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HOPF-TYPE ESTIMATES FOR SOLUTIONS TO HAMILTON-JACOBI EQUATIONS WITH CONCAVE-CONVEX INITIAL DATA

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Abstract. We consider the Cauchy problem for the Hamilton-Jacobi equations with concave-convex initial data. A Hopf-type formula for global Lipschitz solutions and estimates for viscosity solutions of this problem are obtained using techniques of multifunctions and convex analysis.

1. Introduction

This paper is a continuation of the works [10] and [8], where the explicit solutions via Hopf-type formulas of the Cauchy problem to the Hamilton-Jacobi equations with concave-convex hamiltonians were considered. Namely, we consider the Cauchy problem for the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + H(t, \frac{\partial u}{\partial x}) = 0 \quad \text{in } U := \{t > 0,\ x \in \mathbb{R}^n\} \tag{1.1}$$

$$u(0, x) = \phi(x) \quad \text{on } \{t = 0,\ x \in \mathbb{R}^n\}. \tag{1.2}$$

Here $\partial/\partial x = (\partial/\partial x_1, \dots, \partial/\partial x_n)$, the Hamiltonian $H = H(t, p)$ and $\phi = \phi(x)$ are given functions, and $u = u(t, x)$ is unknown.

In this paper we shall assume that $n = n_1 + n_2$ and that the variable $x \in \mathbb{R}^n$ is separated as $x = (x', x'')$ with $x' \in \mathbb{R}^{n1}$, $x'' \in \mathbb{R}^{n2}$, similarly for $p, q, \dots \in \mathbb{R}^n$. In particular, the zero-vector in $\mathbb{R}^n$ will be $0 = (0', 0'')$, where $0'$ and $0''$ stand for the zero-vectors in $\mathbb{R}^{n1}$ and $\mathbb{R}^{n2}$, respectively.

Definition. A function $g = g(x', x'')$ is called concave-convex if it is concave in $x' \in \mathbb{R}^{n1}$ for each $x'' \in \mathbb{R}^{n2}$ and convex in $x'' \in \mathbb{R}^{n2}$ for each $x' \in \mathbb{R}^{n1}$.
For results on the concave-convex functions the reader is referred to [7], [8], [10].

In [10, Chapter 10], Van, Tsuji and Thai Son proposed to examine a class of concave-convex functions in a more general framework where the discussion of the global Legendre transformation still make sense.
Bardi and Faggian [2] found explicit pointwise upper and lower bounds of Hopf-type for the viscosity solutions under the following hypotheses: H depends only on

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p and is a concave-convex function given by the difference of convex functions,

$$H(p', p'') := H_1(p') - H_2(p''),$$

and ϕ is uniformly continuous. Also if $H \in C(\mathbb{R}^n)$ and $\phi = \phi(x)$ is concave-convex function given by special representation $\phi(x) = \phi_1(x) - \phi_2(x)$, where ϕ_1, ϕ_2 are convex and Lipschitz continuous.

Barron, Jensen and Liu [3] and Van, Thanh [11] found Hopf-type estimates for viscosity solutions to the corresponding Cauchy problem when the Hamiltonian $H(\gamma, p)$, $(\gamma, p) \in \mathbb{R} \times \mathbb{R}^n$, is a D. C. function in p, i.e.,

$$H(\gamma, p) = H_1(\gamma, p) - H_2(\gamma, p), \quad (\gamma, p) \in \mathbb{R} \times \mathbb{R}^n,$$

where $H_i(\gamma, p)$, $i = 1, 2$, is a convex function in p.

Ngoan [6], Thai Son [8], Van, Tsuji and Thai Son [10] obtained explicit global Lipschitz solutions and upper and lower bounds of viscosity solutions to the Hamilton-Jacobi equations with concave-convex hamiltonians via Hopf-type formulas.

The aim of this paper is to look for explicit global Lipschitz solution of the Cauchy problem (1.1)–(1.2) and to establish pointwise upper and lower bounds of Hopf-type for viscosity solutions when the initial function $\phi = \phi(x) = \phi(x', x'')$ is concave-convex on $\mathbb{R}^{n1} \times \mathbb{R}^{n2}$.

Definition. A function $u = u(t, x)$ in $\mathrm{Lip}(\bar{U})$ will be called a global Lipschitz solution of the Cauchy problem (1.1)–(1.2) if it satisfies (1.1) almost everywhere (a. e.) in U, with $u(0, x) = \phi(x)$ for all $x \in \mathbb{R}^n$.

