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Stability of solutions for nonlinear nonautonomous differential-delay equations in Hilbert spaces *

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Abstract

We consider nonlinear non-autonomous differential-delay equations having separated linear and sublinear parts. We assume that the Green functions of the linear part is selfadjoint and positive definite to obtain solution estimates, explicit conditions for the absolute stability, and input-output stability. Moreover, it is shown that the suggested conditions characterize the equations that satisfy the generalized Aizerman - Myshkis hypothesis.

1 Introduction and definitions

Let H be a real separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\|_H = (.,.)^{1/2}$, and the unit operator I, cf. [1]. As usually, $L^2(\omega, H)$ is the space of H-valued functions defined on a set $\omega \subseteq \mathbb{R}$ and equipped with the norm

$$\|w\|_{L^2(\omega)} = [\int_\omega \|w(x)\|_H^2 dx]^{1/2} \quad (w \in L^2(\omega, H)).$$

Put $R_+ = [0, \infty)$ and $R_h = [-h, \infty)$ for a positive $h < \infty$. Let $A(t)$ and $B(t)$ $(t \in R_+)$ be variable selfadjoint generally unbounded operators in H with the dense constant domains D_A, D_B, respectively. Besides, $D_A \subseteq D_B$. Let μ be a nondecreasing left-continuous bounded scalar function defined on $[0, h]$ with the property $\mu(0) = 0$. In the present paper we establish solution estimates, explicit conditions for the absolute stability and input-output one of the equation

$$\dot u + A(t)u + B(t)\int_0^h u(t-\tau)d\mu(\tau) = F(t, u(.)) \quad (t > 0,\ \dot u = du/dt) \tag{1.1}$$

where $F : R_+ \times L^2(R_h, H) \to H$ is *a causal nonlinearity* in the sense that

$$F(t, u_1(.)) = F(t, u_2(.)) \text{ if } u_1(\tau) = u_2(\tau)$$

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for all $-h \leq \tau \leq t$ and $u_1, u_2 \in L^2(R_h, H)$. Take the initial condition

$$u(t) = \Phi(t) \ \ (-h \leq t \leq 0) \tag{1.2}$$

with a given function $\Phi : [-h, 0] \to D_A$ continuous in the norm of H.

Note that in the available literature, the stability of proble (1.1), (1.2) is investigated mainly under the condition $A(t) \equiv A$. In addition, it is assumed that the terms containing delays are bounded, cf. [13, 16]. In the papers [7, 10] the equations with variable operators were investigated, but the terms containing delays are assumed to be bounded. About linear non-autonomous equations with unbounded terms containing delays see [17, p. 184] and references given therein. The very interesting papers [3] and [4] on nonlinear non-autonomous equations should be noted. But the nonlinearities considered in these papers are different from the nonlinearities considered in this paper. Moreover, to the best of our knowledge, the absolute and input-output stability of abstract nonlinear differential equations were not investigated in the available literature although these notions are very important in theory of systems, cf. [18].

In addition, in Section 5 below we separate the linear parts of equations of the type (1.1) that satisfy the generalized Aizerman - Myshkis hypothesis.

Definition 1.1 Let E_s $(s \in \mathbb{R})$ be an orthogonal resolution of the identity in H. We will say that E_s is a simple resolution of identity (s.r.i.), if there is a (generating) vector $g \in H$, such that for any $v \in H$,

$$v = \int_{-\infty}^{\infty} \tilde{v}_s dE_s g,$$

where $\tilde{v}_s$ is an $(E_s g, g)$-measurable scalar-valued function. Besides v_s will be called the Eg-coordinate function of v.

Similarly, let E_s be a s.r.i. and W be a normal operator defined by

$$W = \int_{-\infty}^{\infty} \tilde{w}_s dE_s$$

where $\tilde{w}_s$ is an E-measurable scalar-valued function. Then $\tilde{w}_s$ will be called the E-coordinate function of operator W.

As it is well-known [1, Section 83, Theorem 2], any selfadjoint operator with the simple spectrum has a s.r.i. Note also that we write that $\tilde{w}_s$ is an E-measurable, since the measurability of $\tilde{w}_s$ with respect to the measure $(E_s g, g)$ does not depend on a generating vector g [1, Section 83].

