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RADIAL MINIMIZER OF A VARIANT OF THE P-GINZBURG-LANDAU FUNCTIONAL

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Abstract. We study the asymptotic behavior of the radial minimizer of a variant of the p-Ginzburg-Landau functional when $p \geq n$. The location of the zeros and the uniqueness of the radial minimizer are derived. We also prove the $W^{1,p}$ convergence of the radial minimizer for this functional.

1. Introduction

Let $n \geq 2$, $B = \{x \in R^n; |x| < 1\}$. Consider the minimizers of the variant for the p-Ginzburg-Landau-type functional

$$E_\varepsilon(u, B) = \frac{1}{p}\int_B |\nabla u|^p + \frac{1}{4\varepsilon^p}\int_B |u|^2(1-|u|^2)^2, \quad (p \geq n)$$

on the class functions

$$W = \{u(x) = f(r)\frac{x}{|x|} \in W^{1,p}(B, R^n); f(1) = 1, r = |x|\}.$$

By the direct method in the calculus of variations we see that the minimizer u_ε exists. It will be called the *radial minimizer*.

When $p = n = 2$, the asymptotic behavior of the minimizer u_ε of $E_\varepsilon(u, B)$ in the class H^1_g were studied in [5]. In this paper, we will study the asymptotic behavior of the radial minimizer u_ε. We will prove the following theorems.

Theorem 1.1. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$. Then for any $\eta \in (0, 1/2)$, there exists a constant $h = h(\eta)$ independent of $\varepsilon \in (0, 1)$ such that $Z_\varepsilon = \{x \in B; |u_\varepsilon(x)| < 1 - \eta\} \subset B(0, h\varepsilon)$. For any given $\varepsilon \in (0, \varepsilon_0)$, the radial minimizers u_ε of $E_\varepsilon(u, B)$ are unique on W.*

Theorem 1.2. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$. Then as $\varepsilon \to 0$,*

$$u_\varepsilon \to \frac{x}{|x|}, \quad \textit{in } W^{1,p}_{\mathrm{loc}}(\overline{B} \setminus \{0\}, R^n).$$

Some basic properties of minimizers are given in §2. The proof of Theorem 1.1 is presented in §3. The proof of Theorem 1.2. is based uniform estimates proved in §4.

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2. Preliminaries

Let

$$V=\left\{f\in W^{1,p}_{\rm loc}(0,1];r^{\frac{n-1}{p}}f_r\in L^p(0,1),r^{(n-1-p)/p}f\in L^p(0,1),f(1)=1\right\}.$$

Then $V=\{f(r);u(x)=f(r)\frac{x}{|x|}\in W\}$. As stated in [6, Proposition 2.1], we have

Proposition 2.1. *The set V defined above is a subset of $\{f\in C[0,1];f(0)=0\}$.*

Proposition 2.2. *The minimizer $u_\varepsilon\in W$ is a weak radial solution of*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u)=\frac{1}{\varepsilon^p}u(1-|u|^2)|u|^2-\frac{1}{2\varepsilon^p}u(1-|u|^2)^2,\quad on\quad B, \tag{2.1}$$

Proof. Denote u_ε by u. For any $t\in[0,1)$ and $\phi=f(r)\frac{x}{|x|}\in W^{1,p}_0(B,R^n)$, we have $u+t\phi\in W$ as long as t is small sufficiently. Since u is a minimizer we obtain $\frac{dE_\varepsilon(u+t\phi,B)}{dt}|_{t=0}=0$, namely,

$$0=\int_B|\nabla u|^{p-2}\nabla u\nabla\phi dx-\frac{1}{\varepsilon^p}\int_B u\phi(1-|u|^2)|u|^2dx+\frac{1}{2\varepsilon^p}\int_B u\phi(1-|u|^2)^2dx. \tag{2.2}$$

□

Proposition 2.3. *Let $u_\varepsilon\in W$ satisfying (2.2). Then $|u_\varepsilon|\le 1$ a.e. on $\overline{B}$.*

Proof. Let $u=u_\varepsilon$ in (2.2) and set $\phi=u(|u|^2-1)_+$, where for a positive constant k, $(|u|^2-1)_+=\min(k,\max(0,|u|^2-1))$. Then

$$\begin{aligned}&\int_B|\nabla u|^p(|u|^2-1)_+ +2\int_B|\nabla u|^{p-2}(u\nabla u)^2\\&+\frac{1}{\varepsilon^p}\int_B|u|^4(|u|^2-1)_+^2+\frac{1}{2\varepsilon^p}\int_B|u|^2(|u|^2-1)_+(|u|^2-1)^2=0\end{aligned}$$

from which it follows that

$$\frac{1}{\varepsilon^p}\int_B|u|^4(|u|^2-1)_+^2=0.$$

Thus $|u|=0$ or $(|u|^2-1)_+=0$ a.e. on B. Using proposition 2.1 we know that $|u|=|u_\varepsilon|\le 1$ a.e. on B. □

By the same argument as in [6, Proposition 2.5], we obtain the following statement.

