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VANISHING NON-LOCAL REGULARIZATION OF A SCALAR CONSERVATION LAW

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Abstract. We prove that the solution to the regularization of a scalar conservation law by a fractional power of the Laplacian converges, as the regularization vanishes, to the entropy solution of the hyperbolic problem. We also give an error estimate when the initial condition has bounded variation.

1. Introduction

We consider the problem

$$\begin{gathered}\partial_t u^\varepsilon(t,x) + \operatorname{div}(f(u^\varepsilon))(t,x) + \varepsilon g[u^\varepsilon(t,\cdot)](x) = 0\,, \quad t>0\,,\ x\in\mathbb{R}^N\,,\\ u^\varepsilon(0,x) = u_0(x)\,, \quad x\in\mathbb{R}^N\,,\end{gathered} \tag{1.1}$$

where $f=(f_1,\dots,f_N)\in(C^\infty(\mathbb{R}))^N$, $u_0\in L^\infty(\mathbb{R}^N)$ and g is the non-local operator defined through the Fourier transform by

$$\mathcal{F}(g[u^\varepsilon(t,\cdot)])(\xi) = |\xi|^\lambda \mathcal{F}(u^\varepsilon(t,\cdot))(\xi)\,, \quad \text{with } \lambda\in]1,2], \tag{1.2}$$

that is to say g is the fractionnal power, of order $\lambda/2$ of the Laplacian.

In the case $\varepsilon=0$, this equation reduces to the classical scalar conservation law

$$\begin{gathered}\partial_t u(t,x) + \operatorname{div}(f(u))(t,x) = 0\,, \quad t>0\,,\ x\in\mathbb{R}^N\,,\\ u(0,x) = u_0(x)\,, \quad x\in\mathbb{R}^N.\end{gathered} \tag{1.3}$$

Existence and uniqueness of a solution to this equation, in the L^∞ framework, has been established by Kruzhkov [8]; it relies on so-called "entropy solutions", which must satisfy particular inequalities. The case $\lambda=2$ and $\varepsilon>0$ in (1.1) corresponds to $g[u^\varepsilon(t,\cdot)](x) = -(2\pi)^2\Delta u^\varepsilon(t,x)$ and is called the parabolic regularization of (1.3). In this situation, existence, uniqueness and regularity of solutions to this equation are well-known (see e.g. [10]), and an entropy solution of (1.3) can be obtained by proving that, as $\varepsilon\to 0$, the solution to this parabolic regularization converges to a function which satisfies the entropy inequalities of (1.3).

For general $\lambda\in]1,2]$ and $\varepsilon>0$, the study of (1.1) less classical, though motivated by physical problems of detonation (see [4], [5] for example), hydrodynamics, molecular biology, etc... (see the introduction of [1] and references therein). A

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number of papers ([1], [2]...) have studied this equation (also called "fractal conservation law"). The existence results in [1] for (1.1) give global solutions in the case $N = 1$, but which are not very regular, or local (in time) solutions for general $N \geq 1$ and small initial data, but still not regular (in Morrey spaces). In [2] or [3], the authors consider a parabolic regularization of (1.1), that is to say they add a Laplacian operator to the equation; thanks to this second order operator, a global solution is obtained and regularity results can be proved. These papers are mainly interested in asymptotic behaviours for this equation.

However, one could consider (1.1) as a (possible) regularization of (1.3), without having to add another term. In this case, a natural space for the initial data is $L^\infty(\mathbb{R}^N)$, and the question is whether or not (1.1) gives rise to a solution which is regular for $t > 0$ (i.e. whether or not g has the same effect on the regularity as $-\Delta$). It has been proved in [6] that this indeed happens: there exists a unique bounded solution to (1.1), in a suitable sense, and this solution belongs to $C^\infty(]0, \infty[\times\mathbb{R}^N)$. It is constructed via a splitting method, and inherits thus all the properties that are common to both the conservation law and the equation $\partial_t v + \varepsilon g[v] = 0$, such as essential bounds, comparison and contraction principles, etc...; its regularity is proved using the Banach fixed point theorem on Duhamel's formula for $\partial_t u^\varepsilon + \varepsilon g[u^\varepsilon] = -\operatorname{div}(f(u^\varepsilon))$.

Once it has been established that (1.1) has the same regularizing effect, with respect to (1.3), as the parabolic equation, the next question is to know if, as in the parabolic case, the solution to (1.1) remains close to the solution of (1.3) for small ε. This is the aim of the present work, and our main results are the following.

Theorem 1.1. *Let $u_0 \in L^\infty(\mathbb{R}^N)$. The solution to* (1.1) *converges, as $\varepsilon \to 0$ and in $C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ for all $T > 0$, to the entropy solution of* (1.3).

Remark 1.2. This theorem (as well as the results of [6]) is valid for more general g (roughly speaking, the methods work for operators whose kernels are approximate units — see subsection 2.1). For example, sums of operators of the kind (1.2) (or more general Lévy operators) can be considered, with, as a special case, the equation $\partial_t u^\varepsilon + \operatorname{div}(f(u^\varepsilon)) + \varepsilon g[u^\varepsilon] - \varepsilon\Delta u^\varepsilon = 0$ (as in [2], [3]).

As a by-product of the proof of Theorem 1.1, we also obtain the following error estimate.

Theorem 1.3. *Let $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$, u^ε be the solution to* (1.1) *and u be the entropy solution to* (1.3). *Then, for all $T > 0$, $\|u^\varepsilon - u\|_{C([0,T];L^1(\mathbb{R}^N))} = \mathcal{O}(\varepsilon^{1/\lambda})$.*

Remark 1.4. This result in the case of parabolic regularization ($\lambda = 2$) has already been proved in [9]. The special feature of Theorem 1.3 is that it establishes an elegant relationship between the rate of convergence and the order of the operator chosen for the regularization of (1.3); in fact, this error estimate is optimal (see Remark 2.1).

Remark 1.5. Note that, since u^ε and u are bounded (by $\|u_0\|_\infty$), a convergence in $L^1(\mathbb{R}^N)$ (respectively in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$) implies, by interpolation, a convergence in $L^p(\mathbb{R}^N)$ (respectively in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$) for all finite p. For example, under the hypotheses of Theorem 1.3, we have, for all $p \in [1, \infty[$ and all $T > 0$, $\|u^\varepsilon - u\|_{C([0,T];L^p(\mathbb{R}^N))} = \mathcal{O}(\varepsilon^{\frac{1}{p\lambda}})$.

This paper is organized as follows. In the next section, we prove approximate entropy inequalities for u^ε; this function has been obtained in [6], using a splitting method, as a limit of explicit functions: we first prove approximate entropy inequalities on those explicit functions, and then deduce the corresponding inequalities for u^ε; it is not clear that these estimates could be inferred from the methods of [1]. In Section 3, we use Kruzhkov's classical doubling variable technique to combine the approximate entropy inequalities on u^ε and the entropy inequalities on u, which gives an estimate on $|u^\varepsilon - u|$ and proves Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.3, which is an easy consequence of the estimates obtained in Section 3. We have gathered, in Section 5, some results concerning g and its kernel, which we use in the rest of the work.

2. Approximate entropy inequalities for the solution of (1.1)

To prove approximate entropy inequalities for u^ε, we need to recall the construction of this function (see [6]).

2.1. **Construction of u^ε.** The solution to $\partial_t v + \varepsilon g[v] = 0$ with initial condition $v(0,\cdot) = v_0$ is (at least formally) given by $v(t,\cdot) = K_\varepsilon(t,\cdot) * v_0$, where

$$K_\varepsilon(t,x) = \mathcal{F}^{-1}(e^{-\varepsilon t|\cdot|^\lambda})(x).$$

The main property of this kernel is that $(K_\varepsilon(t,\cdot))_{t\to 0}$ is an approximate unit. This means that $K_\varepsilon(t,\cdot)$ is non-negative (see [11]), has integral equal to 1 and that, for all $\nu > 0$, $\int_{|y|\geq\nu} K_\varepsilon(t,y)\,dy \to 0$ as $t \to 0$ ([1]).

We assume here that $u_0 \in C_c^\infty(\mathbb{R}^N)$ (though $u_0 \in L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)\cap BV(\mathbb{R}^N)$ would be enough). Let $\delta > 0$ and $u^{\varepsilon,\delta} : [0,\infty[\times\mathbb{R}^N \mapsto \mathbb{R}$ be defined by $u^{\varepsilon,\delta}(0,\cdot) = u_0$ and

- For all even p, $u^{\varepsilon,\delta}$ is, on $]p\delta,(p+1)\delta]\times\mathbb{R}^N$, the solution to $\partial_t u^{\varepsilon,\delta} + 2\varepsilon g[u^{\varepsilon,\delta}] = 0$ with initial condition $u^{\varepsilon,\delta}(p\delta,\cdot)$, that is to say $u^{\varepsilon,\delta}(t,x) = K_\varepsilon(2(t-p\delta),\cdot) * u^{\varepsilon,\delta}(p\delta,\cdot)(x)$ for $(t,x) \in]p\delta,(p+1)\delta]\times\mathbb{R}^N$.
- For all odd p, $u^{\varepsilon,\delta}$ is, on $]p\delta,(p+1)\delta]\times\mathbb{R}^N$, the entropy solution to $\partial_t u^{\varepsilon,\delta} + 2\operatorname{div}(f(u^{\varepsilon,\delta})) = 0$ with initial condition $u^{\varepsilon,\delta}(p\delta,\cdot)$.

We have then $u^{\varepsilon,\delta} \in C([0,\infty[;L^1(\mathbb{R}^N))$ with $u^{\varepsilon,\delta}(0,\cdot) = u_0$ and for all $t \geq 0$,

$$\begin{gathered} \|u^{\varepsilon,\delta}(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}, \quad \|u^{\varepsilon,\delta}(t,\cdot)\|_{L^1(\mathbb{R}^N)} \leq \|u_0\|_{L^1(\mathbb{R}^N)}, \\ |u^{\varepsilon,\delta}(t,\cdot)|_{BV(\mathbb{R}^N)} \leq \|\nabla u_0\|_{L^1(\mathbb{R}^N)}. \end{gathered} \tag{2.1}$$

It has been proved in [6] that $u^{\varepsilon,\delta}$ converges, as $\delta \to 0$ and in $C([0,T];L^1_{\rm loc}(\mathbb{R}^N))$ for all $T > 0$, to the solution u^ε of (1.1). It has also been noticed that, for δ small enough, $u^{\varepsilon,\delta}$ is in fact, on $[p\delta,(p+1)\delta]\times\mathbb{R}^N$ for all odd p, a regular solution to $\partial_t u^{\varepsilon,\delta} + 2\operatorname{div}(f(u^{\varepsilon,\delta})) = 0$; moreover, for such δ and all $t \geq 0$, $u^{\varepsilon,\delta}(t,\cdot)$ is regular.

The results of [6] are stated in dimension $N = 1$ and with g instead of εg (with K_1 instead of K_ε) but, as indicated in this reference, they are valid in any dimension $N \geq 1$ and substituting K_ε for K_1 (K_ε has the same properties, for a fixed $\varepsilon > 0$, as K_1: in fact, $K_\varepsilon(t,x) = K_1(\varepsilon t,x)$), they also hold with εg.