2 Statement of the main result

It is assumed that

$$A(t) = \int_{-\infty}^{\infty} a_s(t) dE_s \text{ and } B(t) = \int_{-\infty}^{\infty} b_s(t) dE_s, \tag{2.1}$$

where E_s is a s.r.i. and $a_s(t), b_s(t)$ are real functions continuous in $t \in R_+$ for almost all $s \in \mathbb{R}$ and E_s-measurable in s for all $t \geq 0$. In addition, $a_s(t)$ *is positive*. Moreover, there are nonnegative constants q and l_0, such that

$$\|F(., v(.))\|_{L^2(R_+)} \leq q\|v\|_{L^2(R_h)} + l_0 \quad (v \in L^2(R_h, H)). \tag{2.2}$$

Let us consider the equation

$$\dot{w}(t) + A(t)w(t) + B(t)\int_0^h w(t-\tau)d\mu(\tau) = 0 \quad (t > t_1 \geq 0) \tag{2.3}$$

Below we will check that under conditions pointed below, problem (2.3), (1.2) has a solution ϕ differentiable almost everywhere on R_+. Moreover, we will show that (2.3) has the Green function $G(t, t_1)$ $(t, t_1 \geq -h)$. That is G is the operator-valued function whose values are bounded in H operators satisfying (2.3) almost everywhere on R_+ and the initial conditions

$$G(t_1, t_1) = I \text{ and } G(t, t_1) = 0 \quad (t_1 - h \leq t < t_1, \ -h \leq t_1 < \infty). \tag{2.4}$$

Let G be the Green function of equation (2.3) and ϕ be a solution of problem (1.4), (1.2). Then a continuous function u defined on R_H, satisfying the equation

$$u(t) = \phi(t) + \int_0^t G(t,s)F(s, u(.))ds \quad (t \geq 0) \tag{2.5}$$

and condition (1.2) will be called the mild solution of problem (1.1), (1.2). The existence of mild solutions is assumed. About various existence results see [16, 10], etc.

Finally denote by γ_E the set of points of the growth of E_s and put

$$a_s^- = \inf_{t\geq 0} a_s(t) \text{ and } b_s^- = \inf_{t \geq 0} b_s(t).$$

Now we are in a position to formulate the main result of the paper

Theorem 2.1 *Let the conditions (2.1), (2.2),*

$$\inf_{s\in\gamma_E} a_s^- + \mu(h)\, b_s^- > q \tag{2.6}$$

and

$$\beta := \sup_{s\in\gamma_E} \frac{\mu(h)\, |b_s^-|}{a_s^-} < 1 \tag{2.7}$$

hold. In addition, let the Green function $G(t, t_1)$ to equation (2.3) be positive definite for all $t \geq t_1 \geq 0$. Then there is a constant $c_1 > 0$, independent of the initial conditions, such that any solution u of problem (1.1), (1.2) satisfis the inequality

$$\|u\|_{L^2(R_+)} \leq c_1(\|B_0\Phi\|_{L^2[-h,0]} + \|\Phi(0)\|_H + l_0), \tag{2.8}$$

where

$$B_0 = \int_{-\infty}^{\infty} b_s^- dE_s.$$

The proof of this theorem is divided into a series of lemmas presented in the next two sections.

We also will check that the Green function to equation (2.3) is positive definite, provided

$$e\,h \sup_{s\in\gamma_E, t\geq 0} b_s(t) \int_0^h exp\Big[\int_{t-\tau}^t a_s(t_1)dt_1\Big]\, d\mu(\tau) < 1 \tag{2.9}$$

(see Corollary 3.5 below). Now Theorem 2.1 implies

Corollary 2.2 *Let conditions (2.1), (2.2), (2.6), (2.8) and (2.9) hold. Then inequality (2.8) is valid for any solution $u(t)$ to problem (1.1), (1.2).*