2. Hopf-type formula for global Lipschitz solutions

We consider the Cauchy problem for the Hamilton-Jacobi equation

$$u_t + H(t, Du) = 0 \quad \text{in } U := \{t > 0,\ x \in \mathbb{R}^n\} \tag{2.1}$$

$$u(0, x) = \phi(x) \quad \text{on } \{t = 0,\ x \in \mathbb{R}^n\}, \tag{2.2}$$

where the Hamiltonian H depends on the variable t and the spatial derivatives Du.

We note that Van, Tsuji, Hoang and Thai Son [9], [10] have obtained a Hopf-type formula with the initial function $\phi = \phi(x)$ nonconvex and H merely continuous. Moreover, a global Lipschitz solution of (2.1)–(2.2) is given by an explicit Hopf-type formula in the following case (see Chap. 9, [10]): The Hamiltonian (depends explicitly on t) $H = H(t, p)$ is continuous in $U_G := \{(t, p) : t \in (0, +\infty) \backslash G,\ p \in \mathbb{R}^n\}$ where G is closed subset of $\mathbb{R}$ with Lebesgue measure zero; and, for each $N \in (0, +\infty)$ corresponds a function $g_N := g_N(t) \in L^{\infty}_{\mathrm{loc}}(\mathbb{R})$ so that

$$\sup_{|p| \le N} |H(t, p)| \le g_N(t) \quad \text{for almost } t \in (0, +\infty);$$

while the initial function $\phi = \phi(x)$ satisfies one of the following two conditions:

(1) $\phi = \phi_1 - \phi_2$, where ϕ_1, ϕ_2 are convex functions;
(2) ϕ is minimum of a family of convex functions.

In this section, we look for explicit global Lipschitz solutions of problem (2.1)–(2.2), where $x \in \mathbb{R}^n$, $n = n_1 + n_2$, $x = (x', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and the initial-valued function $\phi = \phi(x) := \phi(x', x'')$ is a strictly concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$

satisfying the following conditions:

$$\lim_{|x''|\to+\infty}\frac{\phi(x',x'')}{|x''|}=+\infty \text{ for each } x'\in\mathbb{R}^{n_1}, \tag{2.3}$$

$$\lim_{|x'|\to+\infty}\frac{\phi(x',x'')}{|x'|}=-\infty \text{ for each } x''\in\mathbb{R}^{n_2}. \tag{2.4}$$

We now consider the Cauchy problem (2.1)–(2.2) with the following hypotheses:

(M1) The Hamiltonian $H=H(t,p)$ is continuous in

$$U_G:=\{(t,p):t\in(0,+\infty)\backslash G, p\in\mathbb{R}^n\}$$

with G be a closed subset of $\mathbb{R}$ with Lebesgue measure 0. Moreover, for each $N\in(0,+\infty)$ there corresponds a function $g_N:=g_N(t)\in L^\infty_{\rm loc}(\mathbb{R})$ so that

$$\sup_{|p|\le N}|H(t,p)|\le g_N(t)\quad\text{for almost } t\in(0,+\infty);$$

(M2) The equality

$$\sup_{p''\in\mathbb{R}^{n_2}}\inf_{p'\in\mathbb{R}^{n_1}}\varphi(t,x,p)=\inf_{p'\in\mathbb{R}^{n_1}}\sup_{p''\in\mathbb{R}^{n_2}}\varphi(t,x,p)$$

is satisfied in U, where

$$\varphi(t,x,p):=\langle p,x\rangle-\phi^*(p)-\int_0^t H(\tau,p)d\tau \tag{2.5}$$

for $(t,x)=(t,x',x'')\in U$, $p=(p',p'')\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. Here, ϕ^* denotes the conjugate of ϕ which is defined as in Section 3 later.

(M3) To each bounded subset V of U there corresponds a positive number $N(V)$ so that

$$\max_{\substack{|q''|\le N(V)\\ q''\in\mathbb{R}^{n_2}}}\inf_{q'\in\mathbb{R}^{n_1}}\varphi(t,x,q',q'')>\inf_{q'\in\mathbb{R}^{n_1}}\varphi(t,x,q',p''),$$

$$\min_{\substack{|q'|\le N(V)\\ q'\in\mathbb{R}^{n_1}}}\sup_{q''\in\mathbb{R}^{n_2}}\varphi(t,x,q',q'')<\sup_{q''\in\mathbb{R}^{n_2}}\varphi(t,x,p',q''),$$

whenever $(t,x)\in V$, $p=(p',p'')\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ and $\min\{|p'|,|p''|\}>N(V)$.

The main result of this Section is as follows.

Theorem 2.1. *Let ϕ be a strictly concave-convex function on $\mathbb{R}^n$ with* (2.3)–(2.4) *and assume M1–M3. Then the formula*

$$u(t,x):=\sup_{p''\in\mathbb{R}^{n_2}}\inf_{p'\in\mathbb{R}^{n_1}}\varphi(t,x,p)=\inf_{p'\in\mathbb{R}^{n_1}}\sup_{p''\in\mathbb{R}^{n_2}}\varphi(t,x,p), \tag{2.6}$$

for $(t,x)\in U$, determines a global Lipschitz solution of the Cauchy problem (2.1)–(2.2).