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[1] This comes from $K_\varepsilon(t,x) = t^{-N/\lambda}K_\varepsilon(1,t^{-1/\lambda}x)$ (change of variable in the definition of K_ε) and from $K_\varepsilon(1,\cdot) \in L^1(\mathbb{R}^N)$ (because the $N+1$ first derivatives of $\xi \mapsto e^{-\varepsilon|\xi|^\lambda}$ are integrable on $\mathbb{R}^N$).

Remark 2.1. If $f = 0$, the solution to (1.1) is $u^\varepsilon(t,x) = K_\varepsilon(t,\cdot) * u_0(x)$ and the solution to (1.3) is $u(t,x) = u_0(x)$. Taking, for example, u_0 the characteristic function of $[-1,1]^N$, some easy computations and the homogeneity property $K_\varepsilon(1,x) = K_1(\varepsilon,x) = \varepsilon^{-N/\lambda} K_1(1,\varepsilon^{-1/\lambda}x)$ (see footnote 1 on page 3) show that $\|u^\varepsilon(1,\cdot)\|_{L^1(\mathbb{R}^N\setminus[-1,1]^N)} \geq c\varepsilon^{1/\lambda}$ for some $c > 0$. Hence, the estimate of Theorem 1.3 is optimal.

2.2. **The approximate entropy inequalities.** We now establish the following approximate entropy inequalities for the solution to (1.1).

Proposition 2.2. *Assume that $u_0 \in L^\infty(\mathbb{R}^N)$ and let u^ε be the solution to* (1.1)*. Let $\eta : \mathbb{R} \mapsto \mathbb{R}$ be a regular convex function and $\phi = (\phi_1,\dots,\phi_N)$ such that $\phi_i' = \eta' f_i'$. Then, for all non-negative $\varphi \in C_c^\infty([0,\infty[\times\mathbb{R}^N)$, we have*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^N} \eta(u^\varepsilon(t,x))\partial_t\varphi(t,x) + \phi(u^\varepsilon(t,x))\cdot\nabla\varphi(t,x)\,dt\,dx \\ &+ \int_{\mathbb{R}^N} \eta(u_0(x))\varphi(0,x)\,dx \\ &\geq \varepsilon \int_0^\infty \int_{\mathbb{R}^N} \eta(u^\varepsilon(t,x)) g[\varphi(t,\cdot)](x)\,dt\,dx. \end{aligned} \tag{2.2}$$

Remark 2.3. If $\varphi \in C_c^\infty([0,\infty[\times\mathbb{R}^N)$, then $t \in [0,\infty[\mapsto \nabla\varphi(t,\cdot) \in L^1(\mathbb{R}^N)^N$ and $t \in [0,\infty[\mapsto \Delta\varphi(t,\cdot) \in L^1(\mathbb{R}^N)$ are continuous; hence, by Lemma 5.1 and the linearity of g, the function $t \in [0,\infty[\mapsto g[\varphi(t,\cdot)] \in L^1(\mathbb{R}^N)$ is continuous. In particular, since $\varphi(t,\cdot) = 0$ for t large enough, $(t,x) \mapsto g[\varphi(t,\cdot)](x)$ is integrable on $]0,\infty[\times\mathbb{R}^N$.

Proof of Proposition 2.2. Note that (2.2) with η or $\eta - \eta(0)$ are the same inequalities. Indeed, the entropy fluxes ϕ associated to η and $\eta - \eta(0)$ are identical,

$$\int_0^\infty \int_{\mathbb{R}^N} \partial_t\varphi(t,x)\,dt\,dx + \int_{\mathbb{R}^N} \varphi(0,x)\,dx = 0$$

and, since $g[\varphi(t,\cdot)] \in L^1(\mathbb{R}^N)$ for all $t \geq 0$ (see Lemma 5.1),

$$\int_{\mathbb{R}^N} g[\varphi(t,\cdot)](x)\,dx = \mathcal{F}(g[\varphi(t,\cdot)])(0) = (|\cdot|^\lambda \mathcal{F}(\varphi(t,\cdot)))(0) = 0.$$

Hence, there is no loss in generality if we assume that $\eta(0) = 0$, which we do from now on.

The proof is done in two steps. We first suppose that the initial condition is regular, in which case we establish approximate entropy inequalities for the functions $u^{\varepsilon,\delta}$ constructed in subsection 2.1, and we deduce the result of the proposition by letting $\delta \to 0$. We then prove the proposition for general initial conditions.

Step 1: Assume that $u_0 \in C_c^\infty(\mathbb{R}^N)$. We take δ small enough so that $u^{\varepsilon,\delta}$ is, on $[p\delta,(p+1)\delta] \times \mathbb{R}^N$ for all odd p, a regular solution to $\partial_t u^{\varepsilon,\delta} + 2\operatorname{div}(f(u^{\varepsilon,\delta})) = 0$. For odd p, we therefore have

$$\begin{aligned} &\int_{p\delta}^{(p+1)\delta} \int_{\mathbb{R}^N} \eta(u^{\varepsilon,\delta}(t,x))\partial_t\varphi(t,x) + 2\phi(u^{\varepsilon,\delta}(t,x))\cdot\nabla\varphi(t,x)\,dt\,dx \\ &= \int_{\mathbb{R}^N} \eta(u^{\varepsilon,\delta}((p+1)\delta,x))\varphi((p+1)\delta,x)\,dx - \int_{\mathbb{R}^N} \eta(u^{\varepsilon,\delta}(p\delta,x))\varphi(p\delta,x)\,dx. \end{aligned}$$

Summing on odd p's (note that, since the support of φ is compact, this sum is finite), and defining χ_δ as the characteristic function of $\cup_{\text{odd } p}]p\delta,(p+1)\delta]$, we find

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}\big(\eta(u^{\varepsilon,\delta}(t,x))\partial_t\varphi(t,x)+2\phi(u^{\varepsilon,\delta}(t,x))\cdot\nabla\varphi(t,x)\big)\,\chi_\delta(t)\,dt\,dx\\&=\sum_{\text{odd } p}(a_{p+1}-a_p)\end{aligned}$$

where $a_p=\int_{\mathbb{R}^N}\eta(u^{\varepsilon,\delta}(p\delta,x))\varphi(p\delta,x)\,dx$. Since

$$\sum_{\text{odd } p}(a_{p+1}-a_p)=\sum_{\text{even } p\,,\,p\geq 2}a_p-\sum_{\text{odd } p}a_p=\sum_{\text{even } p}(a_p-a_{p+1})-a_0\,,$$

we deduce that

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}\big(\eta(u^{\varepsilon,\delta}(t,x))\partial_t\varphi(t,x)+2\phi(u^{\varepsilon,\delta}(t,x))\cdot\nabla\varphi(t,x)\big)\,\chi_\delta(t)\,dt\,dx\\&+\int_{\mathbb{R}^N}\eta(u_0(x))\varphi(0,x)\,dx\\&=\sum_{\text{even } p}(a_p-a_{p+1}).\end{aligned}\tag{2.3}$$

If p is even, we have, by definition, $u^{\varepsilon,\delta}((p+1)\delta)=K_\varepsilon(2\delta)*u^{\varepsilon,\delta}(p\delta)$ (it is convenient, because of the convolution product, to omit the space variable). Since η is convex and $K_\varepsilon(2\delta)$ is positive with integral equal to 1, Jensen's inequality gives then $\eta(u^{\varepsilon,\delta}((p+1)\delta))\leq K_\varepsilon(2\delta)*\eta(u^{\varepsilon,\delta}(p\delta))$. The function φ being non-negative, we deduce that $\eta(u^{\varepsilon,\delta}((p+1)\delta))\varphi((p+1)\delta)\leq K_\varepsilon(2\delta)*\eta(u^{\varepsilon,\delta}(p\delta))\varphi((p+1)\delta)$ and thus

$$\begin{aligned}a_{p+1}-a_p&\leq\int_{\mathbb{R}^N}K_\varepsilon(2\delta)*\eta(u^{\varepsilon,\delta}(p\delta))\varphi((p+1)\delta)-\eta(u^{\varepsilon,\delta}(p\delta))\varphi(p\delta)\\&=\int_{\mathbb{R}^N}F((p+1)\delta)\varphi((p+1)\delta)-F(p\delta)\varphi(p\delta),\end{aligned}$$

where $F(p\delta)=\eta(u^{\varepsilon,\delta}(p\delta))$ and, for $t\in]p\delta,(p+1)\delta]$, $F(t)=K_\varepsilon(2(t-p\delta))*F(p\delta)$ (i.e. F satisfies $\partial_tF+2\varepsilon g[F]=0$ on $]p\delta,(p+1)\delta]$). We have $\eta(0)=0$, so that, letting C_0 be the Lipschitz constant of η on $[-\|u_0\|_\infty,\|u_0\|_\infty]$ and using (2.1),

$$\|F(p\delta)\|_{L^1(\mathbb{R}^N)}\leq C_0\|u^{\varepsilon,\delta}(p\delta)\|_{L^1(\mathbb{R}^N)}\leq C_0\|u_0\|_{L^1(\mathbb{R}^N)}\tag{2.4}$$

$$\|\nabla F(p\delta)\|_{L^1(\mathbb{R}^N)}\leq C_0\|\nabla u^{\varepsilon,\delta}(p\delta)\|_{L^1(\mathbb{R}^N)}\leq C_0\|\nabla u_0\|_{L^1(\mathbb{R}^N)}\tag{2.5}$$

(recall that δ is small enough so that $u^{\varepsilon,\delta}(t,\cdot)$ is regular for all $t\geq 0$). Lemma 5.2 in the appendix enables then to write

$$\begin{aligned}a_{p+1}-a_p&\leq\int_{\mathbb{R}^N}F((p+1)\delta)\varphi((p+1)\delta)-F(p\delta)\varphi(p\delta)\\&=\int_{p\delta}^{(p+1)\delta}\int_{\mathbb{R}^N}F(t,x)\partial_t\varphi(t,x)-2\varepsilon F(t,x)g[\varphi(t,\cdot)](x)\,dt\,dx.\end{aligned}\tag{2.6}$$

We have, by Lemma 5.3 in the appendix, for all $\nu > 0$ and all $t \in]p\delta, (p+1)\delta]$,

$$\begin{aligned}
&\|F(t) - \eta(u^{\varepsilon,\delta}(p\delta))\|_{L^1(\mathbb{R}^N)} \\
&= \|K_\varepsilon(2(t-p\delta)) * F(p\delta) - F(p\delta)\|_{L^1(\mathbb{R}^N)} \\
&\leq 2\|F(p\delta)\|_{L^1(\mathbb{R}^N)} \int_{|y|\geq\nu} K_\varepsilon(2(t-p\delta), y)\, dy + \nu \|\nabla F(p\delta)\|_{L^1(\mathbb{R}^N)}.
\end{aligned}$$

Using (2.4) and (2.5), we deduce

$$\begin{aligned}
&\|F(t) - \eta(u^{\varepsilon,\delta}(p\delta))\|_{L^1(\mathbb{R}^N)} \\
&\leq 2C_0\|u_0\|_{L^1(\mathbb{R}^N)} \sup_{0<s\leq 2\delta} \int_{|y|\geq\nu} K_\varepsilon(s,y)\, dy + C_0\nu\|\nabla u_0\|_{L^1(\mathbb{R}^N)}.
\end{aligned} \tag{2.7}$$

