu(|u_t|^2+|v_t|^2)d\Gamma_1 - \theta\int_\Omega(|u_t|^2+|v_t|^2)dx \\
&+\int_{\Gamma_1}(M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})\Big\{m\cdot\nabla u+(\frac{n}{2}-\theta)u\Big\}d\Gamma_1 \\
&+\int_{\Gamma_1}(M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\frac{\partial v}{\partial\nu}+\frac{\partial v_t}{\partial\nu})\Big\{m\cdot\nabla v+(\frac{n}{2}-\theta)v\Big\}d\Gamma_1 \\
&-(1-\theta)M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx \\
&+\epsilon cM(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_\Omega(|\nabla u|^2dx+|\nabla v|^2)dx \\
&+c_\epsilon\int_\Omega(|\nabla u_t|^2+|\nabla v_t|^2)dx-(\frac{n}{2}-\theta)(2+\delta)\int_\Omega F_1(u)dx \\
&-(\frac{n}{2}-\theta)(2+\delta)\int_\Omega F_2(v)dx-\frac{1}{2}M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_{\Gamma_1} m\cdot\nu|\nabla u|^2d\Gamma_1 \\
&-\frac{1}{2}M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_{\Gamma_1} m\cdot\nu|\nabla v|^2d\Gamma_1.
\end{aligned}
$$

Using Poincaré's inequality and taking θ and ϵ small enough we obtain

$$
\begin{aligned}
\frac{d}{dt}\psi(t) \le &-\theta\mathcal{N}(t)+c_\epsilon\int_\Omega(|\nabla u_t|^2+|\nabla v_t|^2)dx+\frac{1}{2}\int_{\Gamma_1} m\cdot\nu(|u_t|^2+|v_t|^2)d\Gamma_1 \\
&-\frac{1}{2}M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\int_{\Gamma_1} m\cdot\nu(|\nabla u|^2+|\nabla v|^2)d\Gamma_1 \\
&+\int_{\Gamma_1}(M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})\Big\{m\cdot\nabla u+(\frac{n}{2}-\theta)u\Big\}d\Gamma_1 \\
&+\int_{\Gamma_1}(M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\frac{\partial v}{\partial\nu}+\frac{\partial v_t}{\partial\nu})\Big\{m\cdot\nabla v+(\frac{n}{2}-\theta)v\Big\}d\Gamma_1.
\end{aligned}
\tag{3.2}
$$

Now, we analyze some boundary term of the above inequality. Applying Young and Poincaré's inequalities we have, for $\epsilon_1 > 0$

$$
\begin{aligned}
&\int_{\Gamma_1}(M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})\big\{m\cdot\nabla u+(\frac{n}{2}-\theta)u\big\}d\Gamma_1 \\
&\le \epsilon_1\int_{\Gamma_1}\big\{|m\cdot\nabla u|^2+\big(\frac{n}{2}-\theta\big)^2|u|^2\big\}d\Gamma_1 \\
&\quad + c_{\epsilon_1}\int_{\Gamma_1}|(M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})|^2d\Gamma_1 \\
&\le \epsilon_1 c\Big\{\int_{\Gamma_1} m\cdot\nu|\nabla u|^2d\Gamma_1+\mathcal{N}(t)\Big\} \\
&\quad + c_{\epsilon_1}\int_{\Gamma_1}|(M(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})|^2d\Gamma_1.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\int_{\Gamma_1} (M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu}) \Big\{ m \cdot \nabla v + (\frac{n}{2} - \theta) v \Big\} \, d\Gamma_1 \\
&\le \epsilon_1 \int_{\Gamma_1} \big\{ |m \cdot \nabla v|^2 + \big(\frac{n}{2} - \theta\big)^2 |v|^2 \big\} d\Gamma_1 \\
&\quad + c_{\epsilon_1} \int_{\Gamma_1} |(M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu})|^2 d\Gamma_1 \\
&\le \epsilon_1 c \Big\{ \int_{\Gamma_1} m \cdot \nu |\nabla v|^2 d\Gamma_1 + \mathcal{N}(t) \Big\} \\
&\quad + c_{\epsilon_1} \int_{\Gamma_1} |(M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu})|^2 d\Gamma_1.
\end{aligned}
$$

Substituting the two inequalities above into (3.2), choosing ϵ_1 small snough and taking into account that the boundary conditions (2.1)-(2.2) can be written as

$$
\begin{aligned}
M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu}) &= -\tau_1 \{ u_t + k_1(t) u - k_1' \diamond u - k_1(t) u_0 \}, \\
M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu}) &= -\tau_2 \{ v_t + k_2(t) v - k_2' \diamond v - k_2(t) v_0 \},
\end{aligned}
$$

our conclusion follows. □

To show that the energy decay exponentially we need of the following Lemma whose proof can be found in [16].