Proposition 2.4. *Assume u_ε is a weak radial solution of (2.1). Then there exist positive constants C_1,ρ which are both independent of ε such that*

$$\|\nabla u_\varepsilon(x)\|_{L(B(x,\rho\varepsilon/8))}\le C_1\varepsilon^{-1},\quad if\quad x\in B(0,1-\rho\varepsilon), \tag{2.3}$$

$$|u_\varepsilon(x)|\ge\frac{29}{30},\quad if\quad x\in\overline{B}\setminus B(0,1-2\rho\varepsilon). \tag{2.4}$$

Proposition 2.5. *Let u_ε be a radial minimizer of $E_\varepsilon(u,B)$. Then there exists a constant C independent of $\varepsilon\in(0,1)$ such that*

$$E_\varepsilon(u_\varepsilon,B)\le C\varepsilon^{n-p}+C;\quad for\ p>n, \tag{2.5}$$

$$E_\varepsilon(u_\varepsilon,B)\le C|\ln\varepsilon|+C,\quad for\ p=n. \tag{2.6}$$

Proof. Let

$$I(\varepsilon,R)=\min\Big\{\int_{B(0,R)}[\frac{1}{p}|\nabla u|^p+\frac{1}{\varepsilon^p}(1-|u|^2)^2];u\in W_R\Big\},$$

where $W_R=\{u(x)=f(r)\frac{x}{|x|}\in W^{1,p}(B(0,R),R^n);r=|x|,f(R)=1\}$. Then

$$\begin{aligned}I(\varepsilon,1)&=E_\varepsilon(u_\varepsilon,B)\\&=\frac{1}{p}\int_B|\nabla u_\varepsilon|^pdx+\frac{1}{4\varepsilon^p}\int_B(1-|u_\varepsilon|^2)^2|u_\varepsilon|^2dx\\&=\varepsilon^{n-p}[\frac{1}{p}\int_{B(0,\varepsilon^{-1})}|\nabla u_\varepsilon|^pdy+\frac{1}{4}\int_{B(0,\varepsilon^{-1})}(1-|u_\varepsilon|^2)^2|u_\varepsilon|^2dy\\&=\varepsilon^{n-p}I(1,\varepsilon^{-1}).\end{aligned}\tag{2.7}$$

Let u_1 be a solution of $I(1,1)$ and define

$$u_2=\begin{cases}u_1, & \text{if } 0<|x|<1\\ \frac{x}{|x|}, & \text{if } 1\le|x|\le\varepsilon^{-1}.\end{cases}$$

Thus $u_2\in W_{\varepsilon^{-1}}$, and

$$\begin{aligned}I(1,\varepsilon^{-1})&\le\frac{1}{p}\int_{B(0,\varepsilon^{-1})}|\nabla u_2|^p+\frac{1}{4}\int_{B(0,\varepsilon^{-1})}(1-|u_2|^2)^2|u_2|^2\\&=\frac{1}{p}\int_B|\nabla u_1|^p+\frac{1}{4}\int_B(1-|u_1|^2)^2|u_1|^2+\frac{1}{p}\int_{B(0,\varepsilon^{-1})\setminus B}|\nabla\frac{x}{|x|}|^p\\&=I(1,1)+\frac{(n-1)^{p/2}|S^{n-1}|}{p}\int_1^{\varepsilon^{-1}}r^{n-p-1}dr\end{aligned}$$

Hence

$$I(1,\varepsilon^{-1})\le I(1,1)+\frac{(n-1)^{p/2}|S^{n-1}|}{p(p-n)}(1-\varepsilon^{p-n})\le C,\quad\text{for } p>n;$$

$$I(1,\varepsilon^{-1})\le I(1,1)+\frac{(n-1)^{p/2}|S^{n-1}|}{p}|\ln\varepsilon|,\quad\text{for}\quad p=n.$$

Substituting this into (2.7) yields (2.5) and (2.6). □

3. Proof of Theorem 1.1

Proposition 3.1. *Let u_ε be a radial minimizer of $E_\varepsilon(u,B)$. Then there exists a positive constant ε_0 such that as $\varepsilon\in(0,\varepsilon_0)$,*

$$\frac{1}{\varepsilon^n}\int_B|u_\varepsilon|^2(1-|u_\varepsilon|^2)^2\le C,\tag{3.1}$$

where C is independent of ε.

Proof. When $p>n$, the conclusion follows from multiplying (2.5) by ε^{p-2}. When $p=n$, the proof is similar to the proof in [7, Theorem 1]. Thus we can obtain this proposition by using (2.6). □

Proposition 3.2. *Let u_ε be a radial minimizer of $E_\varepsilon(u,B)$. Assume $p>n$. Then for any $\eta\in(0,1/2)$, there exist positive constants λ,μ independent of $\varepsilon\in(0,1)$ such that if*

$$\frac{1}{\varepsilon^p}\int_{B\cap B^{2l\varepsilon}}|u_\varepsilon|^2(1-|u_\varepsilon|^2)^2\le\mu,\tag{3.2}$$

where $B^{2l\varepsilon}$ is some ball of radius $2l\varepsilon$ with $l \geq \lambda$, then

$$|u_\varepsilon(x)| \in [0, 1-\eta] \cup [1-\eta/2, 1], \quad \forall x \in B \cap B^{l\varepsilon}.$$

Proof. First we observe that there exists a constant $\beta > 0$ such that for any $x \in B$ and $0 < \rho \leq 1$, $|B \cap B(x,\rho)| \geq \beta\rho^2$.