We have $|\eta(u^{\varepsilon,\delta}(t)) - \eta(u^{\varepsilon,\delta}(p\delta))| \leq C_0|u^{\varepsilon,\delta}(t) - u^{\varepsilon,\delta}(p\delta)|$ and, for $t \in]p\delta, (p+1)\delta]$, $u^{\varepsilon,\delta}(t) = K_\varepsilon(2(t-p\delta)) * u^{\varepsilon,\delta}(p\delta)$. Hence, Lemma 5.3 and (2.1) give, for all $t \in]p\delta, (p+1)\delta]$ and all $\nu > 0$,

$$\begin{aligned}
&\|\eta(u^{\varepsilon,\delta}(t)) - \eta(u^{\varepsilon,\delta}(p\delta))\|_{L^1(\mathbb{R}^N)} \\
&\leq 2C_0\|u_0\|_{L^1(\mathbb{R}^N)} \sup_{0<s\leq 2\delta} \int_{|y|\geq\nu} K_\varepsilon(s,y)\, dy + C_0\nu\|\nabla u_0\|_{L^1(\mathbb{R}^N)}.
\end{aligned} \tag{2.8}$$

Gathering (2.7) and (2.8), we find, for all $t \in]p\delta, (p+1)\delta]$ and all $\nu > 0$,

$$\begin{aligned}
&\|F(t) - \eta(u^{\varepsilon,\delta}(t))\|_{L^1(\mathbb{R}^N)} \\
&\leq 4C_0\|u_0\|_{L^1(\mathbb{R}^N)} \sup_{0<s\leq 2\delta} \int_{|y|\geq\nu} K_\varepsilon(s,y)\, dy + 2C_0\nu\|\nabla u_0\|_{L^1(\mathbb{R}^N)} = \omega_\varepsilon(\delta,\nu)
\end{aligned}$$

with $\lim_{\nu\to 0}(\lim_{\delta\to 0} \omega_\varepsilon(\delta,\nu)) = 0$ (because $(K_\varepsilon(t,\cdot))_{t\to 0}$ is an approximate unit). Using this inequality in (2.6), we obtain, for all $\nu > 0$,

$$\begin{aligned}
&a_{p+1} - a_p \\
&\leq \int_{p\delta}^{(p+1)\delta} \int_{\mathbb{R}^N} \eta(u^{\varepsilon,\delta}(t,x))\partial_t\varphi(t,x) - 2\varepsilon\eta(u^{\varepsilon,\delta}(t,x))g[\varphi(t,\cdot)](x)\, dt\, dx \\
&\quad + \omega_\varepsilon(\delta,\nu) \int_{p\delta}^{(p+1)\delta} \left(\|\partial_t\varphi(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} + 2\varepsilon\|g[\varphi(t,\cdot)]\|_{L^\infty(\mathbb{R}^N)} \right) dt.
\end{aligned} \tag{2.9}$$

Note that, by definition of g, we can write

$$\begin{aligned}
&\|g[\varphi(t,\cdot)]\|_{L^\infty(\mathbb{R}^N)} \\
&\leq \| |\cdot|^\lambda \mathcal{F}(\varphi(t,\cdot))\|_{L^1(\mathbb{R}^N)} \\
&= \Big\| \frac{|\cdot|^\lambda}{1 + (2\pi|\cdot|)^{2(N+1)}} \mathcal{F}\left(\varphi(t,\cdot) + (-\Delta)^{N+1}\varphi(t,\cdot)\right) \Big\|_{L^1(\mathbb{R}^N)} \\
&\leq \Big\| \frac{|\cdot|^\lambda}{1 + (2\pi|\cdot|)^{2(N+1)}} \Big\|_{L^1(\mathbb{R}^N)} \|\mathcal{F}\left(\varphi(t,\cdot) + (-\Delta)^{N+1}\varphi(t,\cdot)\right)\|_{L^\infty(\mathbb{R}^N)} \\
&\leq \Big\| \frac{|\cdot|^\lambda}{1 + (2\pi|\cdot|)^{2(N+1)}} \Big\|_{L^1(\mathbb{R}^N)} \|\varphi(t,\cdot) + (-\Delta)^{N+1}\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)}
\end{aligned} \tag{2.10}$$

with $\lambda - 2(N+1) \leq -2N < -N$. Hence, $t \mapsto \|g[\varphi(t,\cdot)]\|_{L^\infty(\mathbb{R}^N)}$ is integrable on $[0,\infty[$ (in fact, this function is continuous and null for t large).

Summing (2.9) on even p's and coming back to (2.3), we deduce

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}\big(\eta(u^{\varepsilon,\delta}(t,x))\partial_t\varphi(t,x)+2\phi(u^{\varepsilon,\delta}(t,x))\cdot\nabla\varphi(t,x)\big)\,\chi_\delta(t)\,dt\,dx\\&+\int_{\mathbb{R}^N}\eta(u_0(x))\varphi(0,x)\,dx\\&\geq-\int_0^\infty\int_{\mathbb{R}^N}\big(\eta(u^{\varepsilon,\delta}(t,x))\partial_t\varphi(t,x)-2\varepsilon\eta(u^{\varepsilon,\delta}(t,x))g[\varphi(t,\cdot)](x)\big)\,(1-\chi_\delta(t))\,dt\,dx\\&\quad-\omega_\varepsilon(\delta,\nu)\int_0^\infty\big(\|\partial_t\varphi(t,\cdot)\|_{L^\infty(\mathbb{R}^N)}+2\varepsilon\|g[\varphi(t,\cdot)]\|_{L^\infty(\mathbb{R}^N)}\big)\,dt\end{aligned}$$

(note that $1-\chi_\delta$ is the characteristic function of $\cup_{\text{even }p}]p\delta,(p+1)\delta]$), that is to say

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}\eta(u^{\varepsilon,\delta}(t,x))\partial_t\varphi(t,x)+2\phi(u^{\varepsilon,\delta}(t,x))\cdot\nabla\varphi(t,x)\chi_\delta(t)\,dt\,dx\\&+\int_{\mathbb{R}^N}\eta(u_0(x))\varphi(0,x)\,dx\\&\geq 2\varepsilon\int_0^\infty\int_{\mathbb{R}^N}\eta(u^{\varepsilon,\delta}(t,x))g[\varphi(t,\cdot)](x)(1-\chi_\delta(t))\,dt\,dx\\&\quad-\omega_\varepsilon(\delta,\nu)\int_0^\infty\big(\|\partial_t\varphi(t,\cdot)\|_{L^\infty(\mathbb{R}^N)}+2\varepsilon\|g[\varphi(t,\cdot)]\|_{L^\infty(\mathbb{R}^N)}\big)\,dt.\end{aligned}\tag{2.11}$$

As $\delta\to0$, we have $u^{\varepsilon,\delta}\to u^\varepsilon$ in $C([0,T];L^1_{\rm loc}(\mathbb{R}^N))$ for all $T>0$; hence, η and ϕ being Lipschitz-continuous on $[-\|u_0\|_\infty,\|u_0\|_\infty]$ and $u^{\varepsilon,\delta}$ taking its values in this interval, we deduce that $\eta(u^{\varepsilon,\delta})\to\eta(u^\varepsilon)$ and $\phi(u^{\varepsilon,\delta})\to\phi(u^\varepsilon)$, as $\delta\to0$ and in $C([0,T];L^1_{\rm loc}(\mathbb{R}^N))$ for all $T>0$. This allows to see that, as $\delta\to0$,

$$\int_0^\infty\int_{\mathbb{R}^N}\eta(u^{\varepsilon,\delta}(t,x))\partial_t\varphi(t,x)\,dt\,dx\to\int_0^\infty\int_{\mathbb{R}^N}\eta(u^\varepsilon(t,x))\partial_t\varphi(t,x)\,dt\,dx.\tag{2.12}$$

We also deduce that

$$\int_{\mathbb{R}^N}\phi(u^{\varepsilon,\delta}(\cdot,x))\cdot\nabla\varphi(\cdot,x)\,dx\to\int_{\mathbb{R}^N}\phi(u^\varepsilon(\cdot,x))\cdot\nabla\varphi(\cdot,x)\,dx\quad\text{ in }L^\infty_{\rm loc}([0,\infty[),$$

and thus in $L^1(]0,\infty[)$ (these functions are null for t large). Since $\chi_\delta\to1/2$ in $L^\infty(]0,\infty[)$ weak-$*$, this implies

$$\int_0^\infty\int_{\mathbb{R}^N}2\phi(u^{\varepsilon,\delta}(t,x))\cdot\nabla\varphi(t,x)\chi_\delta(t)\,dt\,dx\to\int_0^\infty\int_{\mathbb{R}^N}\phi(u^\varepsilon(t,x))\cdot\nabla\varphi(t,x)\,dt\,dx.\tag{2.13}$$

For all $M\geq0$, we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\eta(u^{\varepsilon,\delta}(t,x))g[\varphi(t,\cdot)](x)\,dx-\int_{\mathbb{R}^N}\eta(u^\varepsilon(t,x))g[\varphi(t,\cdot)](x)\,dx\Big|\\&\leq\|g[\varphi(t,\cdot)]\|_{L^\infty(\mathbb{R}^N)}\int_{|x|\leq M}|\eta(u^{\varepsilon,\delta}(t,x))-\eta(u^\varepsilon(t,x))|\,dx\\&\quad+2C_1\int_{|x|\geq M}|g[\varphi(t,\cdot)](x)|\,dx\end{aligned}\tag{2.14}$$

with $C_1=\sup\{|\eta(z)|\,,\ |z|\leq\|u_0\|_\infty\}$. By Remark 2.3, the function $t\in[0,\infty[\mapsto g[\varphi(t,\cdot)]\in L^1(\mathbb{R}^N)$ is continuous and null for t large enough; this implies that

$\{g[\varphi(t,\cdot)]\,,\ t\geq 0\}$ is compact in $L^1(\mathbb{R}^N)$, and thus, by Vitali's theorem, that

$$\lim_{M\to\infty}\int_{|x|\geq M}|g[\varphi(t,\cdot)](x)|\,dx=0\quad\text{uniformly with respect to } t\geq 0.$$

For a fixed M, we have $\int_{|x|\leq M}|\eta(u^{\varepsilon,\delta}(t,x))-\eta(u^{\varepsilon}(t,x))|\,dx\to 0$ as $\delta\to 0$, locally uniformly with respect to $t\geq 0$; since $\sup_{t\geq 0}\|g[\varphi(t,\cdot)]\|_{L^\infty(\mathbb{R}^N)}<\infty$ (see (2.10)), these considerations and (2.14) show that, as $\delta\to 0$,

$$\int_{\mathbb{R}^N}\eta(u^{\varepsilon,\delta}(\cdot,x))g[\varphi(\cdot,\cdot)](x)dx\to\int_{\mathbb{R}^N}\eta(u^{\varepsilon}(\cdot,x))g[\varphi(\cdot,\cdot)](x)\,dx\quad\text{in } L^\infty_{\rm loc}([0,\infty[),$$

and thus also in $L^1(]0,\infty[)$ (because $\varphi(t,\cdot)=0$ for t large). We have $1-\chi_\delta\to 1/2$ in $L^\infty(]0,\infty[)$ weak-$*$, which implies

$$\begin{aligned}&2\varepsilon\int_0^\infty\int_{\mathbb{R}^N}\eta(u^{\varepsilon,\delta}(t,x))g[\varphi(t,\cdot)](x)(1-\chi_\delta(t))\,dt\,dx\\&\to\varepsilon\int_0^\infty\int_{\mathbb{R}^N}\eta(u^{\varepsilon}(t,x))g[\varphi(t,\cdot)](x)\,dt\,dx.\end{aligned}\tag{2.15}$$