Lemma 3.3. *Let f be a real positive function of class C^1. If there exists positive constants γ_0, γ_1 and c_0 such that*

$$
f'(t) \le -\gamma_0 f(t) + c_0 e^{-\gamma_1 t},
$$

then there exist positive constants γ and c such that $f(t) \le (f(0) + c) e^{-\gamma t}$.

Next, we shall show inequality (2.8). We shall prove this result for strong solutions, that is, for solutions with initial data $(u_0, v_0) \in \big(H^2(\Omega) \cap V\big)^2$ and $(u_1, v_1) \in \big(H^2(\Omega) \cap V\big)^2$ satisfying the compatibility conditions (2.4)- (2.5). Our conclusion follow by standard density arguments. Using hypothesis (2.6) in Lemma 3.1 we get

$$
\begin{aligned}
\frac{d}{dt} E(t) \le &-\frac{\tau_1}{2} \int_{\Gamma_1} \big(|u_t|^2 - b_2 k_1' \Box u + b_1 k_1(t) |u|^2 - |k_1(t) u_0|^2 \big) d\Gamma \\
&- \frac{\tau_2}{2} \int_{\Gamma_1} \big(|v_t|^2 - b_2 k_2' \Box v + b_1 k_2(t) |v|^2 - |k_2(t) v_0|^2 \big) d\Gamma \\
&- \int_\Omega (|\nabla u_t|^2 + |\nabla v_t|^2) dx.
\end{aligned}
$$

On the other hand applying inequality (3.1) with $\mu = 1/2$ in Lemma 3.2 we obtain

$$\begin{aligned}\frac{d}{dt}\psi(t) \le -\frac{\theta}{2}\mathcal{N}(t) + C\int_{\Gamma_1}\big(|u_t|^2 + k_1(t)|u|^2 - k_1'\Box u + |k_1(t)u_0|^2\big)d\Gamma\\ + C\int_{\Gamma_1}\big(|v_t|^2 + k_2(t)|v|^2 - k_2'\Box v + |k_2(t)v_0|^2\big)d\Gamma\\ + c_\epsilon\int_\Omega(|\nabla u_t|^2 + |\nabla v_t|^2)dx.\end{aligned}$$

Let us introduce the Lyapunov functional

$$\mathcal{L}(t) := NE(t) + \psi(t), \tag{3.3}$$

with $N > 0$. Taking N large, the previous inequalities imply that

$$\frac{d}{dt}\mathcal{L}(t) \le -\frac{\theta}{2}E(t) + 2NR^2(t)E(0),$$

where $R(t) = k_1(t) + k_2(t)$. Moreover, using Young's inequality and taking N large we find that

$$\frac{N}{2}E(t) \le \mathcal{L}(t) \le 2NE(t). \tag{3.4}$$

From this inequality we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \le -\frac{\theta}{2}\mathcal{L}(t) + 2NR^2(t)E(0),$$

from where follows, in view of Lemma 3.3 and of the exponential decay of k_1, k_2, that

$$\mathcal{L}(t) \le \{\mathcal{L}(0) + c\}e^{-\gamma_1 t},$$

for some positive constants c, γ. From the inequality (3.4) our conclusion follows.

4. Uniform Rate of Polynomial Decay

Our attention will be focused on the uniform rate of decay when the resolvent kernels k_1 and k_2 satisfy (2.7). First of all we will prove the following three lemmas that will be used in the sequel.