3 Preliminaries

Let σ_h be the σ-algebra of the Borel sets of $[0,h]$ and $\nu(t,.), \nu_+(t,.)$ nonnegative measure defined on σ_h and continuously dependent on $t \geq 0$. Consider the equations

$$\dot{x}(t) + \int_0^h x(t-\tau)\nu(t,d\tau) = 0 \ (t\geq 0) \tag{3.1}$$

and

$$\dot{y}(t) + \int_0^h y(t-\tau)\nu_+(t,d\tau) = 0 \ (t\geq 0). \tag{3.2}$$

Denote by $G_\nu(t,t_1)$ and $G_\nu^+(t,t_1)$ the Green function to equations (3.1) and (3.2), respectively. So they are the solutions of (3.1) and (3.2), respectively, with the initial conditions

$$G_\nu^+(t_1,t_1) = G_\nu(t_1,t_1) = 1, \quad G_\nu(t,t_1) = G_\nu^+(t,t_1) = 0, \quad (t_1 - h \leq t < t_1). \tag{3.3}$$

Lemma 3.1 *Let $G_\nu^+(t,t_1) \geq 0$ $(t > t_1 \geq 0)$ and*

$$\nu_+(t,\tau) \geq \nu(t,\tau) \ \ (\tau \in \sigma_h,\ t\geq 0).$$

Then

$$G_\nu(t,t_1) \geq G_\nu^+(t,t_1) \geq 0 \ (t > t_1 \geq 0). \tag{3.4}$$

Proof. From (3.1) with $x(t) = G_\nu(t,0)$ it follows

$$\dot{x}(t) + \int_0^h x(t-\tau)\nu_+(t,d\tau) = f(t) \tag{3.5}$$

where

$$f(t) = \int_0^h x(t-\tau)(\nu_+(t,d\tau) - \nu(t,d\tau)).$$

According to the initial conditions (3.3), for a sufficiently small $t_0 > h$,

$$x(t) \geq 0 \text{ and } f(t) \geq 0 \ \ (0 \leq t \leq t_0).$$

Hence, by virtue of the Variation of Constants Formula, we get

$$x(t) = G_\nu^+(t,0) + \int_0^t G_\nu^+(t,s)f(s)ds \geq G_\nu^+(t,0) \ (0 \leq t \leq t_0).$$

Extending this inequality to the whole half-line, we arrive at the required result if $t_1 = 0$. Similarly inequality (3.4) can be proved in the general case. As claimed. Q.E.D.

It is simple to check that according to (2.1), equation (2.3) has Green's function represented by

$$G(t,t_1) = \int_{-\infty}^{\infty} G_s(t,t_1)dE_s$$

where $G_s(t,t_1)$ is Green's function to the equation

$$\dot{u}(t) + a_s(t)u(t) + b_s(t)\int_0^h u(t-\tau)d\mu \ (t \geq 0). \tag{3.6}$$

Due to the previous lemma we have

Corollary 3.2 *Let $G_s(t,t_1) \geq 0$. Then*

$$G_s(t,t_1) \leq W_s(t-t_1) \ \ (t \geq t_1 \geq 0),$$

where $W_s(t)$ is the Green function to the equation

$$\dot{u}(t) + a_s^- u(t) + b_s^- \int_0^h u(t-\tau)d\mu = 0 \ (t \geq 0). \tag{3.7}$$

To establish the positivity conditions, let us consider the equation

$$\dot{u}(t) + c_0 u(t-h) = 0 \ \ (t \geq 0) \tag{3.8}$$

with a real constant c_0.

Lemma 3.3 *Let the condition $ehc_0 < 1$ hold. Then the Green function to equation (3.8) is nonnegative.*

For the proof see for instance [8]. Due to Lemmas 3.1 and 3.3 we can assert that the Green function to the equation

$$\dot{u}(t) + b(t)\int_0^h u(t-\tau)d\mu(\tau) = 0 \ \ (t \geq 0)$$

with a bounded real function $b(t)$ is positive, provided

$$eh \sup_t b(t)\mu(h) < 1. \tag{3.9}$$

Now let us consider the equation

$$\dot{u}(t) + a(t)u(t) + b(t)\int_0^h u(t-\tau)d\mu(\tau) = 0 \quad (t \geq 0). \tag{3.10}$$

with bounded real functions $a(t), b(t)$. Substituting in this equation the equality,

$$u(t) = v(t)exp\Big[-\int_0^t a(t_1)dt_1\Big]$$

we have the equation

$$\dot{v}(t) + b(t)\int_0^h exp\Big[\int_{t-\tau}^t a(t_1)dt_1\Big]v(t-\tau)d\mu(\tau) = 0.$$

Now (3.9) and Lemma 3.1 imply

Lemma 3.4 *Let the condition*

$$e\, h \sup_{t\geq 0} b(t)\int_0^h exp\Big[\int_{t-\tau}^t a(t_1)dt_1\Big]\, d\mu(\tau) < 1$$

hold. Then the Green function to equation (3.10) is nonnegative.