Passing to the limit $\delta\to 0$ in (2.11), thanks to (2.12), (2.13) and (2.15), we deduce

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}\eta(u^\varepsilon(t,x))\partial_t\varphi(t,x)+\phi(u^\varepsilon(t,x))\cdot\nabla\varphi(t,x)\,dt\,dx\\&+\int_{\mathbb{R}^N}\eta(u_0(x))\varphi(0,x)\,dx\\&\geq\varepsilon\int_0^\infty\int_{\mathbb{R}^N}\eta(u^\varepsilon(t,x))g[\varphi(t,\cdot)](x)\,dt\,dx\\&\quad-\big(\lim_{\delta\to 0}\omega_\varepsilon(\delta,\nu)\big)\int_0^\infty\big(\|\partial_t\varphi(t,\cdot)\|_{L^\infty(\mathbb{R}^N)}+2\varepsilon\|g[\varphi(t,\cdot)]\|_{L^\infty(\mathbb{R}^N)}\big)\,dt.\end{aligned}$$

Since this is satisfied for all $\nu>0$, we can let $\nu\to 0$ and use the property of $\omega_\varepsilon(\delta,\nu)$ to see that (2.2) holds.

Step 2: We now only assume that $u_0\in L^\infty(\mathbb{R}^N)$. Let $u_{0,n}\in C_c^\infty(\mathbb{R}^N)$ which converges a.e. to u_0 and is bounded by $\|u_0\|_\infty$; we define u_n^ε as the solution to (1.1) with $u_{0,n}$ as initial datum. As in Section 6.4 of [6], we can see that $(u_n^\varepsilon)_{n\geq 1}$ is bounded ([2]) and converges pointwise to u^ε as $n\to\infty$ ([3]).

u_n^ε satisfies (2.2), with $u_{0,n}$ instead of u_0. Hence, using the dominated convergence theorem, we let $n\to\infty$ in this inequality to see that it is also satisfied by u^ε, and the proof is complete. □

3. Proof of convergence

Proposition 3.1. *Let $u_0\in L^\infty(\mathbb{R}^N)$, u^ε be the solution to* (1.1) *and u be the entropy solution to* (1.3). *Let L be a Lipschitz constant of f on* $[-\|u_0\|_\infty,\|u_0\|_\infty]$

[2]This can be deduced from (2.1) by letting $\delta\to 0$, and using $\|u_{0,n}\|_\infty\leq\|u_0\|_\infty$.

[3]This is a consequence of estimates in [6] which show that all the derivatives of u_n^ε are bounded on $]t_0,\infty[\times\mathbb{R}^N$, for all $t_0>0$, uniformly with respect to n; there is thus a subsequence of $(u_n^\varepsilon)_{n\geq 1}$ which converges pointwise and, to prove that the limit is a solution to (1.1), we let $n\to\infty$ in Duhamel's formula which defines these solutions.

and $T > 0$. If B is a subset of $\mathbb{R}^N$, we define $\widetilde{B} = \{x \in \mathbb{R}^N \mid \operatorname{dist}(x, B) \le 1\}$ and, for $(\mu, \nu) \in]0, 1[^2$,

$$\omega_1^B(\mu,\nu) = \sup_{0<t<T} \Big(\sup_{0<r<\mu,|z|<\nu} \int_B |u(t,x) - u(t+r, x+z)|\, dx \Big) \tag{3.1}$$

$$\omega_2^B(\mu,\nu) = \sup_{|z|<\nu} \int_B |u_0(x) - u_0(x+z)|\, dx + \sup_{0<s<\mu} \int_{\widetilde{B}} |u_0(x) - u(s,x)|\, dx. \tag{3.2}$$

Here $B(R)$ denotes the ball in $\mathbb{R}^N$ of center 0 and radius R. Then, for all $M > LT$, there exists $C_1 > 0$ such that, for all $t_0 \in [0, T]$, for all $\varepsilon > 0$, for all $\mu \in]0, 1[$ and for all $\nu \in]0, 1[$,

$$\begin{aligned} &\int_{B(M-LT)} |u^\varepsilon(t_0,x) - u(t_0,x)|\, dx \\ &\le C_1 \omega_1^{B(M+1)}(\mu,\nu) + \omega_2^{B(M+1)}(\mu,\nu) \\ &\quad + 2\varepsilon \|u_0\|_\infty \int_{\mathbb{R}^N} \int_0^T \int_{\mathbb{R}^N} |g[h_{\nu,M}(y,t,\cdot)](x)|\, dy\, dt\, dx \end{aligned} \tag{3.3}$$

for some $h_{\nu,M} \in C_c^\infty(\mathbb{R}^N \times [0, T] \times \mathbb{R}^N)$ only depending on ν and M.

Remark 3.2. As in Remark 2.3, the regularity of $h_{\nu,M}$ enables us to see that $(y, t) \in \mathbb{R}^N \times [0, T] \mapsto g[h_{\nu,M}(y, t, \cdot)] \in L^1(\mathbb{R}^N)$ is continuous and that the mapping $(y, t, x) \mapsto g[h_{\nu,M}(y, t, \cdot)](x)$ is integrable on $\mathbb{R}^N \times [0, T] \times \mathbb{R}^N$.

Proof of Proposition 3.1. We use the doubling variable technique of Kruzhkov (see [8]). Equation (2.2) has been obtained for regular convex η but it is easy, thanks to an approximation technique, to see that it also holds with the entropy $\eta_\kappa(z) = |z - \kappa|$, associated to the flux $\phi_\kappa(z) = f(z\top\kappa) - f(z\bot\kappa)$ (where $z\top\kappa = \max(z, \kappa)$ and $z\bot\kappa = \min(z, \kappa)$).

Let $\varphi \in C_c^\infty([0, \infty[\times\mathbb{R}^N \times [0, \infty[\times\mathbb{R}^N)$ be non-negative. Applying, for fixed $(s, y) \in]0, \infty[\times\mathbb{R}^N$, (2.2) to $\eta_{u(s,y)}$ and $\varphi(\cdot, \cdot, s, y)$, and integrating on $(s, y) \in]0, \infty[\times\mathbb{R}^N$, we find

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - u(s,y)| \partial_t \varphi(t,x,s,y) \\ &\qquad + F(u^\varepsilon(t,x), u(s,y)) \cdot \nabla_x \varphi(t,x,s,y)\, ds\, dy\, dt\, dx \\ &+ \int_0^\infty \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_0(x) - u(s,y)| \varphi(0,x,s,y)\, ds\, dy\, dx \\ &\ge \varepsilon \int_0^\infty \int_{\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - u(s,y)|\, g[\varphi(t,\cdot,s,y)](x)\, ds\, dy\, dt\, dx \end{aligned} \tag{3.4}$$

where $F(z, w) = f(z\top w) - f(z\bot w)$ is symmetric. We can see, as in Remarks 2.3 and 3.2, that $(t, x, s, y) \mapsto g[\varphi(t, \cdot, s, y)](x)$ is integrable on $]0, \infty[\times\mathbb{R}^N\times]0, \infty[\times\mathbb{R}^N$, so that all the integral signs in the right-hand side can be manipulated at wish, using Fubini's theorem.

Since u is the entropy solution to (1.3), it satisfies, for all $\kappa \in \mathbb{R}$ and all nonnegative $\psi \in C_c^\infty([0,\infty[\times\mathbb{R}^N)$,

$$\int_0^\infty \int_{\mathbb{R}^N} \eta_\kappa(u(s,y))\partial_s\psi(s,y) + \phi_\kappa(u(s,y)) \cdot \nabla_y\psi(s,y)\,ds\,dy + \int_{\mathbb{R}^N} \eta_\kappa(u_0(y))\psi(0,y)\,dy \geq 0.$$

Applying this inequality to $\kappa = u^\varepsilon(t,x)$ and $\psi = \varphi(t,x,\cdot,\cdot)$, and integrating on $(t,x) \in]0,\infty[\times\mathbb{R}^N$, we obtain

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} & |u(s,y) - u^\varepsilon(t,x)|\partial_s\varphi(t,x,s,y) \\ & + F(u(s,y),u^\varepsilon(t,x)) \cdot \nabla_y\varphi(t,x,s,y)\,ds\,dy\,dt\,dx \\ + \int_0^\infty \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} & |u_0(y) - u^\varepsilon(t,x)|\varphi(t,x,0,y)\,dt\,dx\,dy \geq 0.\end{aligned} \tag{3.5}$$

Summing (3.4) and (3.5), we see that

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - u(s,y)|(\partial_t\varphi(t,x,s,y) + \partial_s\varphi(t,x,s,y))\,ds\,dy\,dt\,dx \\ &+ \int_0^\infty \int_{\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} F(u^\varepsilon(t,x),u(s,y)) \cdot \big(\nabla_x\varphi(t,x,s,y) \\ &\qquad + \nabla_y\varphi(t,x,s,y)\big)\,ds\,dy\,dt\,dx \\ &+ \int_0^\infty \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u_0(x) - u(s,y)|\varphi(0,x,s,y)\,ds\,dy\,dx \\ &+ \int_0^\infty \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u_0(y) - u^\varepsilon(t,x)|\varphi(t,x,0,y)\,dt\,dx\,dy \\ &\geq \varepsilon \int_0^\infty \int_{\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - u(s,y)|\, g[\varphi(t,\cdot,s,y)](x)\,ds\,dy\,dt\,dx.\end{aligned} \tag{3.6}$$

Let $\rho_\nu \in C_c^\infty(\mathbb{R}^N)$ and $\theta_\mu \in C_c^\infty(\mathbb{R})$ be smoothing kernels such that $\mathrm{supp}(\rho_\nu) \subset \{x \in \mathbb{R}^N \mid |x| < \nu\}$ and $\mathrm{supp}(\theta_\mu) \subset]0,\mu[$. We take $\psi \in C_c^\infty([0,\infty[\times\mathbb{R}^N)$ a nonnegative function and we let $\varphi(t,x,s,y) = \psi(t,x)\rho_\nu(y-x)\theta_\mu(s-t)$; we have

$$\begin{gathered}\partial_t\varphi(t,x,s,y) + \partial_s\varphi(t,x,s,y) = \partial_t\psi(t,x)\rho_\nu(y-x)\theta_\mu(s-t), \\ \nabla_x\varphi(t,x,s,y) + \nabla_y\varphi(t,x,s,y) = \nabla_x\psi(t,x)\rho_\nu(y-x)\theta_\mu(s-t), \\ \varphi(t,x,0,y) = 0 \text{ for } t \geq 0.\end{gathered}$$