From Proposition 2.3 and (2.5) it follows that $\|u_\varepsilon\|_{W^{1,p}(B)} \leq C\varepsilon^{\frac{2-p}{2}}$. By embedding theorem we know that there exists a positive constant C_0 which is independent of ε, such that for any $x, x_0 \in B$,

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C_0\varepsilon^{\frac{2-p}{p}}|x - x_0|^{1-\frac{2}{p}}.$$

To obtain the conclusion, we choose

$$\lambda = \frac{\eta}{4C_0}, \quad \mu = \frac{\beta}{16}\eta^2(1-\eta)^2\lambda^n. \tag{3.3}$$

Suppose that there is a point $x_0 \in B \cap B^{l\varepsilon}$ such that $1-\eta < |u_\varepsilon(x_0)| < 1-\eta/2$. Then

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C_0\varepsilon^{\frac{2-p}{p}}|x - x_0|^{1-\frac{2}{p}} \leq C_0\lambda = \frac{\eta}{4}, \quad \forall x \in B(x_0, \lambda\varepsilon)$$

Hence $(1 - |u_\varepsilon(x)|^2)^2 > (\frac{\eta}{4})^2$, for all $x \in B(x_0, \lambda\varepsilon)$, and

$$\int_{B(x_0,\lambda\varepsilon)\cap B} |u_\varepsilon|^2(1-|u_\varepsilon|^2)^2 > \frac{\eta^2}{16}(1-\eta)^2|B\cap B(x_0,\lambda\varepsilon)| \geq \beta\frac{\eta^2}{16}(1-\eta)^2(\lambda\varepsilon)^n = \mu\varepsilon^n \tag{3.4}$$

Since $x_0 \in B^{l\varepsilon} \cap B$, and $(B(x_0, \lambda\varepsilon) \cap B) \subset (B^{2l\varepsilon} \cap B)$, (3.4) implies

$$\int_{B^{2l\varepsilon}\cap B} |u_\varepsilon|^2(1 - |u_\varepsilon|^2)^2 > \mu\varepsilon^n$$

which contradicts (3.2) and thus proposition 3.2 is proved. □

Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$, $p > n$. Given $\eta \in (0, 1/2)$. Let λ, μ be constants in Proposition 3.2 corresponding to η. If

$$\frac{1}{\varepsilon^n}\int_{B(x^\varepsilon,2\lambda\varepsilon)\cap B} |u_\varepsilon|^2(1 - |u_\varepsilon|^2)^2 \leq \mu \tag{3.5}$$

then $B(x^\varepsilon, \lambda\varepsilon)$ is called good ball. Otherwise $B(x^\varepsilon, \lambda\varepsilon)$ is called bad ball.

Now suppose that $\{B(x_i^\varepsilon, \lambda\varepsilon), i \in I\}$ is a family of balls satisfying

$$\begin{aligned} &(i): \quad x_i^\varepsilon \in B, i \in I; \\ &(ii): \quad B \subset \cup_{i\in I} B(x_i^\varepsilon, \lambda\varepsilon) \\ &(iii): \quad B(x_i^\varepsilon, \lambda\varepsilon/4) \cap B(x_j^\varepsilon, \lambda\varepsilon/4) = \emptyset, i \neq j \end{aligned} \tag{3.6}$$

Denote $J_\varepsilon = \{i \in I; B(x_i^\varepsilon, \lambda\varepsilon) \text{ is a bad ball}\}$.

Proposition 3.3. *Assume $p > n$, there exists a positive integer N independent of $\varepsilon \in (0, 1)$, such that the number of bad balls satisfies* $\operatorname{Card} J_\varepsilon \leq N$.

Proof. Since (3.6) implies that every point in B can be covered by finite, say m (independent of ε) balls, from Proposition 3.1 and the definition of bad balls, we have

$$\begin{aligned}\mu\varepsilon^n Card J_\varepsilon &\le \sum_{i\in J_\varepsilon}\int_{B(x_i^\varepsilon,2\lambda\varepsilon)\cap B}|u_\varepsilon|^2(1-|u_\varepsilon|^2)^2\\ &\le m\int_{\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,2\lambda\varepsilon)\cap B}|u_\varepsilon|^2(1-|u_\varepsilon|^2)^2\\ &\le m\int_B|u_\varepsilon|^2(1-|u_\varepsilon|^2)^2\le mC\varepsilon^n\end{aligned}$$

and hence Card $J_\varepsilon \le \frac{mC}{\mu} \le N$. □

Similar to the argument in [1, Theorem IV.1], we have the following statement.

Proposition 3.4. *Assume $p > n$, there exist a subset $J \subset J_\varepsilon$ and a constant $h \in [\lambda, \lambda 9^N]$ such that*

$$\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,\lambda\varepsilon)\subset\cup_{i\in J}B(x_j^\varepsilon,h\varepsilon),\quad |x_i^\varepsilon-x_j^\varepsilon|>8h\varepsilon,\quad i,j\in J,\quad i\ne j. \tag{3.7}$$

Applying proposition 3.4, we may modify the family of bad balls such that the new one, denoted by $\{B(x_i^\varepsilon, h\varepsilon); i \in J\}$, satisfies

$$\begin{gathered}\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,\lambda\varepsilon)\subset\cup_{i\in J}B(x_i^\varepsilon,h\varepsilon),\\ \lambda\le h;\quad \text{Card}\, J\le \text{Card}\, J_\varepsilon\\ |x_i^\varepsilon-x_j^\varepsilon|>8h\varepsilon, i,j\in J, i\ne j\,.\end{gathered}$$

The last condition implies that every two balls in the new family are not intersected. Now we prove our main result of this section.