Lemma 4.1. *Let (u, v) be a solution of system (1.1)-(1.6) and let us denote by $(\phi_1, \phi_2) = (u, v)$. Then, for $p > 1$, $0 < r < 1$ and $t \ge 0$, we have*

$$\begin{aligned}&\Big(\int_{\Gamma_1}|k_i'|\Box\phi_i d\Gamma_1\Big)^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}}\\ &\le 2\Big(\int_0^t|k_i'(s)|^r ds\|\phi_i\|^2_{L^\infty(0,t;L^2(\Gamma_1))}\Big)^{\frac{1}{(1-r)(p+1)}}\int_{\Gamma_1}|k_i'|^{1+\frac{1}{p+1}}\Box\phi_i d\Gamma_1\end{aligned}$$

while for $r = 0$ we get

$$\begin{aligned}&\Big(\int_{\Gamma_1}|k_i'|\Box\phi_i d\Gamma_1\Big)^{\frac{p+2}{p+1}}\\ &\le 2\Big(\int_0^t\|\phi_i(s,.)\|^2_{L^2(\Gamma_1)}ds + t\|\phi_i(s,.)\|^2_{L^2(\Gamma_1)}\Big)^{p+1}\int_{\Gamma_1}|k_i'|^{1+\frac{1}{p+1}}\Box\phi_i d\Gamma_1,\end{aligned}$$

for $i = 1, 2$.

For the proof of this lemma see e. g. [15].

Lemma 4.2. *Let $f \geq 0$ be a differentiable function satisfying*

$$f'(t) \leq -\frac{c_1}{f(0)^{\frac{1}{\alpha}}} f(t)^{1+\frac{1}{\alpha}} + \frac{c_2}{(1+t)^\beta} f(0) \quad \text{for} \quad t \geq 0,$$

for some positive constants c_1, c_2, α and β such that $\beta \geq \alpha + 1$. Then there exists a constant $c > 0$ such that

$$f(t) \leq \frac{c}{(1+t)^\alpha} f(0) \quad \text{for} \quad t \geq 0.$$

For the proof of this lemma see e. g. [16].

Next we show inequality (2.9). We shall prove this result for strong solutions, that is, for solutions with initial data $(u_0, v_0) \in \big(H^2(\Omega) \cap V\big)^2$ and $(u_1, v_1) \in \big(H^2(\Omega) \cap V\big)^2$ satisfying the compatibility conditions (2.4)- (2.5). Our conclusion will follow by standard density arguments. We use some estimates of the previous section which are independent of the behavior of the resolvent kernels k_1, k_2. Using hypothesis (2.7) in Lemma 3.1 yields

$$\begin{aligned}\frac{d}{dt}E(t) \leq & -\frac{\tau_1}{2}\int_{\Gamma_1}\Big(|u_t|^2 + b_2[-k_1']^{1+\frac{1}{p+1}}\Box u + b_1 k_1^{1+\frac{1}{p}}(t)|u|^2 - |k_1(t)u_0|^2\Big)d\Gamma_1 \\ & -\frac{\tau_1}{2}\int_{\Gamma_1}\big(|v_t|^2 + b_2[-k_2']^{1+\frac{1}{p+1}}\Box v + b_1 k_2^{1+\frac{1}{p}}(t)|v|^2 - |k_2(t)v_0|^2\big)d\Gamma_1.\end{aligned}$$

Applying inequality (3.1) with $\mu = \frac{p+2}{2(p+1)}$ and using hypothesis (2.7) we obtain the estimates

$$|k_1' \diamond u|^2 \leq c[-k_1']^{1+\frac{1}{p+1}}\Box u, \quad |k_2' \diamond v|^2 \leq c[-k_2']^{1+\frac{1}{p+1}}\Box v.$$

The above inequalities in Lemma 3.2 yields

$$\begin{aligned}\frac{d}{dt}\psi(t) \leq & -\frac{\theta}{2}\mathcal{N}(t) + c\int_{\Gamma_1}\big(|u_t|^2 + k_1^{1+\frac{1}{p}}(t)|u|^2 + [-k_1']^{1+\frac{1}{p+1}}\Box u + |k_1(t)u_0|^2\big)d\Gamma_1 \\ & + c\int_{\Gamma_1}\big(|v_t|^2 + k_2^{1+\frac{1}{p}}(t)|v|^2 + [-k_2']^{1+\frac{1}{p+1}}\Box v + |k_2(t)v_0|^2\big)d\Gamma_1.\end{aligned}$$

In this conditions, taking N large the Lyapunov functional defined in (3.3) satisfies

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq & -\frac{\theta}{2}\mathcal{N}(t) + 2NR^2(t)E(0) \\ & - \frac{Nc_2}{2}\Big\{\int_{\Gamma_1}[-k_1']^{1+\frac{1}{p+1}}\Box u d\Gamma_1 + \int_{\Gamma_1}[-k_2']^{1+\frac{1}{p+1}}\Box v d\Gamma_1\Big\}.\end{aligned}$$