Hence, (3.6) gives

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - u(s,y)|\partial_t\psi(t,x)\rho_\nu(y-x)\theta_\mu(s-t)\,ds\,dy\,dt\,dx \\ &+ \int_0^\infty \int_{\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} F(u^\varepsilon(t,x),u(s,y)) \cdot \nabla_x\psi(t,x)\rho_\nu(y-x)\theta_\mu(s-t)\,ds\,dy\,dt\,dx \\ &+ \int_0^\infty \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u_0(x) - u(s,y)|\psi(0,x)\rho_\nu(y-x)\theta_\mu(s)\,ds\,dy\,dx \\ &\geq \varepsilon \int_0^\infty \int_{\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} \theta_\mu(s-t)|u^\varepsilon(t,x) - u(s,y)|g[\rho_\nu(y-\cdot)\psi(t,\cdot)](x)\,ds\,dy\,dt\,dx.\end{aligned} \tag{3.7}$$

Let A_1, A_2 and A_3 be the first three lines of this inequality ([4]). We take $T>0$ and B a bounded set in $\mathbb{R}^N$, and we suppose that $\operatorname{supp}(\psi)\subset[0,T]\times B$. Then

$$\begin{aligned}
&\Big|A_1-\int_0^\infty\int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{R}^N}|u^\varepsilon(t,x)-u(t,x)|\partial_t\psi(t,x)\rho_\nu(y-x)\theta_\mu(s-t)\,ds\,dy\,dt\,dx\Big|\\
&\le\int_0^\infty\int_{\mathbb{R}^N}\int_0^\infty\int_B|u(t,x)-u(s,y)|\,|\partial_t\psi(t,x)|\rho_\nu(y-x)\theta_\mu(s-t)\,ds\,dy\,dt\,dx\\
&\le\|\partial_t\psi\|_{L^1(0,T;L^\infty(\mathbb{R}^N))}\\
&\quad\times\sup_{0<t<T}\Big(\int_0^\infty\int_{\mathbb{R}^N}\int_B|u(t,x)-u(s,y)|\rho_\nu(y-x)\theta_\mu(s-t)\,ds\,dy\,dx\Big)\\
&\le\|\partial_t\psi\|_{L^1(0,T;L^\infty(\mathbb{R}^N))}\omega_1^B(\mu,\nu).
\end{aligned}$$

Since $\int_0^\infty\theta_\mu(s-t)\,ds=1$ for all $t>0$ and $\int_{\mathbb{R}^N}\rho_\nu(y-x)\,dy=1$ for all $x\in\mathbb{R}^N$, we find

$$A_1\le\int_0^\infty\int_{\mathbb{R}^N}|u^\varepsilon(t,x)-u(t,x)|\partial_t\psi(t,x)\,dt\,dx+\|\partial_t\psi\|_{L^1(0,T;L^\infty(\mathbb{R}^N))}\omega_1^B(\mu,\nu). \tag{3.8}$$

We have $|F(u^\varepsilon(t,x),u(s,y))|\le L|u^\varepsilon(t,x)-u(s,y)|$ (because both functions u^ε and u take their values in $[-\|u_0\|_\infty,\|u_0\|_\infty]$) and therefore

$$\begin{aligned}
&|A_2|\\
&\le L\int_0^\infty\int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{R}^N}|u^\varepsilon(t,x)-u(s,y)|\,|\nabla_x\psi(t,x)|\rho_\nu(y-x)\theta_\mu(s-t)\,ds\,dy\,dt\,dx\\
&\le L\int_0^\infty\int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{R}^N}|u^\varepsilon(t,x)-u(t,x)|\,|\nabla_x\psi(t,x)|\rho_\nu(y-x)\theta_\mu(s-t)\,ds\,dy\,dt\,dx\\
&\quad+L\int_0^\infty\int_{\mathbb{R}^N}\int_0^\infty\int_B|u(t,x)-u(s,y)|\,|\nabla_x\psi(t,x)|\rho_\nu(y-x)\theta_\mu(s-t)\,ds\,dy\,dt\,dx\\
&\le L\int_0^\infty\int_{\mathbb{R}^N}|u^\varepsilon(t,x)-u(t,x)|\,|\nabla_x\psi(t,x)|\,dt\,dx+L\|\nabla_x\psi\|_{L^1(0,T;L^\infty(\mathbb{R}^N))}\omega_1^B(\mu,\nu).
\end{aligned}\tag{3.9}$$

Note that if $x\in B$ and $\rho_\nu(y-x)\neq0$, then $\operatorname{dist}(y,B)\le\nu\le1$; therefore, for $\nu\le1$,

$$\begin{aligned}
&|A_3|\\
&\le\|\psi(0,\cdot)\|_{L^\infty(\mathbb{R}^N)}\int_0^\infty\int_{\mathbb{R}^N}\int_B|u_0(x)-u_0(y)|\rho_\nu(y-x)\theta_\mu(s)\,ds\,dy\,dx\\
&\quad+\|\psi(0,\cdot)\|_{L^\infty(\mathbb{R}^N)}\int_0^\infty\int_{\mathbb{R}^N}\int_B|u_0(y)-u(s,y)|\rho_\nu(y-x)\theta_\mu(s)\,ds\,dy\,dx\\
&\le\|\psi(0,\cdot)\|_{L^\infty(\mathbb{R}^N)}\omega_2^B(\mu,\nu)\,.
\end{aligned}\tag{3.10}$$

[4] We keep the precise expression of the fourth term up to the end, since it will be useful in the proof of Theorem 1.3.

Gathering (3.8), (3.9) and (3.10) in (3.7), we deduce

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^N}|u^\varepsilon(t,x)-u(t,x)|(\partial_t\psi(t,x)+L|\nabla_x\psi(t,x)|)\,dt\,dx\\
&+\big(\|\partial_t\psi\|_{L^1(0,T;L^\infty(\mathbb{R}^N))}+L\|\nabla_x\psi\|_{L^1(0,T;L^\infty(\mathbb{R}^N))}\big)\,\omega_1^B(\mu,\nu)\\
&+\|\psi(0,\cdot)\|_{L^\infty(\mathbb{R}^N)}\omega_2^B(\mu,\nu)\\
&\geq\varepsilon\int_0^\infty\int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{R}^N}\theta_\mu(s-t)|u^\varepsilon(t,x)-u(s,y)|\,g[\rho_\nu(y-\cdot)\psi(t,\cdot)](x)\,ds\,dy\,dt\,dx.
\end{aligned}\tag{3.11}$$

Let $M>LT$ and $w_M\in C_c^\infty([0,\infty[)$ be non-increasing, with values in $[0,1]$, such that $w_M\equiv 1$ on $[0,M]$ and $\mathrm{supp}(w_M)\subset[0,M+1]$. Let $\Theta\in C_c^\infty([0,T[)$ with values in $[0,1]$. Then $\psi(t,x)=w_M(|x|+Lt)\Theta(t)$ is non-negative, belongs to $C_c^\infty([0,\infty[\times\mathbb{R}^N)$ (the function Θ has its support in $[0,T[$ and $(t,x)\mapsto w_M(|x|+Lt)$ is regular on $[0,T]\times\mathbb{R}^N$ since, in the neighborhood of $[0,T]\times\{0\}$, we have $w_M(|x|+Lt)=1$) and $\mathrm{supp}(\psi)\subset[0,T[\times B(M+1)$. We have

$$\begin{gathered}
\partial_t\psi(t,x)=Lw_M'(|x|+Lt)\Theta(t)+w_M(|x|+Lt)\Theta'(t)\\
|\nabla_x\psi(t,x)|=\big|w_M'(|x|+Lt)\Theta(t)\frac{x}{|x|}\big|=(-w_M'(|x|+Lt))\Theta(t)
\end{gathered}$$

(recall that w_M is non-increasing). Hence, $\partial_t\psi(t,x)+L|\nabla_x\psi(t,x)|=w_M(|x|+Lt)\Theta'(t)$. Moreover,

$$\begin{gathered}
\|\partial_t\psi\|_{L^1(0,T;L^\infty(\mathbb{R}^N))}\leq LT\|w_M'\|_\infty+\|\Theta'\|_{L^1(0,T)}\,,\\
\|\nabla_x\psi\|_{L^1(0,T;L^\infty(\mathbb{R}^N))}\leq T\|w_M'\|_\infty.
\end{gathered}$$

Therefore, (3.11) gives

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^N}|u^\varepsilon(t,x)-u(t,x)|w_M(|x|+Lt)\Theta'(t)\,dt\,dx\\
&+(2LT\|w_M'\|_\infty+\|\Theta'\|_{L^1(0,T)})\omega_1^{B(M+1)}(\mu,\nu)+\omega_2^{B(M+1)}(\mu,\nu)\\
&-\varepsilon\int_0^\infty\int_{\mathbb{R}^N}\int_0^T\int_{\mathbb{R}^N}\Theta(t)\theta_\mu(s-t)|u^\varepsilon(t,x)-u(s,y)|\\
&\qquad\times g\big[\rho_\nu(y-\cdot)w_M(|\cdot|+Lt)\big](x)\,ds\,dy\,dt\,dx\geq 0.
\end{aligned}\tag{3.12}$$

Let $t_0\in[0,T[$ and take $\Theta(t)=\Theta_\beta(t)=\int_t^\infty\theta_\beta(s-t_0)\,ds$. Then, for β small enough, $\Theta_\beta\in C_c^\infty([0,T[)$, has its values in $[0,1]$ and $\|\Theta_\beta'\|_{L^1(0,T)}\leq 1$. Since, for all $t\in[0,T]$, $w_M(|\cdot|+Lt)\equiv 1$ on $B(M-LT)$ and $\Theta_\beta'(t)=-\theta_\beta(t-t_0)$, we deduce from (3.12) that

$$\begin{aligned}
&\int_0^T\int_{B(M-LT)}|u^\varepsilon(t,x)-u(t,x)|\theta_\beta(t-t_0)\,dt\,dx\\
&\leq(2LT\|w_M'\|_\infty+1)\omega_1^{B(M+1)}(\mu,\nu)+\omega_2^{B(M+1)}(\mu,\nu)\\
&\quad-\varepsilon\int_0^\infty\int_{\mathbb{R}^N}\int_0^T\int_{\mathbb{R}^N}\Theta_\beta(t)\theta_\mu(s-t)|u^\varepsilon(t,x)-u(s,y)|\\
&\qquad\times g\big[\rho_\nu(y-\cdot)w_M(|\cdot|+Lt)\big](x)\,ds\,dy\,dt\,dx\,.
\end{aligned}$$