Theorem 3.5. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$. Assume $p \ge n$. Then for any $\eta \in (0, 1/2)$, there exists a constant $h = h(\eta)$ independent of $\varepsilon \in (0, 1)$ such that $Z_\varepsilon = \{x \in B; |u_\varepsilon(x)| < 1 - \eta\} \subset B(0, h\varepsilon)$. In particular the zeroes of u_ε are contained in $B(0, h\varepsilon)$.*

Proof. When $p > n$. Denote $Y_\varepsilon = \{x \in B; 1 - \eta \le |u_\varepsilon(x)| \le 1 - \eta/2\}$. Suppose there exists a point $x_0 \in Y_\varepsilon$ such that $x_0 \overline{\in} B(0, h\varepsilon)$. Then all points on the circle $S_0 = \{x \in B;\ |x| = |x_0|\}$ satisfy $|u_\varepsilon(x)| < 1 - \eta$ and hence by virtue of Proposition 3.3 all points on S_0 are contained in bad balls. However, since $|x_0| \ge h\varepsilon, S_0$ can not be covered by a single bad ball. S_0 can be covered by at least two bad balls. However this is impossible. This means $Y_\varepsilon \subset B(0, h\varepsilon)$.

Furthermore, for any given y_0 satisfying $|u_\varepsilon(y_0)| = f(r_0) < 1-\eta$, where $|y_0| = r_0$, we claim $y_0 \in B(0, h\varepsilon)$. In fact, From $f(r_0) < 1 - \eta$, $f(1) = 1 > 1 - \eta/2$, and the continuity of f, it follows that there exists $\xi \in (r_0, 1)$ such that $1 - \eta < f(\xi) < 1 - \eta/2$, so $\xi \in Y_\varepsilon \subset (0, h\varepsilon)$ which implies $r_0 \in (0, h\varepsilon)$.

When $p = n$, The space $W^{1,n}(B)$ does not embed into $C^\alpha(\overline{B})$. Hence in the proof of Proposition 3.2 we can not derive the similar conclusion in $\overline{B}$ globally. Now, by virtue of Proposition 2.4, we may do argument *on* $B(0, 1 - \rho\varepsilon)$ instead of *on* B in the proof of Proposition 3.2 by using (2.3) and it is also true that we may take

$$\frac{1}{\varepsilon^n}\int_{B(x^\varepsilon,2\lambda\varepsilon)\cap B(0,1-\rho\varepsilon)}|u_\varepsilon|^2(1-|u_\varepsilon|^2)^2\le\mu$$

as a ruler to distinguish the bad balls in $B(0, 1 - \rho\varepsilon)$. Similarly, we also obtain that the set $\{x \in B(0, 1 - \rho\varepsilon); 1 - \eta \le |u_\varepsilon(x)| \le 1 - \eta/2\}$ must be covered by finite disintersected bad balls for any $\eta \in (0, 1/2)$. Moreover, it follows that the set

$\{x \in B(0, 1-\rho\varepsilon); |u_\varepsilon(x)| \le 1-\eta\} \subset B(0, h\varepsilon)$ by the same argument above. Noting (2.4), we can see that the theorem holds. □

By Proposition 2.4, Proposition 3.2 and Theorem 3.5 we can see that

$$|u_\varepsilon(x)| \ge \min(\frac{29}{30}, 1-2\eta), \quad \forall x \in \overline{B} \setminus B(0, h\varepsilon). \tag{3.8}$$

Theorem 3.6. *For any given $\varepsilon \in (0, \varepsilon_0)$, the radial minimizers u_ε of $E_\varepsilon(u, B)$ are unique on W.*

Proof. Fix $\varepsilon \in (0, 1)$. Suppose $u_1(x) = f_1(r)\frac{x}{|x|}$ and $u_2(x) = f_2(r)\frac{x}{|x|}$ are both radial minimizers of $E_\varepsilon(u, B)$ on W, then they are both weak radial solutions of (2.1). Namely, they satisfy

$$\int_B |\nabla u|^{p-2}\nabla u \nabla\phi + \frac{1}{2\varepsilon^p}\int_B [(1+3|u|^4) - 4|u|^2]\phi = 0$$

Taking $\phi = u_1 - u_2 = (f_1 - f_2)\frac{x}{|x|}$, we have

$$\begin{aligned} &\int_B (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla(u_1 - u_2)dx \\ &+ \frac{1}{2\varepsilon^p}\int_B (f_1 - f_2)^2[1 + 3(f_1^4 + f_1^3 f_2 + f_1^2 f_2^2 + f_1 f_2^3 + f_2^4) \\ &- 4(f_1^2 + f_2^2 + f_1 f_2)]dx = 0 \end{aligned}$$

Letting η in (3.8) be sufficiently small such that

$$1 \ge f_1, \quad f_2 \ge \frac{29}{30}, \quad \text{on } B \setminus B(0, h(\eta)\varepsilon)$$

for any given $\varepsilon \in (0, 1)$. Hence

$$\int_B (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla(u_1 - u_2)dx \le \frac{C}{\varepsilon^p}\int_{B(0,h\varepsilon)} (f_1 - f_2)^2 dx.$$

Applying (2.11) of [8], we can see that there exists a positive constant γ independent of ε and h such that

$$\gamma \int_B |\nabla(u_1 - u_2)|^2 dx \le \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)} (f_1 - f_2)^2 dx, \tag{3.9}$$

which implies

$$\int_B |\nabla(f_1 - f_2)|^2 dx \le \frac{1}{\gamma\varepsilon^p}\int_{B(0,h\varepsilon)} (f_1 - f_2)^2 dx. \tag{3.10}$$