To prove this theorem, we need the following lemmas, which are similar to the lemmas 10.5 and 10.6 in [10].

Lemma 2.2. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^m$, and $\eta=\eta(\xi,p)=\eta(\xi,p',p'')$ be a continuous function on $\mathcal{O}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ with the following properties:*

(1) *The equality*

$$\sup_{p''\in\mathbb{R}^{n_2}}\inf_{p'\in\mathbb{R}^{n_1}}\eta(\xi,p)=\inf_{p'\in\mathbb{R}^{n_1}}\sup_{p''\in\mathbb{R}^{n_2}}\eta(\xi,p)$$

is satisfied in $\mathcal{O}$;

(2) *There is a nonempty subset $E\subset\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ such that $\eta(\xi,p)$ is finite on $\mathcal{O}\times E$ and $\eta(\xi,p)\equiv-\infty$ on $\mathcal{O}\times E^c$, where $E^c=\mathbb{R}^n\setminus E$. Moreover, for each bounded subset V of $\mathcal{O}$, corresponds a positive number $N(V)$ such that*

$$\max_{\substack{|q''|\le N(V)\\ q''\in\mathbb{R}^{n_2}}}\inf_{q'\in\mathbb{R}^{n_1}}\eta(\xi,q',q'')>\inf_{q'\in\mathbb{R}^{n_1}}\eta(\xi,q',p''),$$

and

$$\min_{\substack{|q'|\le N(V)\\ q'\in\mathbb{R}^{n_1}}}\sup_{q''\in\mathbb{R}^{n_2}}\eta(\xi,q',q'')<\sup_{q''\in\mathbb{R}^{n_2}}\eta(\xi,p',q''),$$

whenever $\xi\in V$, $p=(p',p'')\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ and $\min\{|p'|,\,|p''|\}>N(V)$;

(3) *For each fixed p of E, $\eta=\eta(\xi,p)$ is differentiable in $\xi\in\mathcal{O}$ with continuous gradient*

$$\partial\eta/\partial\xi=\ \partial\eta(\xi,p)/\partial\xi$$

on $\mathcal{O}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$.

Then we have:

i. *The function*

$$\psi=\psi(\xi):=\sup_{p''\in\mathbb{R}^{n_2}}\inf_{p'\in\mathbb{R}^{n_1}}\eta(\xi,p)=\inf_{p'\in\mathbb{R}^{n_1}}\sup_{p''\in\mathbb{R}^{n_2}}\eta(\xi,p)$$

is a locally Lipschitz continuous on $\mathcal{O}$.

ii. *$\psi=\psi(\xi)$ is directionally differentiable in $\mathcal{O}$ with*

$$\begin{aligned}\partial_e\psi(\xi)&=\max_{p''\in L''(\xi)}\min_{p'\in L'(\xi)}\langle\partial\eta(\xi,p',p'')/\partial\xi,e\rangle\\&=\min_{p'\in L'(\xi)}\max_{p''\in L''(\xi)}\langle\partial\eta(\xi,p',p'')/\partial\xi,e\rangle,\quad \xi\in\mathcal{O},\ e\in\mathbb{R}^m\end{aligned}$$

where

$$L'(\xi):=\{p'\in\mathbb{R}^{n_1}:\sup_{p''\in\mathbb{R}^{n_2}}\eta(\xi,p',p'')=\psi(\xi)\}\tag{2.7}$$

$$L''(\xi):=\{p''\in\mathbb{R}^{n_2}:\inf_{p'\in\mathbb{R}^{n_1}}\eta(\xi,p',p'')=\psi(\xi)\}.\tag{2.8}$$

Lemma 2.3. *Suppose that the conditions 1–2 in Lemma 2.2 are satisfied for a continuous function $\eta=\eta(\xi,p',p'')$ on $\mathcal{O}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. Then (2.7)–(2.8) determines the non-empty valued, closed, locally bounded multifunction $L=L(\xi):=L'(\xi)\times L''(\xi)$, $\xi\in\mathcal{O}$.*

Proof of Theorem 2.1. . We can verify that the function

$$\eta=\eta(\xi,p):=\varphi(t,x,p)$$

satisfies all the assumptions of Lemma 2.2, where

$$E:=\operatorname{dom}\phi^*\neq\emptyset,\quad m:=1+n=1+n_1+n_2,\quad \xi:=(t,x).$$

