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A REMARK ON THE EXISTENCE OF LARGE SOLUTIONS VIA SUB AND SUPERSOLUTIONS

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Abstract. We study the boundary blow-up elliptic problem $\Delta u = a(x)f(u)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, with $u|_{\partial\Omega} = +\infty$. Under suitable growth assumptions on a near $\partial\Omega$ and on f both at zero and at infinity, we prove the existence of at least a positive solution. Our proof is based on the method of sub and supersolutions, which permits on the one hand oscillatory behaviour of $f(u)$ at infinity and on the other hand positive weights $a(x)$ which are unbounded and/or oscillatory near the boundary.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain. In this work we consider the boundary blow-up elliptic problem

$$\begin{gathered} \Delta u = a(x)f(u) \quad \text{in } \Omega \\ u = +\infty \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where a is a Hölder continuous positive function defined in Ω and f is locally Hölder in $(0, +\infty)$. We are interested in the existence of positive classical solutions to (1.1), that is solutions $u \in C^2(\Omega)$ to $\Delta u = a(x)f(u)$ such that $u(x) \to +\infty$ as $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0+$.

Boundary blow-up problems like (1.1) have received a great deal of attention in the recent years. Without being exhaustive with the references, let us quote [2] (as the starting point for these problems), [7], [1], [4], [9], [10], [11], [12] and [5] (see also references therein).

In the reference situation $f(u) = u^p$ (see hypotheses (1.3) below), existence and uniqueness of positive solutions to problem (1.1) have been obtained before under different kinds of assumptions on the weight $a(x)$. For instance in [1] and [11] when a is bounded and bounded away from zero, [5] when a is bounded, but *is* zero on $\partial\Omega$, with a prescribed behaviour or [12] and [3] where a goes to $+\infty$ also in a completely determined way. Our existence result (Theorem 1.1) covers all these situations, and also more general weights which can behave as the three cases above in different parts of $\partial\Omega$. The advantage of our approach is that all possible cases

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are treated together with a very simple proof, based on the case $a(x) \equiv 1$. Also, at the best of our knowledge, this seems to be the first time where the method of sub and supersolutions is used to prove existence of solutions to boundary blow-up problems.

We start by quoting our hypotheses on a and f. We will assume that $a \in C^\nu(\Omega)$ for some $0 < \nu < 1$, $a > 0$ in Ω, and that $f \in C^\nu(0, +\infty)$. In addition there exist constants $C_1, C_2 > 0$ and $\gamma_2 \geq \gamma_1 > -2$ such that

$$C_2 d(x)^{\gamma_2} \leq a(x) \leq C_1 d(x)^{\gamma_1}, \quad x \in \Omega. \tag{1.2}$$

Note that γ_1 and γ_2 can have different signs, and so a is permitted to be bounded in some parts of $\partial\Omega$, and to go to $+\infty$ or even oscillate in some others (it can also go to zero).

For the nonlinearity f we further assume that there exist $p_1 \geq p_2 > 1$ such that

$$f(u) \leq C_1 u^{p_1} \quad u \in \mathbb{R}^+, \quad f(u) \geq C_2 u^{p_2} \quad \text{for large } u\,. \tag{1.3}$$

Note that we can take the constants C_1 and C_2 to be the same as in (1.2). These assumptions allow $f(u)$ to be oscillating for large u, i.e. f does not need to be increasing at infinity. We remark that $\gamma_2 \geq \gamma_1$ and $p_1 \geq p_2$ are nothing else but compatibility conditions, and $\gamma_1 > -2$ is necessary in order to have positive solutions to (1.1) (compare with [3] in the radial case and $f(u) = u^p$). We now state our Theorem.

Theorem 1.1. *Assume a and f verify hypotheses* (1.2) *and* (1.3)*. Then problem (1.1) has at least a positive solution u which verifies*

$$D_1 d(x)^{-\alpha_1} \leq u(x) \leq D_2 d(x)^{-\alpha_2} \quad \text{in } \Omega, \tag{1.4}$$

where $\alpha_i = (2 + \gamma_i)/(p_i - 1)$, and D_1, D_2 are positive constants.

Remark 1.2. (a) Note that $p_1 \geq p_2$, $\gamma_1 \leq \gamma_2$ imply that $\alpha_1 \leq \alpha_2$, and thus (1.4) makes sense.
b) In the light of the possible oscillatory behaviour of a and f, according to hypotheses (1.2) and (1.3), one would expect that estimates (1.4) can not be improved in general, even if $p_1 = p_2$ and $\gamma_1 = \gamma_2$ ($\alpha_1 = \alpha_2$), in contrast with the case when a has a prescribed asymptotics near $\partial\Omega$ and f near $+\infty$.
c) If $f(u) = u^p$, $p > 1$, uniqueness of positive solutions verifying (1.4) can be achieved for positive weights a satisfying (1.2) with $\gamma_1 = \gamma_2$, through an adaptation of the proof of Theorem 3.4 in [8].
d) The regularity assumptions on a and f can of course be relaxed to continuity, obtaining weak solutions to (1.1) in that case.
e) Theorem 1.1 can be adapted to nonlinearities with a different type of growth, for instance exponential:

$$f(u) \leq C_1 e^{p_1 u} \quad u \in \mathbb{R}^+, \quad f(u) \geq C_2 e^{p_2 u} \quad \text{for large } u, \tag{1.5}$$

and we obtain the existence of at least a classical solution u such that $2\lambda_2 \log d + C \leq u \leq 2\lambda_1 \log d + C'$, where $\lambda_i = (\gamma_i + 2)/p_i$.