For all $t_0\in[0,T[$, $\theta_\beta(\cdot-t_0)$ converges, as $\beta\to 0$ and in the weak-$*$ sense of the measures on $[0,T]$, to the Dirac mass at t_0; as $\beta\to 0$, we also have $\Theta_\beta\to\mathbf{1}_{[0,t_0]}$

everywhere and $|\Theta_\beta| \leq 1$. Since both u and u^ε are continuous $[0,T] \mapsto L^1_{\rm loc}(\mathbb{R}^N)$ and

$$t \mapsto \int_0^\infty \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \theta_\mu(s-t)|u^\varepsilon(t,x) - u(s,y)|\, g[\rho_\nu(y-\cdot)w_M(|\cdot|+Lt)](x)\, ds\, dy\, dx$$

is integrable on $[0,T]$ (see Remark 3.2), we can let $\beta \to 0$ to find

$$\begin{aligned} &\int_{B(M-LT)} |u^\varepsilon(t_0,x) - u(t_0,x)|\, dx \\ &\leq (2LT\|w_M'\|_\infty + 1)\omega_1^{B(M+1)}(\mu,\nu) + \omega_2^{B(M+1)}(\mu,\nu) + \varepsilon T_{\varepsilon,\mu,\nu,M}(t_0) \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} T_{\varepsilon,\mu,\nu,M}(t_0) = -\int_0^\infty \int_{\mathbb{R}^N} \int_0^{t_0} \int_{\mathbb{R}^N} &\theta_\mu(s-t)|u^\varepsilon(t,x) - u(s,y)| \\ &\times g[\rho_\nu(y-\cdot)w_M(|\cdot|+Lt)](x)\, ds\, dy\, dt\, dx \end{aligned} \tag{3.14}$$

satisfies

$$|T_{\varepsilon,\mu,\nu,M}(t_0)| \leq 2\|u_0\|_\infty \int_{\mathbb{R}^N} \int_0^T \int_{\mathbb{R}^N} |g[h_{\nu,M}(y,t,\cdot)](x)|\, dy\, dt\, dx$$

with $h_{\nu,M}(y,t,x) = \rho_\nu(y-x)w_M(|x|+Lt) \in C_c^\infty(\mathbb{R}^N \times [0,T] \times \mathbb{R}^N)$. This concludes the proof of the proposition for $t_0 < T$, and the estimate for $t_0 = T$ is obtained by letting $t_0 \to T$ in (3.3). $\square$

The result in Theorem 1.1 is then an easy consequence of the following lemma.

Lemma 3.3. *Let $u \in C([0,\infty[; L^1_{\rm loc}(\mathbb{R}^N))$ and $T > 0$. If B is a bounded subset of $\mathbb{R}^N$, we define $\omega_1^B(\mu,\nu)$ and $\omega_2^B(\mu,\nu)$ from u by (3.1) and (3.2), with $u_0 = u(0,\cdot)$. Then, as $(\mu,\nu) \to (0,0)$, $\omega_1^B(\mu,\nu)$ and $\omega_2^B(\mu,\nu)$ approach* 0.

Proof of Theorem 1.1. Let $T > 0$ and $M > LT$, with L a Lipschitz constant of f on $[-\|u_0\|_\infty, \|u_0\|_\infty]$. Let C_1 and $h_{\nu,M}$ be given by Proposition 3.1. Take $\alpha > 0$. Since u is the entropy solution to (1.3), it is in $C([0,\infty[; L^1_{\rm loc}(\mathbb{R}^N))$. Hence, applying Lemma 3.3, we fix $\mu \in]0,1[$ and $\nu \in]0,1[$ small enough so that

$$C_1\omega_1^{B(M+1)}(\mu,\nu) + \omega_2^{B(M+1)}(\mu,\nu) \leq \alpha.$$

By Remark 3.2, we can choose $\varepsilon_0 > 0$ (depending on ν and M) such that

$$2\varepsilon_0\|u_0\|_\infty \int_{\mathbb{R}^N} \int_0^T \int_{\mathbb{R}^N} |g[h_{\nu,M}(y,t,\cdot)](x)|\, dy\, dt\, dx \leq \alpha\,,$$

and Proposition 3.1 shows that for all $\varepsilon \leq \varepsilon_0$

$$\sup_{t\in[0,T]} \int_{B(M-LT)} |u^\varepsilon(t,x) - u(t,x)|\, dx \leq 2\alpha.$$

This reasoning can be made for all $T > 0$ and all $M > LT$, which proves that $u^\varepsilon \to u$ in $C([0,T]; L^1_{\rm loc}(\mathbb{R}^N))$ for all $T > 0$. $\square$

Proof of Lemma 3.3. The convergence of $\omega_2^B(\mu,\nu)$ is quite easy. Indeed, since $u_0 = u(0,\cdot) \in L^1_{\rm loc}(\mathbb{R}^N)$ and B is bounded, we know that

$$\int_B |u_0(x) - u_0(x+z)|\, dx \to 0 \quad \text{as } z \to 0.$$

By continuity of $u : [0,\infty[\mapsto L^1_{\rm loc}(\mathbb{R}^N)$ and since $\widetilde{B}$ is bounded, we also have $\|u(s,\cdot)-u_0\|_{L^1(\widetilde{B})}\to 0$ as $s\to 0$. Hence, this proves that $\omega_2^B(\mu,\nu)\to 0$ as $(\mu,\nu)\to 0$.

The convergence of $\omega_1^B(\mu,\nu)$ is a bit more tricky. We split it in two parts:

$$
\begin{aligned}
\omega_1^B(\mu,\nu) &\le \sup_{0<t<T}\Big(\sup_{|z|<\nu}\int_B |u(t,x)-u(t,x+z)|\,dx\Big)\\
&\quad + \sup_{0<t<T}\Big(\sup_{0<r<\mu\,,|z|<\nu}\int_B |u(t,x+z)-u(t+r,x+z)|\,dx\Big)\\
&\le \sup_{0<t<T}\Big(\sup_{|z|<\nu}\int_B |u(t,x)-u(t,x+z)|\,dx\Big) \qquad (3.15)\\
&\quad + \sup_{0<t<T}\Big(\sup_{0<r<\mu}\int_{\widetilde{B}} |u(t,y)-u(t+r,y)|\,dy\Big). \qquad (3.16)
\end{aligned}
$$

By hypothesis, $u\in C([0,T+1];L^1(\widetilde{B}))$; hence, u is uniformly continuous $[0,T+1]\mapsto L^1(\widetilde{B})$ and

$$
\begin{aligned}
&\sup_{0<t<T}\Big(\sup_{0<r<\mu}\int_{\widetilde{B}} |u(t,y)-u(t+r,y)|\,dy\Big)\\
&\le \sup_{(t,s)\in[0,T+1]^2\,,\ 0<s-t<\mu}\|u(t,\cdot)-u(s,\cdot)\|_{L^1(\widetilde{B})}\to 0
\end{aligned} \qquad (3.17)
$$

as $\mu\to 0$. Moreover, since $u\in C([0,T];L^1(\widetilde{B}))$, the set $\mathcal{K}=\{u(t,\cdot)\,,\ 0\le t\le T\}$ is compact in $L^1(\widetilde{B})$; therefore, by Kolmogorov's compactness theorem, the translations are equicontinuous on $\mathcal{K}$, that is to say

$$
\sup_{v\in\mathcal{K}}\Big(\sup_{|z|<\nu}\int_B |v(x)-v(x+z)|\,dx\Big)\to 0
$$

as $\nu\to 0$. This quantity bounds (3.15), which proves, together with (3.17), that $\omega_1^B(\mu,\nu)\to 0$ as $(\mu,\nu)\to 0$. □

4. Proof of the error estimate

In this section, we prove Theorem 1.3 beginning with a stronger version of Lemma 3.3 in the case of more regular functions.

Lemma 4.1. *Let $u\in \mathrm{Lip}([0,\infty[;L^1(\mathbb{R}^N))$ such that $\sup_{t\ge 0}|u(t,\cdot)|_{BV(\mathbb{R}^N)}<\infty$. We define $\omega_1^{\mathbb{R}^N}(\mu,\nu)$ and $\omega_2^{\mathbb{R}^N}(\mu,\nu)$ from u by* (3.1) *and* (3.2)*, with $T=\infty$, $u_0=u(0,\cdot)$ and $B=\mathbb{R}^N$. Then $\omega_1^{\mathbb{R}^N}(\mu,\nu)=\mathcal{O}(\mu+\nu)$ and $\omega_2^{\mathbb{R}^N}(\mu,\nu)=\mathcal{O}(\mu+\nu)$.*

Proof. It is well-known (see e.g. [7] or (5.7)) that if $v\in BV(\mathbb{R}^N)$, then

$$
\int_{\mathbb{R}^N}|v(x+h)-v(x)|\,dx\le |h|\,|v|_{BV(\mathbb{R}^N)}. \qquad (4.1)
$$

Thus,

$$
\sup_{|z|<\nu}\int_{\mathbb{R}^N}|u_0(x)-u_0(x+z)|\,dx=\mathcal{O}(\nu)
$$

and, since $u:[0,\infty[\mapsto L^1(\mathbb{R}^N)$ is Lipschitz continuous, we deduce that $\omega_2^{\mathbb{R}^N}(\mu,\nu)=\mathcal{O}(\mu+\nu)$. We split $\omega_1^{\mathbb{R}^N}(\mu,\nu)$ as in the proof of Lemma 3.3 (with $\widetilde{B}=\mathbb{R}^N$ here). By the Lipschitz continuity of u, (3.16) is a $\mathcal{O}(\mu)$; by (4.1) and the bound on $|u(t,\cdot)|_{BV(\mathbb{R}^N)}$, (3.15) is a $\mathcal{O}(\nu)$. This concludes the proof of the lemma. □

Proof of Theorem 1.3. Since $u_0 \in L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)\cap BV(\mathbb{R}^N)$, it is classical that $|u(t,\cdot)|_{BV(\mathbb{R}^N)} \le |u_0|_{BV(\mathbb{R}^N)}$. The function f being regular, the BV semi-norm of $f(u(t,\cdot))$ is also bounded and, thanks to $\partial_t u + \mathrm{div}(f(u)) = 0$, we see that u is Lipschitz continuous $[0,\infty[\mapsto L^1(\mathbb{R}^N)$. Hence, Lemma 4.1 and (3.13) show that, for all $T>0$, for all $M>LT$ and all $t_0\in[0,T]$, if $\mu\in]0,1[$ and $\nu\in]0,1[$,

$$\int_{B(M-LT)} |u^\varepsilon(t_0,x)-u(t_0,x)|\,dx \le C_0(2LT\|w_M'\|_\infty+2)(\mu+\nu)+\varepsilon T_{\varepsilon,\mu,\nu,M}(t_0), \tag{4.2}$$

where we recall that $T_{\varepsilon,\mu,\nu,M}(t_0)$ is defined by (3.14).