Since the Green function $G_s(t, t_1)$ to equation (3.6) is the E_s-coordinate function to the Green function $G_s(t, t_1)$ to equation (2.3), the latter result implies

Corollary 3.5 *Let condition (2.9) hold. Then the Green function to equation (2.3) is positive definite.*

4 Proof of Theorem 2.1

Consider the equation

$$\dot{v}(t) + A(t)v(t) + B(t)\int_0^h v(t-\tau)d\mu(\tau) = f(t) \tag{4.1}$$

with a given $f \in L^2(R_+, H)$. Since E_s is a m.r.i., there is a vector $g \in H$ with $\|g\| = 1$, such that function f and *the solution w to equation (4.1) with the zero initial condition* can be represented as

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}_s(t)dE_s g \tag{4.2}$$

and

$$w(t) = \int_{-\infty}^{\infty} y_s(t)dE_s g. \tag{4.3}$$

Here $\tilde{f}_s(t)$ and $y_s(t)$ are the Eg-coordinate functions of $f(t)$ and $w(t)$, respectively.

Lemma 4.1 *Let f be defined by (4.2) and*
$$\int_0^\infty |\tilde f_s(t)|^2 dt \le c_0 \ \ (s\in\gamma_E)$$
where constant c_0 does not depend on s. Then
$$\|f\|^2_{L^2(R_+)} \le c_0.$$

Proof. We have
$$\int_0^\infty \|f(t)\|^2_H dt = \int_0^\infty \int_{-\infty}^\infty |\tilde f(t)|^2 d(E_s g, g)\, dt =$$
$$\int_{-\infty}^\infty \int_0^\infty |\tilde f_s(t)|^2 dt\; d(E_s g, g) \le c_0 \int_{-\infty}^\infty d\,(E_s g, g) = c_0.$$
As claimed. Q.E.D.

Due to (2.1) and (4.3)
$$\dot y_s(t) + a_s(t) y_s(t) + b_s(t)\int_0^h y_s(t-\tau)d\mu(\tau) = \tilde f_s(t).$$
Hence, under the zero initial condition we have
$$y_s(t) = \int_0^t G_s(t,t_1)\tilde f_s(t_1) d\tau.$$
If $G_s(t,t_1)\ge 0$, due to Corollary 3.2,
$$|y_s(t)| \le \int_0^t W_s(t-t_1)|\tilde f_s(t_1)| d\tau.$$
Consequently,
$$\int_0^\infty |y_s(t)|^2 dt \le \int_0^\infty |\tilde f_s(t)|^2 dt [\int_0^\infty W_s(t)dt]^2.$$
Clearly the Laplace transform to $W_s(t)$ is
$$\int_0^\infty e^{-\lambda t} W_s(t) dt = (\lambda + a_s^- + b_s^- \int_0^h e^{-\lambda\tau} d\mu(\tau))^{-1}.$$
Hence due to (2.6)
$$\theta_\mu := \sup_{s\in\gamma_E} \int_0^\infty W_s(t)dt = \sup_{s\in\gamma_E} (a_s^- + \int_0^h d\mu(\tau) b_s^-)^{-1} < 1/q < \infty. \tag{4.4}$$
Thus,
$$\int_0^\infty |y_s(t)|^2 dt \le \theta_\mu^2 \int_0^\infty |\tilde f_s(t)|^2 dt.$$

Now Lemma 4.1 implies

$$\|w\|_{L^2(R_+)} \le \theta_\mu \|f\|_{L^2(R_+)}.$$

But due to (2.5)

$$w(t) = \int_0^t G(t,t_1) f(t_1) dt_1.$$

Thus, we have proved

Lemma 4.2 *Under the condition (2.6), let G be positive definite. Then*

$$\| \int_0^t G(t,t_1) f(t_1) dt_1 \|_{L^2(R_+)} \le \theta_\mu \|f\|_{L^2(R_+)},$$

where θ_μ is defined by (4.4).