Let us fix $0 < r < 1$ such that $\frac{1}{p+1} < r < \frac{p}{p+1}$. From (2.7) we have that

$$\int_0^\infty |k_i'|^r \leq c\int_0^\infty \frac{1}{(1+t)^{r(p+1)}} < \infty \quad \text{for} \quad i = 1, 2.$$

Using this estimate in Lemma 4.1 we get

$$\int_{\Gamma_1}[-k_1']^{1+\frac{1}{p+1}}\Box u d\Gamma_1 \geq cE(0)^{-\frac{1}{(1-r)(p+1)}}\Big(\int_{\Gamma_1}[-k_1']\Box u d\Gamma_1\Big)^{1+\frac{1}{(1-r)(p+1)}}, \tag{4.1}$$

$$\int_{\Gamma_1}[-k_2']^{1+\frac{1}{p+1}}\Box v\, d\Gamma_1 \geq cE(0)^{-\frac{1}{(1-r)(p+1)}}\Big(\int_{\Gamma_1}[-k_2']\Box v d\Gamma_1\Big)^{1+\frac{1}{(1-r)(p+1)}}. \tag{4.2}$$

On the other hand, from the Trace theorem we have

$$E(t)^{1+\frac{1}{(1-r)(p+1)}} \leq cE(0)^{\frac{1}{(1-r)(p+1)}}\mathcal{N}(t). \tag{4.3}$$

Substitution of (4.1)-(4.3) into (4.1) we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\le -cE(0)^{-\frac{1}{(1-r)(p+1)}}E(t)^{1+\frac{1}{(1-r)(p+1)}} + 2NR^2(t)E(0)\\ &\quad - cE(0)^{-\frac{1}{(1-r)(p+1)}}\Big\{\Big(\int_{\Gamma_1}[-k_1']\Box u d\Gamma_1\Big)^{1+\frac{1}{(1-r)(p+1)}}\\ &\quad + \Big(\int_{\Gamma_1}[-k_2']\Box v d\Gamma_1\Big)^{1+\frac{1}{(1-r)(p+1)}}\Big\}.\end{aligned}$$

Taking into account the inequality (3.4) we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \le -\frac{c}{\mathcal{L}(0)^{\frac{1}{(1-r)(p+1)}}}\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + 2NR^2(t)E(0),$$

for some $c > 0$, from where follows, applying Lemma 4.2, that

$$\mathcal{L}(t) \le \frac{c}{(1+t)^{(1-r)(p+1)}}\mathcal{L}(0).$$

Since $(1-r)(p+1) > 1$ we get, for $t \ge 0$, the following bounds

$$t\|u\|^2_{L^2(\Gamma_1)} + t\|v\|^2_{L^2(\Gamma_1)} \le t\mathcal{L}(t) < \infty,$$

$$\int_0^t \big(\|u\|^2_{L^2(\Gamma_1)} + \|v\|^2_{L^2(\Gamma_1)}\big)\,ds \le c\int_0^t \mathcal{L}(t)\,ds < \infty.$$

Using the above estimates in Lemma 4.1 with $r = 0$ we get

$$\int_{\Gamma_1}[-k_1']^{1+\frac{1}{p+1}}\Box u d\Gamma_1 \ge \frac{c}{E(0)^{\frac{1}{p+1}}}\Big(\int_{\Gamma_1}[-k_1']\Box u d\Gamma\Big)^{1+\frac{1}{p+1}},$$

$$\int_{\Gamma_1}[-k_2']^{1+\frac{1}{p+1}}\Box v d\Gamma_1 \ge \frac{c}{E(0)^{\frac{1}{p+1}}}\Big(\int_{\Gamma_1}[-k_2']\Box v d\Gamma\Big)^{1+\frac{1}{p+1}}.$$

Using these inequalities instead of (4.1)-(4.2) and reasoning in the same way as above we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \le -\frac{c}{\mathcal{L}(0)^{\frac{1}{p+1}}}\mathcal{L}(t)^{1+\frac{1}{p+1}} + 2NR^2(t)E(0).$$

Applying Lemma 4.2 again, we obtain

$$\mathcal{L}(t) \le \frac{c}{(1+t)^{p+1}}\mathcal{L}(0).$$

Finally, from (3.4) we conclude

$$E(t) \le \frac{c}{(1+t)^{p+1}}E(0),$$

which completes the present proof.

Remark. We would like to mention that for 3 or more variables:u,v,..., the same procedure can be used to obtain similar conclusions.

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