Here we put $\mathcal{O}:=\bar{U}$ and conclude that

$$L(t,x)=L'(t,x)\times L''(t,x)=\{p\in E:\varphi(t,x,p)=u(t,x)\}$$

determines a nonempty-valued, locally bounded, closed multifunction $L = L(t,x)$ of $(t,x) \in \bar{U}$. Take arbitrary an $r \in (0,+\infty)$ and denote

$$V_r = \{(t,x) \in \bar{U} : t + |x| < r\}, \quad N_r = N(V_r).$$

Let $g_{N_r} = g_{N_r}(t)$ as be in the condition M1. Then for any two points (t^1,x^1) and (t^2,x^2) are in V_r, we may choose an element $p = (p^{'1}, p^{''2}) \in L'(t^1,x^1) \times L''(t^2,x^2)$ of the nonempty set

$$L'(t^1,x^1) \times L''(t^2,x^2) \subset \bar{B}^{n_1}(0', N_r) \times \bar{B}^{n_2}(0'', N_r)$$

and get

$$\begin{aligned} u(t^2,x^2) - u(t^1,x^1) &= \inf_{p' \in \mathbb{R}^{n_1}} \varphi(t^2,x^2,p',p^{''2}) - \sup_{p'' \in \mathbb{R}^{n_2}} \varphi(t^1,x^1,p^{'1},p'') \\ &\le \varphi(t^2,x^2,p^{'1},p^{''2}) - \varphi(t^1,x^1,p^{'1},p^{''2}) \\ &= \varphi(t^2,x^2,p) - \varphi(t^1,x^1,p) \\ &= \langle p, x^2 - x^1 \rangle + \int_{t_2}^{t_1} H(\tau,p)d\tau \\ &\le N_r \cdot |x^2 - x^1| + s_r \cdot |t^2 - t^1| \end{aligned}$$

where $s_r = \operatorname{ess\,sup}_{t \in (0,r)} g_{N_r}(t)$. Dually,

$$u(t^1,x^1) - u(t^2,x^2) \le N_r \cdot |x^2 - x^1| + s_r \cdot |t^2 - t^1|.$$

Hence, $u = u(t,x)$ is a locally Lipschitz continuous in $\bar{U}$ and thus it be long to $Lip(\bar{U})$. Next, let $e^o := (1,0,0,\dots,0,0)$, $e^1 := (0,1,0,\dots,0,0)$, $\dots$, $e^n := (0,0,0,\dots,0,1) \in \mathbb{R}^{n+1}$. We now replace in Lemma 2.2 the set $\mathcal{O} := U_G$. From this lemma we see that $u = u(t,x)$ is directionally differentiable in U_G with

$$\begin{aligned} \partial_{e^o} u(t,x) &= \max_{p'' \in L''(t,x)} \min_{p' \in L'(t,x)} \{-H(t,p),\, p \in L(t,x)\} \\ &= \min_{p' \in L'(t,x)} \max_{p'' \in L''(t,x)} \{-H(t,p),\, p \in L(t,x)\}, \\ \partial_{-e^o} u(t,x) &= \max_{p'' \in L''(t,x)} \min_{p' \in L'(t,x)} \{H(t,p),\, p \in L(t,x)\} \\ &= \min_{p' \in L'(t,x)} \max_{p'' \in L''(t,x)} \{H(t,p),\, p \in L(t,x)\}; \end{aligned}$$

and for $1 \le i \le n$:

$$\begin{aligned} \partial_{e^i} u(t,x) &= \max_{p'' \in L''(t,x)} \min_{p' \in L'(t,x)} \{p_i,\; p \in L(t,x)\} \\ &= \min_{p' \in L'(t,x)} \max_{p'' \in L''(t,x)} \{p_i,\; p \in L(t,x)\}, \\ \partial_{-e^i} u(t,x) &= \max_{p'' \in L''(t,x)} \min_{p' \in L'(t,x)} \{-p_i,\; p \in L(t,x)\} \\ &= \min_{p' \in L'(t,x)} \max_{p'' \in L''(t,x)} \{-p_i,\; p \in L(t,x)\}. \end{aligned} \tag{2.9}$$

Since $u = u(t,x)$ is locally Lipschitz continuous in $\bar{U}$, according to Rademacher's Theorem, there exists a set $\mathcal{Q} \subset U$ of ($(n+1)$ dimensional) Lebesgue measure 0 such that $u = u(t,x)$ is differentiable with