When $n > 2$. Applying [4, Theorem 2.1], we have $\|f\|_{\frac{2n}{n-2}} \le \beta\|\nabla f\|_2$, where $\beta = \frac{2(n-1)}{n-2}$. Taking $f = f_1 - f_2$ and applying (3.10), we obtain $f(|x|) = 0$ as $x \in \partial B$ and

$$\Big[\int_B |f|^{\frac{2n}{n-2}} dx\Big]^{\frac{n-2}{n}} \le \beta^2 \int_B |\nabla f|^2 dx \le \beta^2\gamma^{-1}\int_G |f|^2 dx\varepsilon^{-p},$$

where $G = B(0, h\varepsilon)$. Using Holder inequality, we derive

$$\int_G |f|^2 dx \le |G|^{1-\frac{n-2}{n}}[\int_G |f|^{\frac{2n}{n-2}} dx]^{\frac{n-2}{n}} \le |B|^{1-\frac{n-2}{n}} h^2\varepsilon^{2-p}\frac{\beta^2}{\gamma}\int_G |f|^2 dx.$$

Hence for any given $\varepsilon \in (0,1)$,

$$\int_G |f|^2 dx \le C(\beta, |B|, \gamma, \varepsilon) h^2 \int_G |f|^2 dx. \tag{3.11}$$

Denote $F(\eta) = \int_{B(0,h(\eta)\varepsilon)} |f|^2 dx$, then $F(\eta) \ge 0$ and (3.11) implies that

$$F(\eta)(1 - C(\beta, |B|, \gamma, \varepsilon)h^2) \le 0. \tag{3.12}$$

On the other hand, since $C(\beta, |B|, \gamma, \varepsilon)$ is independent of η, we may take η so small that $h = h(\eta) \le \lambda 9^N = 9^N \frac{\eta}{2C_0}$ (which is implied by (3.3)) satisfies

$$0 < 1 - C(\beta, |B|, \gamma, \varepsilon)h^2$$

for the fixed $\varepsilon \in (0,1)$, which and (3.12) imply that $F(\eta) = 0$. Namely $f = 0$ a.e. on G, or

$$f_1 = f_2, \quad a.e. \quad on \quad B(0, h\varepsilon).$$

Substituting this into (3.9), we know that $u_1 - u_2 = C$ a.e. on B. Noticing the continuity of u_1, u_2 which is implied by Proposition 2.1, and $u_1 = u_2 = x$ on ∂B, we can see at last that

$$u_1 = u_2, \quad \text{on } \overline{B}.$$

When $n = 2$, applying [4, Theorem 2.1], we have $\|f\|_6 \le \beta \|\nabla f\|_{2/3}$, where β does not depend on η. By the similar argument above, we may see the same conclusion. □

4. Proof of Theorem 1.2

Let $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u, B_1)$, namely f_ε be a minimizer of $E_\varepsilon(f)$ in V. From Proposition 2.5, we have

$$E_\varepsilon(f_\varepsilon) \le C\varepsilon^{n-p}, \quad \text{for } p > n; \quad E_\varepsilon(f_\varepsilon) \le C|\ln \varepsilon|, \quad \text{for } p = n \tag{4.1}$$

for some constant C independent of $\varepsilon \in (0,1)$. In this section we further prove that for any given $R \in (0,1)$, there exists a constant $C(R)$ such that

$$E_\varepsilon(f_\varepsilon; R) \le C(R) \tag{4.2}$$

for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ sufficiently small, where

$$E_\varepsilon(f; R) = \frac{1}{p}\int_R^1 (f_r^2 + (n-1)r^{-2}f^2)^{p/2} r^{n-1}\, dr + \frac{1}{4\varepsilon^p}\int_R^1 f^2(1-f^2)^2 r^{n-1}\, dr.$$

Proposition 4.1. *Assume $p > n$. Given $T \in (0,1)$. There exist constants $T_j \in [\frac{(j-1)T}{N+1}, \frac{jT}{N+1}]$, $(N = [p])$ and C_j, such that*

$$E_\varepsilon(f_\varepsilon; T_j) \le C_j \varepsilon^{j-p} \tag{4.3}$$

for $j = n, n+1, \dots, N$, where $\varepsilon \in (0, \varepsilon_0)$ with ε_0 sufficiently small.

Proof. For $j = n$, the inequality (4.3) can be obtained by (4.1) easily. Suppose that (4.3) holds for all $j \le m$. Then we have, in particular,

$$E_\varepsilon(f_\varepsilon; T_m) \le C_m \varepsilon^{m-p}. \tag{4.4}$$

If $m = N$ then we are done. Suppose $m < N$, we want to prove (4.3) for $j = m+1$.

From (4.4) and integral mean value theorem, we can see that there exists $T_{m+1} \in [\frac{mT}{N+1}, \frac{(m+1)T}{N+1}]$ such that

$$\frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2|_{r=T_{m+1}} \le \frac{C}{f_\varepsilon^2(T_{m+1})} E_\varepsilon(u_\varepsilon, \partial B(0,T_{m+1})) \le C_m \varepsilon^{m-p} \tag{4.5}$$

It is used that $f_\varepsilon(T_{m+1}) \ge \frac{29}{30}$ by virtue of (3.8) as long as ε_0 and η sufficiently small. Consider the minimizer ρ_1 of the functional

$$E(\rho, T_{m+1}) = \frac{1}{p}\int_{T_{m+1}}^1 (\rho_r^2+1)^{p/2} dr + \frac{1}{2\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho)^2 dr$$