2. Results

This section is devoted to the proof of Theorem 1.1 and the method of sub and supersolutions. First we state and prove an adaptation of the method of sub and supersolutions to problem (1.1) (Lemma 2.1 below is indeed a slight generalization of Lemma 4 in [5], which was not proved there), then we introduce an auxiliary problem which will turn out to be very important for our purposes, and we finally will proceed to the proof of Theorem 1.1.

A function $\underline{u} \in C^2(\Omega)$ is a (classical) subsolution to problem (1.1) if $\underline{u} = +\infty$ on $\partial\Omega$ and $\Delta\underline{u} \geq a(x)f(\underline{u})$ in Ω. Similarly, $\bar{u}$ is a supersolution if $\bar{u} = +\infty$ on $\partial\Omega$ and $\Delta\bar{u} \leq a(x)f(\bar{u})$ in Ω. When $\underline{u}$ and $\bar{u}$ are ordered we have the next result.

Lemma 2.1. *Assume there exist a subsolution $\underline{u}$ and a supersolution $\bar{u}$ to the problem (1.1) such that $\underline{u} \leq \bar{u}$. Then there exists at least a classical solution u such that $\underline{u} \leq u \leq \bar{u}$.*

Proof. For $n \in \mathbb{N}$, we introduce the domain $\Omega_n := \{x \in \Omega : d(x) > 1/n\}$, and consider the problem

$$\begin{aligned} \Delta u = a(x)f(u) \quad &\text{in } \Omega_n \\ u = \underline{u} \quad &\text{on } \partial\Omega_n \,. \end{aligned} \tag{2.1}$$

Since $\underline{u}$ is a subsolution and $\bar{u}$ a supersolution, this problem has at least a positive classical solution u_n such that $\underline{u} \leq u_n \leq \bar{u}$. This in particular gives local bounds for the sequence $\{u_n\}$ which in turn leads to local bounds in $C^{2,\nu}$ (cf. [6]). Thus for every $k \in \mathbb{N}$, we can select a subsequence $\{u_n^k\}$ which converges in $C^2(\overline{\Omega}_k)$. A diagonal procedure gives a subsequence (denoted again by $\{u_n\}$) which converges to a function u in $C^2_{\rm loc}(\Omega)$. Passing to the limit in (2.1) we see that u is a classical solution of the equation in (1.1), verifying $\underline{u} \leq u \leq \bar{u}$. In particular, we deduce that $u = +\infty$ on $\partial\Omega$. This proves the Lemma. □

As already remarked, a fundamental role in our approach is played by the well-known blow-up problem:

$$\begin{aligned} \Delta U = U^{r_i} \quad &\text{in } \Omega \\ U = +\infty \quad &\text{on } \partial\Omega \,, \end{aligned}$$

where $r_i = 1 + 2/\alpha_i > 1$. This problem has a unique positive solution U_i such that $Cd(x)^{-\alpha_i} \leq U_i(x) \leq C'd(x)^{-\alpha_i}$, for some positive constants C and C' (see [1]).

Proof of Theorem 1.1. The proof consists in choosing adequate ordered sub and supersolutions in terms of the functions U_1 and U_2 defined above. Indeed, we set $\underline{u} = \lambda U_1$. Then $\underline{u}$ will be a subsolution provided that

$$\lambda U_1^{r_1} \geq a(x)f(\lambda U_1)\,.$$

By hypothesis (1.3) on f, this is a consequence of $\lambda \leq (C_1 \sup_\Omega a(x)U_1(x)^{p_1-r_1})^{-\frac{1}{p_1-1}}$, which holds for small λ if the supremum is finite. But note that $a(x)U_1^{p_1-r_1} \leq Cd(x)^{\gamma_1-\alpha_1(p_1-r_1)} = C$ in virtue of hypotheses (1.2), and the claim follows. In a similar way we can see that $\bar{u} = \Lambda U_2$ is a supersolution for large Λ. Since $\alpha_1 \leq \alpha_2$, it also follows that $\lambda U_1 \leq \Lambda U_2$, and Lemma 2.1 shows that there exists at least a positive classical solution to (1.1), which in addition verifies the estimates (1.4). This proves the Theorem. □

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