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \partial_{e^o} u(t,x) = -\partial_{-e^o} u(t,x), \\ \frac{\partial u(t,x)}{\partial x_i} &= \partial_{e^i} u(t,x) = -\partial_{-e^i} u(t,x) \end{aligned} \tag{2.10}$$

at any point $(t,x) \in U \setminus \mathcal{Q}$. Hence, (2.9)–(2.10) show that the equalities for $1 \le i \le n$,

$$\begin{aligned}\frac{\partial u(t,x)}{\partial x_i} &= \max_{p''\in L''(t,x)} \min_{p'\in L'(t,x)} \{p_i,\ p \in L(t,x)\}\\ &= \min_{p'\in L'(t,x)} \max_{p''\in L''(t,x)} \{p_i,\ p \in L(t,x)\}\\ &= \min_{p''\in L''(t,x)} \max_{p'\in L'(t,x)} \{p_i,\ p \in L(t,x)\}\\ &= \max_{p'\in L'(t,x)} \min_{p''\in L''(t,x)} \{p_i,\ p \in L(t,x)\}\end{aligned}$$

hold for all $(t,x) \in U \setminus \{\mathcal{P} := (G \times \mathbb{R}^n) \cup \mathcal{Q}\} =: U_{\mathcal{P}}$, this implies

$$L(t,x) = \big\{\frac{\partial u(t,x)}{\partial x}\big\}, \quad (t,x) \in U_{\mathcal{P}};$$

and we obtain

$$\frac{\partial u(t,x)}{\partial t} = \{-H(t,p),\ p \in L(t,x)\}.$$

Thus,

$$\frac{\partial u(t,x)}{\partial t} + H(t, \frac{\partial u(t,x)}{\partial x}) = -H(t, \frac{\partial u(t,x)}{\partial x}) + H(t, \frac{\partial u(t,x)}{\partial x}) = 0$$

hold almost everywhere in U. Furthermore

$$\begin{aligned}u(0,x) &= u(0,x',x'')\\ &= \sup_{p''\in\mathbb{R}^{n_2}} \inf_{p'\in\mathbb{R}^{n_1}} \{\langle p',x'\rangle + \langle p'',x''\rangle - \phi^*(p',p'')\}\\ &= \inf_{p'\in\mathbb{R}^{n_1}} \sup_{p''\in\mathbb{R}^{n_2}} \{\langle p',x'\rangle + \langle p'',x''\rangle - \phi^*(p',p'')\}\\ &= \big(\phi^*(p',p'')\big)^* = \phi(x',x'') = \phi(x)\end{aligned}$$

for all $x = (x',x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. From what has already been proved, we conclude that $u = u(t,x)$ is a global Lipschitz solution of the Cauchy problem (2.1)–(2.2). □

Remark 2.4. If $n_2 = 0$, we obtain the Hopf-type formulas of the Cauchy problem for the convex initial data as in Chapter 8 [10].

Remark 2.5. Assume (M1), (M2). Then (M3) is satisfied if

$$\inf_{p'\in\mathbb{R}^{n_1}} \varphi(t,x,p',p'') \to -\infty \quad \text{locally uniformly in } (t,x) \in \bar{U} \text{ as } |p''| \to +\infty$$

and

$$\sup_{p''\in\mathbb{R}^{n_2}} \varphi(t,x,p',p'') \to +\infty \quad \text{locally uniformly in } (t,x) \in \bar{U} \text{ as } |p'| \to +\infty$$

i.e, if the following statement holds:
For any λ and $\mu \in \mathbb{R}$ and any bounded subset V of $\bar{U}$, there exists positive numbers $N(\lambda,V)$ and $N(\mu,V)$, respectively, so that

$$\inf_{q'\in\mathbb{R}^{n_1}} \varphi(t,x,q',p'') < \lambda \quad \text{whenever } (t,x) \in V,\ |p''| > \ N(\lambda,V)$$

and

$$\sup_{q''\in\mathbb{R}^{n_2}} \varphi(t,x,p',q'') > \mu \quad \text{whenever } (t,x) \in V,\ |p'| > \ N(\mu,V).$$

Indeed, fix an arbitrary $q^0 = (q^{0'}, q^{0''})$ in the domain of ϕ^*, which is not empty. Since the finite function $\bar{U} \ni (t,x) \mapsto \varphi(t,x,q^0)$ is continuous, it follows that: for any bounded subset V of $\bar{U}$,

$$\lambda_V := \inf_{(t,x)\in V} \varphi(t,x,q^0) > -\infty,$$

$$\mu_V := \sup_{(t,x)\in V} \varphi(t,x,q^0) < +\infty.$$

Under the hypothesis above, we certainly find a number $N(\lambda, V) \geq |q^{0''}|$ (for each such V) so that

$$\inf_{q'\in\mathbb{R}^{n_1}} \varphi(t,x,q',p'') < \lambda_V = \inf_{(t,x)\in V} \varphi(t,x,q^{0'},q^{0''})$$

when $(t,x) \in V$ and $|p''| > N(\lambda, V)$,

$$\inf_{q'\in\mathbb{R}^{n_1}} \varphi(t,x,q',p'') < \varphi(t,x,q^{0'},q^{0''})$$