To bound $T_{\varepsilon,\mu,\nu,M}(t_0)$, we use (5.1). We handle the case $\lambda\in]1,2[$, the other one being easier (and, anyway, well-known). We define $\beta=-N-(\lambda-2)$. It is not hard to check, differentiating under the integral sign, that

$$\begin{aligned} g[h_{\nu,M}(y,t,\cdot)](x) &= E_\lambda|\cdot|^\beta * (\Delta_x h_{\nu,M}(y,t,\cdot))(x)\\ &= E_\lambda \mathrm{div}\left(|\cdot|^\beta * \nabla_x h_{\nu,M}(y,t,\cdot)\right)(x) \end{aligned}$$

(recall that $h_{\nu,M}(y,t,x)=\rho_\nu(y-x)w_M(|x|+Lt)\in C_c^\infty(\mathbb{R}^N\times[0,T]\times\mathbb{R}^N)$). Let A be such that the support of $h_{\nu,M}(y,t,\cdot)$ is contained in the ball of center 0 and radius A. From the definition of the convolution product, we see that, for $|x|>A$, $||\cdot|^\beta*\nabla_x h_{\nu,M}(y,t,\cdot)(x)|\le\Lambda(|x|-A)^\beta$; hence, $|\cdot|^\beta*\nabla_x h_{\nu,M}(y,t,\cdot)(x)$ goes to 0, as $|x|\to\infty$, quicker than $|x|^{-N+1}$ (because $\beta=-N-(\lambda-2)<-N+1$). We know that $u^\varepsilon(t,\cdot)$ is regular for all $t>0$ (see [6]), and that $\|\nabla u^\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)}\le|u_0|_{BV(\mathbb{R}^N)}$ (this can be easily seen letting $\delta\to0$ in (2.1) — we have noticed that the construction of u^ε in subsection 2.1 is valid for initial data in $L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)\cap BV(\mathbb{R}^N)$). We can therefore use Stokes formula on a ball of radius R and let $R\to\infty$ to find

$$\begin{aligned} &\int_{\mathbb{R}^N} |u^\varepsilon(t,x)-u(s,y)|\,g[h_{\nu,M}(y,t,\cdot)](x)\,dx\\ &= -E_\lambda\int_{\mathbb{R}^N}\nabla_x(|u^\varepsilon(t,\cdot)-u(s,y)|)(x)\cdot(|\cdot|^\beta*\nabla_x h_{\nu,M}(y,t,\cdot))(x)\,dx\\ &= -E_\lambda\int_{\mathbb{R}^N}\mathrm{sgn}(u^\varepsilon(t,x)-u(s,y))\nabla u^\varepsilon(t,x)\cdot(|\cdot|^\beta*\nabla_x h_{\nu,M}(y,t,\cdot))(x)\,dx\,, \end{aligned}$$

which implies

$$\begin{aligned} &\Big|\int_{\mathbb{R}^N}|u^\varepsilon(t,x)-u(s,y)|\,g[h_{\nu,M}(y,t,\cdot)](x)\,dx\Big|\\ &\le |E_\lambda|\int_{\mathbb{R}^N}|\nabla u^\varepsilon(t,x)|\,|(|\cdot|^\beta*\nabla_x h_{\nu,M}(y,t,\cdot))(x)|\,dx. \end{aligned}$$

Therefore, by (3.14),

$$\begin{aligned} &|T_{\varepsilon,\mu,\nu,M}(t_0)|\\ &\le |E_\lambda|\int_{\mathbb{R}^N}\int_0^{t_0}\int_{\mathbb{R}^N}|\nabla u^\varepsilon(t,x)|\,|(|\cdot|^\beta*\nabla_x h_{\nu,M}(y,t,\cdot))(x)|\,dy\,dt\,dx. \end{aligned} \tag{4.3}$$

We choose w_M such that $(w_M')_{M\ge1}$ is bounded by C_1. Let $\delta\in]0,1[$ and B_δ be the ball of center 0 and radius δ; cutting as in the proof of Lemma 5.1 and using Stokes formula, we have

$$\begin{aligned}
&(|\cdot|^\beta * \nabla_x h_{\nu,M}(y,t,\cdot))(x)\\
&= \int_{B_\delta} |z|^\beta \nabla_x h_{\nu,M}(y,t,x-z)\,dz - \int_{B_\delta^c} |z|^\beta \nabla_z (h_{\nu,M}(y,t,x-z))\,dz\\
&= \int_{B_\delta} |z|^\beta \nabla_x h_{\nu,M}(y,t,x-z)\,dz - \delta^\beta \int_{\partial B_\delta^c} h_{\nu,M}(y,t,x-z)\mathbf{n}(z)\,d\sigma_\delta(z)\\
&\quad + \beta \int_{B_\delta^c} h_{\nu,M}(y,t,x-z)|z|^{\beta-1}\frac{z}{|z|}\,dz
\end{aligned} \tag{4.4}$$

(σ_δ is the $(N-1)$-dimensional measure on ∂B_δ^c and $\mathbf{n}$ is the unit normal to ∂B_δ^c outward to B_δ^c). Since

$$|h_{\nu,M}(y,t,x)| = |\rho_\nu(y-x)w_M(|x|+Lt)| \le \rho_\nu(y-x)$$

and

$$\begin{aligned}
|\nabla_x h_{\nu,M}(y,t,x)| &= \Big| -\nabla\rho_\nu(y-x)w_M(|x|+Lt) + \rho_\nu(y-x)w_M'(|x|+Lt)\frac{x}{|x|}\Big|\\
&\le |\nabla\rho_\nu(y-x)| + C_1\rho_\nu(y-x),
\end{aligned}$$

Equation (4.4) shows that

$$\begin{aligned}
&|(|\cdot|^\beta * \nabla_x h_{\nu,M}(y,t,\cdot))(x)|\\
&\le \int_{B_\delta} |z|^\beta \left(|\nabla\rho_\nu(y-x+z)| + C_1\rho_\nu(y-x+z)\right)\,dz\\
&\quad + \delta^\beta \int_{\partial B_\delta^c} \rho_\nu(y-x+z)\,d\sigma_\delta(z) + |\beta| \int_{B_\delta^c} \rho_\nu(y-x+z)|z|^{\beta-1}\,dz.
\end{aligned}$$

By Fubini-Tonelli's theorem and (4.3), we obtain

$$\begin{aligned}
|T_{\varepsilon,\mu,\nu,M}(t_0)| &\le |E_\lambda| \int_{\mathbb{R}^N}\int_0^{t_0}\int_{\mathbb{R}^N} |\nabla u^\varepsilon(t,x)| \Big(\int_{B_\delta} |z|^\beta \big(|\nabla\rho_\nu(y-x+z)|\\
&\qquad + C_1\rho_\nu(y-x+z)\big)\,dz + \delta^\beta \int_{\partial B_\delta^c} \rho_\nu(y-x+z)\,d\sigma_\delta(z)\\
&\qquad + |\beta| \int_{B_\delta^c} \rho_\nu(y-x+z)|z|^{\beta-1}\,dz\Big)\,dy\,dt\,dx\\
&\le |E_\lambda| \left(\|\nabla\rho_\nu\|_{L^1(\mathbb{R}^N)} + C_1\right) \||\cdot|^\beta\|_{L^1(B_\delta)} \|\nabla u^\varepsilon\|_{L^1(]0,t_0[\times\mathbb{R}^N)}\\
&\quad + |E_\lambda|\delta^\beta \sigma_\delta(\partial B_\delta^c)\|\nabla u^\varepsilon\|_{L^1(]0,t_0[\times\mathbb{R}^N)}\\
&\quad + |E_\lambda|\,|\beta|\,\||\cdot|^{\beta-1}\|_{L^1(B_\delta^c)}\|\nabla u^\varepsilon\|_{L^1(]0,t_0[\times\mathbb{R}^N)}.
\end{aligned}$$

By change of variable, $\||\cdot|^\beta\|_{L^1(B_\delta)} = C_2\delta^{N+\beta}$, $\||\cdot|^{\beta-1}\|_{L^1(B_\delta^c)} = C_3\delta^{N+\beta-1}$ and $\sigma_\delta(\partial B_\delta^c) = C_4\delta^{N-1}$, where C_2, C_3 and C_4 do not depend on δ (recall that $\beta-1 < -N < \beta$). Since $\|\nabla u^\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)} \le |u_0|_{BV(\mathbb{R}^N)}$, we deduce that

$$|T_{\varepsilon,\mu,\nu,M}(t_0)| \le C_5 T(\|\nabla\rho_\nu\|_{L^1(\mathbb{R}^N)} + 1)\delta^{N+\beta} + C_5 T\delta^{N+\beta-1} \tag{4.5}$$

where C_5 does not depend on t_0, ε, μ, ν, M or δ.

Choosing smoothing kernels $(\rho_\nu)_{\nu>0}$ of the kind $\rho_\nu(x) = \nu^{-N}\rho(\nu^{-1}x)$, we have $\|\nabla\rho_\nu\|_{L^1(\mathbb{R}^N)} = C_6\nu^{-1}$. Since $(w_M')_{M\ge1}$ is bounded by C_1, (4.2) and (4.5) give, for all $T>0$, for all $M>LT$ and all $t_0\in[0,T]$,

$$\begin{aligned}&\int_{B(M-LT)}|u^\varepsilon(t_0,x)-u(t_0,x)|\,dx\\&\le C_0(2LTC_1+2)(\mu+\nu)+\varepsilon\Big(\frac{C_5C_6T\delta^{2-\lambda}}{\nu}+C_5T\delta^{2-\lambda}+C_5T\delta^{1-\lambda}\Big)\end{aligned}$$

(we have $N+\beta=2-\lambda$). We let $M\to\infty$ and $\mu\to 0$; since $\nu<1$, this gives

$$\|u^\varepsilon(t_0,\cdot)-u(t_0,\cdot)\|_{L^1(\mathbb{R}^N)}=\mathcal{O}\Big(\nu+\varepsilon\big(\frac{\delta^{2-\lambda}}{\nu}+\delta^{1-\lambda}\big)\Big).$$

Minimizing on δ and then on ν, we see that the best choices are (up to multiplicative constants) $\delta=\nu$ and $\nu=\varepsilon^{1/\lambda}$, which proves Theorem 1.3. □

5. Appendix

5.1. An expression and an estimate of $g[\varphi]$.

Lemma 5.1. *Let $\lambda\in]1,2]$. There exist $E_\lambda\in\mathbb{R}$ and $C_\lambda>0$ such that, for all $\varphi\in\mathcal{S}(\mathbb{R}^N)$,*

$$g[\varphi]=\begin{cases}E_\lambda|\cdot|^{-N-(\lambda-2)}*\Delta\varphi & \text{for } \lambda\in]1,2[\\ E_\lambda\Delta\varphi & \text{for } \lambda=2\end{cases}\tag{5.1}$$

and $\|g[\varphi]\|_{L^1(\mathbb{R}^N)}\le C_\lambda\big(\|\nabla\varphi\|_{L^1(\mathbb{R}^N)}+\|\Delta\varphi\|_{L^1(\mathbb{R}^N)}\big)$.

Proof. If $\lambda=2$, the result is obvious since, up to a multiplicative constant, $g[\varphi]$ is $\Delta\varphi$. We thus assume that $\lambda\in]1,2[$ and we have

$$g[\varphi]=\mathcal{F}^{-1}(|\cdot|^\lambda\mathcal{F}(\varphi))=(2i\pi)^{-2}\mathcal{F}^{-1}(|\cdot|^{\lambda-2}\mathcal{F}(\Delta\varphi))$$

(note that $|\cdot|^{\lambda-2}\in L^1_{\rm loc}(\mathbb{R}^N)$, as $\lambda-2>-N$, and that $\mathcal{F}(\Delta\varphi)\in\mathcal{S}(\mathbb{R}^N)$, so that $|\cdot|^{\lambda-2}\mathcal{F}(\Delta\varphi)$ is integrable on $\mathbb{R}^N$). Since $\lambda-2\in]-N,0[$, it is classical that $\mathcal{F}^{-1}(|\cdot|^{\lambda-2})=C_1|\cdot|^{-N-(\lambda-2)}$ in $\mathcal{S}'(\mathbb{R}^N)$, for some $C_1\in\mathbb{R}$. We can then check, using the definition (by duality) of $\mathcal{F}^{-1}$ on $\mathcal{S}'(\mathbb{R}^N)$, that $\mathcal{F}^{-1}(|\cdot|^{\lambda-2}\mathcal{F}(\Delta\varphi))=C_1|\cdot|^{-N-(\lambda-2)}*\Delta\varphi$, which proves (5.1).