It is easy to prove that the minimizer ρ_ε of $E(\rho, T_{m+1})$ on $W^{1,p}_{f_\varepsilon}((T_{m+1},1), R^+)$ exists and satisfies

$$-\varepsilon^p(v^{(p-2)/2}\rho_r)_r = 1-\rho, \quad in\ (T_{m+1},1), \tag{4.6}$$

$$\rho|_{r=T_{m+1}} = f_\varepsilon,\ \rho|_{r=1} = f_\varepsilon(1) = 1, \tag{4.7}$$

where $v = \rho_r^2 + 1$. Since $f_\varepsilon \le 1$, it follows from the maximum principle

$$\rho_\varepsilon \le 1. \tag{4.8}$$

Applying (4.1) we see easily that

$$E(\rho_\varepsilon; T_{m+1}) \le E(f_\varepsilon; T_{m+1}) \le C E_\varepsilon(f_\varepsilon; T_{m+1}) \le C\varepsilon^{m-p}. \tag{4.9}$$

Now choosing a smooth function $0 \le \zeta(r) \le 1$ in (0,1] such that $\zeta = 1$ on $(0, T_{m+1}), \zeta = 0$ near $r = 1$ and $|\zeta_r| \le C(T_{m+1})$, multiplying (4.6) by $\zeta\rho_r(\rho = \rho_\varepsilon)$ and integrating over $(T_{m+1}, 1)$ we obtain

$$v^{(p-2)/2}\rho_r^2|_{r=T_{m+1}} + \int_{T_{m+1}}^1 v^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})\,dr = \frac{1}{\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho)\zeta\rho_r\,dr. \tag{4.10}$$

Using (4.9) we have

$$\begin{aligned}
&\Big|\int_{T_{m+1}}^1 v^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})\,dr\Big| \\
&\le \int_{T_{m+1}}^1 v^{(p-2)/2}|\zeta_r|\rho_r^2\,dr + \frac{1}{p}\Big|\int_{T_{m+1}}^1 (v^{p/2}\zeta)_r\,dr - \int_{T_{m+1}}^1 v^{p/2}\zeta_r\,dr\Big| \\
&\le C\int_{T_{m+1}}^1 v^{p/2} + \frac{1}{p}v^{p/2}\big|_{r=T_{m+1}} + \frac{C}{p}\int_{T_{m+1}}^1 v^{p/2} dr \\
&\le C\varepsilon^{m-p} + \frac{1}{p}v^{p/2}\big|_{r=T_{m+1}}
\end{aligned} \tag{4.11}$$

and using (4.5),(4.7) and (4.9) we have

$$\begin{aligned}
&\Big|\frac{1}{\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho)\zeta\rho_r\,dr\Big| \\
&= \frac{1}{2\varepsilon^p}\Big|\int_{T_{m+1}}^1 ((1-\rho)^2\zeta)_r\,dr - \int_{T_{m+1}}^1 (1-\rho)^2\zeta_r\,dr\Big| \\
&\le \Big|\frac{1}{2\varepsilon^p}(1-\rho)^2\big|_{r=T_{m+1}} + \frac{C}{2\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho)^2\,dr\Big| \le C\varepsilon^{m-p}.
\end{aligned} \tag{4.12}$$

Combining (4.10) with (4.11), (4.12) yields

$$v^{(p-2)/2}\rho_r^2|_{r=T_{m+1}} \le C\varepsilon^{m-p} + \frac{1}{p}v^{p/2}|_{r=T_{m+1}}.$$

Hence for any $\delta \in (0,1)$,

$$\begin{aligned} v^{p/2}|_{r=T_{m+1}} &= v^{(p-2)/2}(\rho_r^2+1)|_{r=T_{m+1}} \\ &= v^{(p-2)/2}\rho_r^2|_{r=T_{m+1}} + v^{(p-2)/2}|_{r=T_{m+1}} \\ &\le C\varepsilon^{m-p} + \frac{1}{p}v^{p/2}|_{r=T_{m+1}} + v^{(p-2)/2}|_{r=T_{m+1}} \\ &= C\varepsilon^{m-p} + (\frac{1}{p}+\delta)v^{p/2}|_{r=T_{m+1}} + C(\delta) \end{aligned}$$

from which it follows by choosing $\delta > 0$ small enough that

$$v^{p/2}\big|_{r=T_{m+1}} \le C\varepsilon^{m-p}. \tag{4.13}$$

Now we multiply both sides of (4.6) by $\rho - 1$ and integrate. Then

$$-\varepsilon^p\int_{T_{m+1}}^1 [v^{(p-2)/2}\rho_r(\rho-1)]_r\,dr + \varepsilon^p\int_{T_{m+1}}^1 v^{(p-2)/2}\rho_r^2\,dr + \int_{T_{m+1}}^1(\rho-1)^2\,dr = 0.$$

From this, using(4.5), (4.7) and (4.13), we obtain

$$\begin{aligned} E(\rho_\varepsilon;T_{m+1}) &\le C|\int_{T_{m+1}}^1 [v^{(p-2)/2}\rho_r(\rho-1)]_r\,dr| \\ &= Cv^{(p-2)/2}|\rho_r||\rho-1|_{r=T_{m+1}} \le Cv^{(p-1)/2}|\rho-1|_{r=T_{m+1}} \\ &\le (C\varepsilon^{m-p})^{(p-1)/p}(C\varepsilon^m)^{1/2} \le C\varepsilon^{m-p+1}. \end{aligned} \tag{4.14}$$