when $(t,x) \in V$, $|p''| > N(\lambda, V)$,

$$\inf_{q'\in\mathbb{R}^{n_1}} \varphi(t,x,q',p'') < \inf_{q'\in\mathbb{R}^{n_1}} \varphi(t,x,q',q^{0''})$$

when $(t,x) \in V$, $|p''| > N(\lambda, V)$,

$$\inf_{q'\in\mathbb{R}^{n_1}} \varphi(t,x,q',p'') < \max_{\substack{|q''|\leq N(\lambda,V)\\ q''\in\mathbb{R}^{n_2}}} \inf_{q'\in\mathbb{R}^{n_1}} \varphi(t,x,q',q'')$$

when $(t,x) \in V$, $|p''| > N(\lambda, V)$.

Analogously, we also obtain

$$\sup_{q''\in\mathbb{R}^{n_2}} \varphi(t,x,p',q'') > \min_{\substack{|q'|\leq N(\mu,V)\\ q'\in\mathbb{R}^{n_1}}} \sup_{q''\in\mathbb{R}^{n_2}} \varphi(t,x,q',q'')$$

when $(t,x) \in V$, $|p'| > N(\mu, V)$, where $N(\mu, V) \geq |q^{0'}|$. Hence (M3) is satisfied.

3. Hopf-type estimates for viscosity solutions

Consider the Cauchy problem for the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + H\Big(\frac{\partial u}{\partial x}\Big) = 0 \quad \text{in } U := \{t > 0,\, x \in \mathbb{R}^n\} \tag{3.1}$$

$$u(0,x) = \phi(x) \quad \text{on } \{t = 0,\, x \in \mathbb{R}^n\}. \tag{3.2}$$

When $H = H(p)$ is continuous and $\phi = \phi(x)$ is uniformly continuous, the Cauchy problem (3.1)–(3.2) has a unique viscosity solution $u = u(t,x)$ which is in the space of continuous functions that are uniformly continuous in x uniformly in t, $UC_x([0,+\infty) \times \mathbb{R}^n)$ (see [5]). We also refer the readers to [4,5] for the definition and properties of viscosity solutions.

In the case of Lipschitz continuous and convex (or concave) initial data ϕ and merely continuous Hamiltonian H, or for convex ϕ and Lipschitz continuous H, the formula

$$u(t,x) = \sup_{p\in\mathbb{R}^n} \{\langle p, x\rangle - \phi^*(p) - tH(p)\}$$

determines a (unique) viscosity solution $u = u(t,x) \in UC_x([0,+\infty) \times \mathbb{R}^n)$ of the problem (3.1)–(3.2). Here ϕ^* denotes the Legendre transform of ϕ (see, [1,2]).

In this section we are interested in giving explicit pointwise upper and lower bounds for viscosity solutions where the initial function $\phi = \phi(x', x'')$ is concave-convex. First, we rewrite some main results on the conjugate of the concave-convex functions (for the details, see [10, Chapter 10]). Let $\phi = \phi(x', x'')$ is a concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then

$$\phi^{*1}(p', x'') = \inf_{x' \in \mathbb{R}^{n_1}} \{\langle x', p'\rangle - \phi(x', x'')\}$$
$$\big(\text{resp. } \phi^{*2}(x', p'') = \sup_{x'' \in \mathbb{R}^{n_2}} \{\langle x'', p''\rangle - \phi(x', x'')\}\big)$$

is the Fenchel conjugate of x'-concave (resp. x''-convex) function $\phi(x', x'')$.

If $\phi = \phi(x', x'')$ is concave-convex function with conditions (2.3)–(2.4), then $\phi^{*1}(p', x'')$ (resp. $\phi^{*2}(x', p'')$) is concave (resp. convex) not only in $p' \in \mathbb{R}^{n_1}$ (resp. $p'' \in \mathbb{R}^{n_2}$) but also in the whole variable $(p', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ (resp. $(x', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$) and

$$\lim_{|p'| \to +\infty} \frac{\phi^{*1}(p', x'')}{|p'|} = -\infty \quad (\text{resp. } \lim_{|p''| \to +\infty} \frac{\phi^{*2}(x', p'')}{|p''|} = +\infty)$$

locally uniformly in $x'' \in \mathbb{R}^{n_2}$ (resp. $x' \in \mathbb{R}^{n_2}$). Besides the Fenchel "partial conjugate" ϕ^{*1} and ϕ^{*2}, we consider two "total conjugate" of ϕ:

$$\begin{aligned}\bar{\phi}^*(p', p'') &= \inf_{x' \in \mathbb{R}^{n_1}} \{\langle x', p'\rangle + \phi^{*2}(x', p'')\} \\ &= \inf_{x', \in \mathbb{R}^{n_1}} \sup_{x'' \in \mathbb{R}^{n_2}} \{\langle x', p'\rangle + \langle x'', p''\rangle - \phi(x', x'')\}\end{aligned}$$

and

$$\begin{aligned}\underline{\phi}^*(p', p'') &= \sup_{x'' \in \mathbb{R}^{n_2}} \{\langle x'', p''\rangle + \phi^{*1}(p', x'')\} \\ &= \sup_{x'' \in \mathbb{R}^{n_2}} \inf_{x' \in \mathbb{R}^{n_1}} \{\langle x', p'\rangle + \langle x'', p''\rangle - \phi(x', x'')\}.\end{aligned}$$

Therefore, the functions $\bar{\phi}^*$ and $\underline{\phi}^*$ are usually called the upper and lower conjugate, respectively, of ϕ. Note that

$$\underline{\phi}^* \le \bar{\phi}^*.$$

These functions are also concave-convex, and with (2.3)–(2.4) they coincide. In this situation, the Fenchel conjugate

$$\phi^* := \bar{\phi}^* = \underline{\phi}^*$$

of ϕ will simultaneously have the properties

$$\lim_{|p''| \to +\infty} \frac{\phi^*(p', p'')}{|p''|} = +\infty \quad \text{for each } p' \in \mathbb{R}^{n_1}$$
$$\lim_{|p'| \to +\infty} \frac{\phi^*(p', p'')}{|p'|} = -\infty \quad \text{for each } p'' \in \mathbb{R}^{n_2}.$$

If (2.3)–(2.4) are not assumed, the partial conjugates ϕ^{*1} and ϕ^{*2} are still concave and convex, respectively, but might be infinite somewhere, then the lower and upper conjugates $\underline{\phi}^*$ and $\bar{\phi}^*$ might not coincide. One can claim only that

$$\phi^{*1}(p', x'') < +\infty, \quad \forall (p', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},$$
$$\phi^{*2}(x', p'') > -\infty, \quad \forall (x', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

Now let

$$\begin{aligned} D_1 &:= \{p' \in \mathbb{R}^{n_1} : \phi^{*1}(p', x'') > -\infty \ \forall x'' \in \mathbb{R}^{n_2}\}, \\ D_2 &:= \{p'' \in \mathbb{R}^{n_2} : \phi^{*2}(x', p'') < +\infty \ \forall x' \in \mathbb{R}^{n_1}\}, \end{aligned}$$

hence for all $x'' \in \mathbb{R}^{n_2}$, $\phi^{*1}(p', x'')$ is finite on D_1, and for all $x' \in \mathbb{R}^{n_1}$, $\phi^{*2}(x', p'')$ is finite on D_2.

We now consider the Cauchy problem (3.1)–(3.2) with the hypothesis:

(M4) The Hamiltonian $H = H(p)$ is continuous and the initial function $\phi = \phi(x', x'')$ is concave-convex and Lipschitz continuous (without (2.3)–(2.4)).

For $(t, x) \in U$, we set

$$u_-(t,x) := \sup_{p'' \in D_2} \inf_{p' \in \mathbb{R}^{n_1}} \{\langle p, x\rangle - \bar{\phi}^*(p) - tH(p)\} \tag{3.3}$$

$$u_+(t,x) := \inf_{p' \in D_1} \sup_{p'' \in \mathbb{R}^{n_2}} \{\langle p, x\rangle - \underline{\phi}^*(p) - tH(p)\}. \tag{3.4}$$

Remark 3.1. *The concave-convex function $\phi = \phi(x', x'')$ is Lipschitz continuous in the sense: $\phi(x', x'')$ is Lipschitz continuous in $x' \in \mathbb{R}^{n_1}$ for each $x'' \in \mathbb{R}^{n_2}$ and in $x'' \in \mathbb{R}^{n_2}$ for each $x' \in \mathbb{R}^{n_1}$.*

Our estimates for viscosity solutions in this section read as follows:

Theorem 3.2. *Assume (M4). Then the unique viscosity solution $u = u(t,x) \in UC_x([0,+\infty)\times\mathbb{R}^n)$ of the Cauchy problem* (3.1)–(3.2) *satisfies on $\bar{U}$ the inequalities*

$$u_-(t,x) \le \ u(t,x) \le \ u_+(t,x),$$

where u_- and u_+ are defined by (3.3) *and* (3.4) *respectively.*

Proof. For each $\underline{p}' \in D_1$, let

$$\begin{aligned} \Phi(x; \underline{p}') = \Phi(x', x''; \underline{p}') &:= \langle x', \underline{p}'\rangle - \phi^{*1}(\underline{p}', x'') \\ &= \langle x', \underline{p}'\rangle - \inf_{x' \in \mathbb{R}^{n_1}} \big\{\langle x', \underline{p}'\rangle - \phi(x', x'')\big\} \\ &\ge \phi(x', x'') \quad \text{for all } (x', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \end{aligned}$$