Let $\beta=-N-(\lambda-2)\in]-N,0[$. We now estimate $\|\,|\cdot|^\beta*\Delta\varphi\|_{L^1(\mathbb{R}^N)}$, which will conclude the proof (note that this estimate is not a straightforward consequence of Young's inequalities for convolution, since $|\cdot|^\beta$ is not integrable on $\mathbb{R}^N$). We have, if $\mathbf{1}_B$ is the characteristic function of the ball B of center 0 and radius 1 and $\mathbf{1}_{B^c}$ is the characteristic function of $B^c=\mathbb{R}^N\backslash B$,

$$|\cdot|^\beta*\Delta\varphi=(\mathbf{1}_B|\cdot|^\beta)*\Delta\varphi+(\mathbf{1}_{B^c}|\cdot|^\beta)*\Delta\varphi.\tag{5.2}$$

But $\mathbf{1}_B|\cdot|^\beta\in L^1(\mathbb{R}^N)$ (because $\beta>-N$), and thus

$$\|(\mathbf{1}_B|\cdot|^\beta)*\Delta\varphi\|_{L^1(\mathbb{R}^N)}\le\|\,\mathbf{1}_B|\cdot|^\beta\,\|_{L^1(\mathbb{R}^N)}\|\Delta\varphi\|_{L^1(\mathbb{R}^N)}.\tag{5.3}$$

By Stokes formula, we write

$$\begin{aligned}&(\mathbf{1}_{B^c}|\cdot|^\beta)*\Delta\varphi(x)\\&=\int_{B^c}|y|^\beta\Delta\varphi(x-y)\,dy\\&=-\int_{\partial B^c}\nabla\varphi(x-y)\cdot\mathbf{n}(y)\,d\sigma(y)+\beta\int_{B^c}\nabla\varphi(x-y)\cdot\big(|y|^{\beta-1}\frac{y}{|y|}\big)\,dy\end{aligned}$$

where $\mathbf{n}$ is the outward unit normal to B^c and σ is the measure on ∂B^c. We deduce that

$$|(\mathbf{1}_{B^c}|\cdot|^\beta)*\Delta\varphi(x)| \le \int_{\partial B^c} |\nabla\varphi(x-y)|\,d\sigma(y) + |\beta|\int_{B^c}|\nabla\varphi(x-y)|\,|y|^{\beta-1}\,dy$$

and, integrating this thanks to Fubini-Tonelli's theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |(\mathbf{1}_{B^c}|\cdot|^\beta)*\Delta\varphi(x)|\,dx\\
&\le \int_{\partial B^c}\int_{\mathbb{R}^N}|\nabla\varphi(x-y)|\,dxd\sigma(y) + |\beta|\int_{B^c}\int_{\mathbb{R}^N}|\nabla\varphi(x-y)|\,dx\,|y|^{\beta-1}\,dy \qquad (5.4)\\
&= \Big(\sigma(\partial B^c)+|\beta|\int_{B^c}|y|^{\beta-1}\,dy\Big)\int_{\mathbb{R}^N}|\nabla\varphi(z)|\,dz.
\end{aligned}$$

Since $\beta-1=-N-(\lambda-2)-1=-N-\lambda+1<-N$, $\int_{B^c}|y|^{\beta-1}\,dy$ is finite. Gathering (5.3) and (5.4) in (5.2), we deduce that $\||\cdot|^\beta*\Delta\varphi\|_{L^1(\mathbb{R}^N)} \le C(\|\Delta\varphi\|_{L^1(\mathbb{R}^N)} + \|\nabla\varphi\|_{L^1(\mathbb{R}^N)})$ for some C not depending on φ, and the proof is complete. ☐

5.2. **Technical lemmas on the kernel of g.** The results in the following two lemmas have already been used in [6], but their proofs were left to the reader. We include them here for sake of completeness.

Lemma 5.2. *Let $r>0$, $w_0\in L^1(\mathbb{R}^N)$ and, for $t>0$, $w(t,\cdot)=K_r(t,\cdot)*w_0$. Then, for all $\varphi\in C_c^\infty([0,\infty[\times\mathbb{R}^N)$ and all $t_0>0$,*

$$\begin{aligned}
&\int_0^{t_0}\int_{\mathbb{R}^N} w(t,x)\partial_t\varphi(t,x)-rw(t,x)g[\varphi(t,\cdot)](x)\,dt\,dx\\
&=\int_{\mathbb{R}^N} w(t_0,x)\varphi(t_0,x)\,dx-\int_{\mathbb{R}^N} w_0(x)\varphi(0,x)\,dx.
\end{aligned}$$

Proof. We ignore, as in the proof of Proposition 2.2, the space variable. Since $K_r(t)$ and w_0 are integrable, $w(t)$ is integrable and we have

$$\mathcal{F}^{-1}(w(t))=\mathcal{F}^{-1}(K_r(t))\mathcal{F}(w_0)=e^{-rt|\cdot|^\lambda}\mathcal{F}^{-1}(w_0) \qquad (5.5)$$

(note that, since $K_r(t)$ is even, $\mathcal{F}^{-1}(K_r(t))=\mathcal{F}(K_r(t))$). By Fubini's theorem, for all $(a,b)\in L^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} a\mathcal{F}^{-1}(b)=\int_{\mathbb{R}^N}\mathcal{F}^{-1}(a)b. \qquad (5.6)$$

Thus, writing $g[\varphi(t)]=\mathcal{F}^{-1}(|\cdot|^\lambda\mathcal{F}(\varphi(t)))$ and $\partial_t\varphi(t)=\mathcal{F}^{-1}(\mathcal{F}(\partial_t\varphi(t)))$, we have, thanks to (5.5) and (5.6), for all $t>0$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} w(t)\partial_t\varphi(t)-rw(t)g[\varphi(t)]\\
&=\int_{\mathbb{R}^N} e^{-rt|\xi|^\lambda}\mathcal{F}^{-1}(w_0)(\xi)\mathcal{F}(\partial_t\varphi(t))(\xi)-r|\xi|^\lambda e^{-rt|\xi|^\lambda}\mathcal{F}^{-1}(w_0)(\xi)\mathcal{F}(\varphi(t))(\xi)\,d\xi\\
&=\int_{\mathbb{R}^N}\partial_t\Big(e^{-rt|\xi|^\lambda}\mathcal{F}(\varphi(t))(\xi)\Big)\mathcal{F}^{-1}(w_0)(\xi)\,d\xi.
\end{aligned}$$

The mapping $(t,\xi)\mapsto e^{-rt|\xi|^\lambda}\mathcal{F}^{-1}(w_0)(\xi)$ is bounded on $]0,t_0[\times\mathbb{R}^N$ and, by regularity of φ, $(t,\xi)\mapsto|\xi|^\lambda\mathcal{F}(\varphi(t))(\xi)$ and $(t,\xi)\mapsto\mathcal{F}(\partial_t\varphi(t))(\xi)$ are integrable on

$]0,t_0[\times\mathbb{R}^N$ (see e.g. (2.10)); hence, integrating the preceding equality on $]0,t_0[$ and using Fubini's theorem, we find

$$\begin{aligned}&\int_0^{t_0}\int_{\mathbb{R}^N} w(t)\partial_t\varphi(t) - rw(t)g[\varphi(t)]\,dt\\&= \int_{\mathbb{R}^N}\Big(e^{-rt_0|\xi|^\lambda}\mathcal{F}(\varphi(t_0))(\xi) - \mathcal{F}(\varphi(0))(\xi)\Big)\mathcal{F}^{-1}(w_0)(\xi)\,d\xi.\end{aligned}$$

Using once again (5.5) and (5.6), we get

$$\begin{aligned}&\int_0^{t_0}\int_{\mathbb{R}^N} w(t)\partial_t\varphi(t) - rw(t)g[\varphi(t)]\,dt\\&= \int_{\mathbb{R}^N}\mathcal{F}^{-1}(w(t_0))\mathcal{F}(\varphi(t_0)) - \mathcal{F}^{-1}(w_0)\mathcal{F}(\varphi(0))\\&= \int_{\mathbb{R}^N} w(t_0)\varphi(t_0) - w_0\varphi(0)\end{aligned}$$

which concludes the proof. □

Lemma 5.3. *Let $r>0$ and $w_0\in W^{1,1}(\mathbb{R}^N)\cap C^1(\mathbb{R}^N)$. We define, for $t>0$, $w(t,\cdot)=K_r(t,\cdot)*w_0$. Then, for all $\nu>0$ and all $t>0$,*

$$\|w(t,\cdot)-w_0\|_{L^1(\mathbb{R}^N)} \le 2\|w_0\|_{L^1(\mathbb{R}^N)}\int_{|y|\ge\nu}K_r(t,y)\,dy + \nu\|\nabla w_0\|_{L^1(\mathbb{R}^N)}.$$

Proof. The proof relies on classical cuttings of integration domain when approximate units are involved. Since $K_r(t,\cdot)$ is non-negative with integral equal to 1, we can write

$$\begin{aligned}|w(t,x)-w_0(x)| &= \Big|\int_{\mathbb{R}^N}K_r(t,y)(w_0(x-y)-w_0(x))\,dy\Big|\\&\le \int_{\mathbb{R}^N}K_r(t,y)|w_0(x-y)-w_0(x)|\,dy.\end{aligned}$$

Now,

$$\begin{aligned}&\|w(t,\cdot)-w_0\|_{L^1(\mathbb{R}^N)}\\&\le \int_{|y|\ge\nu}K_r(t,y)\int_{\mathbb{R}^N}|w_0(x-y)-w_0(x)|\,dx\,dy\\&\quad+\int_{|y|<\nu}K_r(t,y)\int_{\mathbb{R}^N}|w_0(x-y)-w_0(x)|\,dx\,dy\\&\le 2\|w_0\|_{L^1(\mathbb{R}^N)}\int_{|y|\ge\nu}K_r(t,y)\,dy + \sup_{|z|<\nu}\int_{\mathbb{R}^N}|w_0(x+z)-w_0(x)|\,dx.\end{aligned}$$

Using Fubini-Tonelli's theorem and a change of variable, we write

$$\begin{aligned}\int_{\mathbb{R}^N}|w_0(x+z)-w_0(x)|\,dx &\le \int_{\mathbb{R}^N}\int_0^1|\nabla w_0(x+\zeta z)|\,|z|\,d\zeta\,dx\\&\le |z|\int_{\mathbb{R}^N}|\nabla w_0(y)|\,dy\,,\end{aligned}\tag{5.7}$$

and the proof is complete. □

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