Define

$$w_\varepsilon = \begin{cases} f_\varepsilon & \text{for } r \in (0,T_{m+1}) \\ \rho_\varepsilon & \text{for } r \in [T_{m+1},1] \end{cases}$$

Since f_ε is a minimizer of $E_\varepsilon(f)$, we have $E_\varepsilon(f_\varepsilon) \le E_\varepsilon(w_\varepsilon)$. Thus, it follows that

$$E_\varepsilon(f_\varepsilon;T_{m+1}) \le \frac{1}{p}\int_{T_{m+1}}^1 (\rho_r^2+(n-1)r^{-2}\rho^2)^{p/2}r^{n-1}\,dr + \frac{1}{4\varepsilon^p}\int_{T_{m+1}}^1 \rho^2(1-\rho^2)^2r^{n-1}\,dr$$

by virtue of $\Gamma \le \varepsilon < T_{m+1}$ since ε is sufficiently small. Noticing that

$$\begin{aligned} &\int_{T_{m+1}}^1(\rho_r^2+(n-1)r^{-2}\rho^2)^{p/2}r^{n-1}dr - \int_{T_{m+1}}^1((n-1)r^2\rho^2)^{p/2}r^{n-1}dr \\ &= \frac{p}{2}\int_{T_{m+1}}^1\int_0^1[\rho_r^2+(n-1)r^{-2}\rho^2)s \\ &\quad + (n-1)r^{-2}\rho^2(1-s)]^{(p-2)/2}ds\rho_r^2r^{n-1}dr \\ &\le C\int_{T_{m+1}}^1(\rho_r^2+(n-1)r^{-2}\rho^2)^{(p-2)/2}\rho_r^2r^{n-1}dr \\ &\quad + C\int_{T_{m+1}}^1((n-1)r^{-2}\rho^2)^{(p-2)/2}\rho_r^2r^{n-1}dr \\ &\le C\int_{T_{m+1}}^1(\rho_r^p+\rho_r^2)dr \end{aligned}$$

and using (4.8) we obtain

$$
\begin{aligned}
&E_\varepsilon(f_\varepsilon; T_{m+1})\\
&\le \frac{1}{p}\int_{T_{m+1}}^1 ((n-1)r^{-2}\rho^2)^{p/2} r^{n-1}\,dr + C\int_{T_{m+1}}^1 (\rho_r^p + \rho_r^2)dr + \frac{C}{4\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho^2)^2 dr\\
&\le \frac{1}{p}\int_{T_{m+1}}^1 ((n-1)r^{-2})^{p/2} r^{n-1}\,dr + CE(\rho_\varepsilon; T_{m+1}).
\end{aligned}
$$

Combining this with (4.14) yields (4.3) for $j = m+1$. It is just (4.3) for $j = m+1$. $\square$

Proposition 4.2. *Assume $p \ge n$. Given $T \in (0,1)$. There exist constants $T_{N+1} \in (0,T]$ and $C > 0$ such that*

$$
\begin{gathered}
E_\varepsilon(u_\varepsilon; T_{N+1}) - (n-1)^{p/2}\frac{|S^{n-1}|}{p}\int_{T_{N+1}}^1 r^{n-p-1}dr \le C\varepsilon^{N+1-p}, (p > n);\\
E_\varepsilon(u_\varepsilon; T_{N+1}) - (n-1)^{p/2}\frac{|S^{n-1}|}{p}\int_{T_{N+1}}^1 r^{n-p-1}dr \le C\varepsilon|\ln\varepsilon|, (p = n),
\end{gathered}
$$

where $N = [p]$.

Proof. From (4.1) and (4.3) we can see $E_\varepsilon(u_\varepsilon; T_N) \le CF(\varepsilon)$, where $F(\varepsilon) = |\ln\varepsilon|$ as $p = n$, and $F(\varepsilon) = \varepsilon^{N-p}$ as $p > n$. Hence by using integral mean value theorem we know that there exists $T_{N+1} \in (0,T]$ such that

$$
\frac{1}{p}\int_{\partial B(0,T_{N+1})} |\nabla u_\varepsilon|^p dx + \frac{1}{4\varepsilon^p}\int_{\partial B(0,T_{N+1})} |u_\varepsilon|^2(1-|u_\varepsilon|^2)^2 dx \le CF(\varepsilon). \tag{4.15}
$$

Note that ρ_2 is a minimizer of the functional

$$
E(\rho, T_{N+1}) = \frac{1}{p}\int_{T_{N+1}}^1 (\rho_r^2 + 1)^{p/2}dr + \frac{1}{2\varepsilon^p}\int_{T_{N+1}}^1 (1-\rho)^2 dr
$$

on $W^{1,p}_{f_\varepsilon}((T_{N+1},1), R^+ \cup \{0\})$. It is not difficult to prove by maximum principle that

$$
\rho_2 \le 1. \tag{4.16}
$$

As in the derivation of (4.14), from (4.3) and (4.15) it can be proved that

$$
E(\rho_2, T_{N+1}) \le C\varepsilon F(\varepsilon). \tag{4.17}
$$

Using that u_ε is a minimizer and $\rho_2\frac{x}{|x|} \in W_2$, we also have

$$
\begin{aligned}
E_\varepsilon(f_\varepsilon; T_{N+1}) &\le E_\varepsilon(\rho_2; T_{N+1})\\
&\le \frac{1}{p}\int_{T_{N+1}}^1 [\rho_{2r}^2 + \rho_2^2(n-1)r^{-2}]^{p/2} r^{n-1}dr + \frac{1}{2\varepsilon^p}\int_{T_{N+1}}^1 \rho^2(1-\rho_2)^2 dr.
\end{aligned} \tag{4.18}
$$