Since $\phi^{*1}(\underline{p}', .)$ is a concave and finite, so $-\phi^{*1}(\underline{p}', .)$ is convex and finite, it is convex and Lipschitz continuous function; therefore , $\Phi(x; \underline{p}')$ is convex and Lipschitz continuous with its Fenchel conjugate given by

$$\begin{aligned} \Phi^*(p; \underline{p}') = \Phi^*(p', p''; \underline{p}') &= \sup_{x \in \mathbb{R}^n} \big\{\langle x, p\rangle - \Phi(x, \underline{p}')\big\} \\ &= \sup_{x \in \mathbb{R}^n} \big\{\langle x', p'\rangle + \langle x'', p''\rangle - \langle x', \underline{p}'\rangle + \phi^{*1}(\underline{p}', x'')\big\} \\ &= \begin{cases} +\infty & \text{if } (p', p'') \ne (\underline{p}', p'') \\ \underline{\phi}^*(\underline{p}', p'') & \text{if } (p', p'') = (\underline{p}', p''). \end{cases} \end{aligned}$$

Next, consider the Cauchy problem

$$\begin{gathered} \frac{\partial v}{\partial t} + H(\frac{\partial v}{\partial x}) = 0 \quad \text{in } U = \{t > 0,\, x \in \mathbb{R}^n\}, \\ v(0, x) = \Phi(x; \underline{p}') \quad \text{on } \{t = 0,\, x \in \mathbb{R}^n\}. \end{gathered}$$

This is the Cauchy problem with the continuous Hamiltonian $H = H(p)$ and the convex and Lipschitz continuous initial function $\Phi = \Phi(x; \underline{p}')$ for each $\underline{p}' \in D_1$, its unique viscosity solution $v = v(t,x) \in UC_x\big([0,+\infty) \times \mathbb{R}^n\big)$ is given by

$$\begin{aligned} v(t,x) &= \sup_{p\in\mathbb{R}^n} \{\langle p, x\rangle - \Phi^*(p; \underline{p}') - tH(p)\} \\ &= \sup_{p''\in\mathbb{R}^{n_2}} \{\langle \underline{p}', x'\rangle + \langle p'', x''\rangle - \underline{\phi}^*(\underline{p}', p'') - tH(\underline{p}', p'')\} \end{aligned}$$

with the initial condition

$$v(0,x) = \Phi(x; \underline{p}') \geq \phi(x) = u(0,x)$$

for each $\underline{p}' \in D_1$ (see [1]). Hence, for each $\underline{p}' \in D_1$, $v = v(t,x)$ is a (continuous) supersolution of the problem (3.1)–(3.2) (according to a standard comparison theorem for unbounded viscosity solutions (see [5])), that means

$$u(t,x) \leq v(t,x) \quad \text{for each } \underline{p}' \in D_1,$$

and then

$$\begin{gathered} u(t,x) \leq \inf_{p'\in D_1} \sup_{p''\in\mathbb{R}^{n_2}} \{\langle p, x\rangle - \underline{\phi}^*(p) - tH(p)\} \\ u(t,x) \leq u_+(t,x) \quad \text{on } \bar{U}. \end{gathered}$$

Dually, we also abtain $u(t,x) \geq u_-(t,x)$ on $\bar{U}$. Therefore, Theorem 3.2 has been proved. □

Corollary 3.3. *Assume (M1), (M2) for the case when $H(t,p)$ is not depending on t. Moreover, assume that $\phi = \phi(x', x'')$ is concave-convex and Lipschitz continuous function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and satisfies the conditions* (2.3)–(2.4). *Then* (2.6) *determines the unique viscosity solution $u(t,x) \in UC_x\big([0,+\infty) \times \mathbb{R}^n\big)$ of the Cauchy problem* (3.1)–(3.2).

Proof. Since $\phi = \phi(x', x'')$ is a concave-convex and Lipschitz continuous function so domϕ^* is a bounded and nonempty set. Independently of $(t,x) \in \bar{U}$, it follows that

$$\begin{aligned} \varphi(t,x,p',p'') &\to -\infty \quad \text{whenever } |p''| \text{ is large enough} \\ \varphi(t,x,p',p'') &\to +\infty \quad \text{whenever } |p'| \text{ is large enough.} \end{aligned}$$

From Remark 2.5 implies that hypothesis (M3) hold. Then the conclusion folows from Theorem 3.2. □

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