On the other hand,

$$
\begin{aligned}
&\int_{T_{N+1}}^1 [\rho_r^2+(n-1)r^{-2}\rho^2]^{p/2}r^{n-1}dr - \int_{T_{N+1}}^1 [(n-1)r^{-2}\rho^2]^{p/2}r^{n-1}dr \\
&= \frac{p}{2}\int_{T_{N+1}}^1\int_0^1 [\rho_r^2+(n-1)r^{-2}\rho^2]^{(p-2)/2}s+(n-1)r^{-2}\rho^2(1-s)ds\rho_r^2 r^{n-1}dr \\
&\le C\int_{T_{N+1}}^1 [\rho_r^2+(n-1)r^{-2}\rho^2]^{(p-2)/2}\rho_r^2 r^{n-1}dr \\
&\quad + C\int_{T_{N+1}}^1 [(n-1)r^{-2}\rho^2]^{(p-2)/2}\rho_r^2 r^{n-1}dr \\
&\le C\int_{T_{N+1}}^1 [\rho_r^p+\rho_r^2]dr.
\end{aligned}
$$

Substituting this into (4.18), we have

$$
\begin{aligned}
&E_\varepsilon(f_\varepsilon; T_{N+1}) \\
&\le \frac{1}{p}\int_{T_{N+1}}^1 (n-1)^{p/2}\rho_2^p r^{n-p-1}dr + C\int_{T_{N+1}}^1 (\rho_{2r}^p+\rho_{2r}^2)dr + \frac{C}{\varepsilon^p}\int_{T_{N+1}}^1 (1-\rho_2)^2 dr \\
&\le \frac{1}{p}\int_{T_{N+1}}^1 (n-1)^{p/2}\rho_2^p r^{n-p-1}dr + C\varepsilon F(\varepsilon) \\
&\le \frac{1}{p}(n-1)^{p/2}\int_{T_{N+1}}^1 r^{n-p-1}dr + C\varepsilon F(\varepsilon),
\end{aligned}
$$

using (4.16) and (4.17). This completes the proof. □

Theorem 4.3. *Let $u_\varepsilon = f_\varepsilon(r)\frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u, B_1)$. Then*

$$\lim_{\varepsilon\to 0} u_\varepsilon = \frac{x}{|x|}, \quad \text{in } W^{1,p}(K, R^n)$$

for any compact subset $K \subset \overline{B_1} \setminus \{0\}$.

Proof. Without loss of generality, we may assume $K = \overline{B_1} \setminus B(0, T_{N+1})$. From Proposition 4.2, we have

$$E_\varepsilon(u_\varepsilon, K) = |S^{n-1}|E_\varepsilon(f_\varepsilon; T_{N+1}) \le C\,, \tag{4.19}$$

where C is independent of ε. This and $|u_\varepsilon| \le 1$ imply the existence of a subsequence u_{ε_k} of u_ε and a function $u_* \in W^{1,p}(K, R^n)$, such that

$$
\begin{gathered}
\lim_{\varepsilon_k\to 0} u_{\varepsilon_k} = u_*, \quad \text{weakly in } W^{1,p}(K, R^n), \\
\lim_{\varepsilon_k\to 0} u_{\varepsilon_k} = u_*, \quad \text{in } L^q(K, R), \quad \forall q > 0, \\
\lim_{\varepsilon_k\to 0} f_{\varepsilon_k}(r) = |u_*|, \quad \text{in } C^\alpha([T_{N+1}, 1], R), \quad \alpha > 1 - 1/p.
\end{gathered}
\tag{4.20}
$$

Inequality (4.19) implies $|u_*| \in \{0, 1\}$. Using also (4.20) and $f_{\varepsilon_k}(1) = 1$ we see that $|u_*| = 1$ or $u_* = \frac{x}{|x|}$. Hence, noticing that any subsequence of u_ε has a convergent subsequence and the limit is always $x/|x|$, we can assert

$$\lim_{\varepsilon\to 0} u_\varepsilon = \frac{x}{|x|}, \quad \text{weakly in } W^{1,p}(K, R^n). \tag{4.21}$$

$$\lim_{\varepsilon\to 0} u_\varepsilon = u_*, \quad \text{in } L^q(K, R), \quad \forall q > 0. \tag{4.22}$$

From this and the weakly lower semicontinuity of $\int_K |\nabla u|^p$, using Proposition 4.2, it follows that

$$\begin{aligned}\int_K |\nabla \frac{x}{|x|}|^p &\le \liminf_{\varepsilon_k\to 0} \int_K |\nabla u_\varepsilon|^p \le \limsup_{\varepsilon_k\to 0} \int_K |\nabla u_\varepsilon|^p \\ &\le |S^{n-1}| \int_{T_{N+1}}^1 ((n-1)r^{-2})^{p/2} r^{n-1}\, dr\end{aligned}$$

and hence

$$\lim_{\varepsilon\to 0} \int_K |\nabla u_\varepsilon|^p = \int_K |\nabla \frac{x}{|x|}|^p$$

since

$$\int_K |\nabla \frac{x}{|x|}|^p = |S^{n-1}| \int_{T_{N+1}}^1 ((n-1)r^{-2})^{p/2} r^{n-1}\, dr.$$

Combining this with (4.21)(4.22) completes the proof. □

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