We need also the following

Lemma 4.3 *Under the condition (2.7), let the Green function W_s to the scalar equation (3.7) be non-negative and*

$$c := \inf_{s \in \gamma_E} (a_s + \mu(h) b_s) > 0. \tag{4.5}$$

Then

$$\sup_{s \in \gamma_E} \int_0^\infty W_s^2(t) dt < \infty.$$

Proof. If $b_s > 0$, then from (3.7) and $W_s(t) \ge 0$ it follows that $\dot W_s(t) \le 0$. So $W_s(t) \le 1$ $(t \ge 0)$. Hence due to (4.4),

$$\int_0^\infty W_s^2(t) dt \le \int_0^\infty W_s(t) dt \le \theta_\mu.$$

So in this case the result is proved. Let now $b_s < 0$. Recall that the Laplace transform to $W_s(t)$ is

$$k_s(\lambda) := \Big(\lambda + a_s^- + b_s^- \int_0^h e^{-\lambda\tau} d\mu(\tau)\Big)^{-1}.$$

For all real ω,

$$\begin{aligned} |i\omega + a_s^- + b_s^- \int_0^h e^{-i\omega\tau} d\mu(\tau)| &\ge [\omega^2 + (a_s^-)^2]^{1/2} - |b_s^-|\mu(h) \\ &\ge a_s^- - |b_s^-|\mu(h) \ge c. \end{aligned}$$

In addition,

$$|i\omega + a_s^- + b_s^- \int_0^h e^{-i\omega\tau} d\mu(\tau)| \ge [\omega^2 + (a_s^-)^2]^{1/2} - |b_s^-|\mu(h) \ge$$

$$[\omega^2 + (a_s^-)^2]^{1/2} - a_s\beta \geq c_3|\omega| \ \ (|\omega| \geq c).$$

Hence,

$$\sup_s \int_{-\infty}^{-c} + \int_c^{\infty} |k_s(i\omega)|^2 d\omega \leq 2c_3^{-1} \int_c^{\infty} \omega^{-2} d\omega < \infty$$

and

$$\int_{-c}^{c} |k_s(i\omega)|^2 d\omega \leq 2c\, c^{-2} = 2c^{-1}.$$

This result and the Parseval equality prove the required result. Q.E.D.

Lemma 4.4 *Under the conditions (4.5), (2.7), let the Green function W_s to the scalar equation (3.7) be non-negative. Then any solution $\tilde{y}_s$ of problem (3.7),*

$$\tilde{y}_s(t) = \tilde{\Phi}_s(t) \ \ (-h \leq t \leq 0) \tag{4.6}$$

with a continuous $\tilde{\Phi}_s$, satisfies the inequality

$$\|\tilde{y}_s\|_{L^2(R_+)} \leq c_4(|b_s|\|\tilde{\Phi}_s\|_{L^2[-h,0]} + |\tilde{\Phi}_s(0)|)$$

where constant c_4 does not depend on $s \in \mathbb{R}$.

Proof. Take into account that

$$\tilde{y}_s(t) = W_s(t)\tilde{\Phi}_s(0) + b_s \int_0^h \int_{-\tau}^0 W_s(t-\tau-z)\tilde{\Phi}_s(\tau) dz\, d\tau,$$

see [11, Section 1.6], [6, Section 8.2]. Since

$$\tilde{y}_s(t) = W_s(t)\tilde{\Phi}_s(0) + \int_0^h \int_{-\tau}^0 W_s(t-\tau-z) b_s \tilde{\Phi}_s(\tau) dz\, d\tau, \tag{4.7}$$

thanks to the previous lemma we have the required result. Q.E.D.

Lemma 4.5 *Under condition (2.7) and (4.5), let the Green function G_s to the scalar equation (3.6) be non-negative. Then any solution ϕ_s of problem (3.6), (4.6) satisfies the inequality*

$$\|\phi_s\|_{L^2(R_+)} \leq c_1[|\tilde{\Phi}_s(0)| + |b_s|\|\tilde{\Phi}_s\|_{L^2[-h,0]}].$$

Proof. Let $\tilde{y}_s$ be a solution of problem (3.7), (4.6). Since delays do net depend on time, if $\tilde{\Phi}_s$ is non-negative, then ϕ_s is positive due to the integral representations of solutions [12, p.140]. Repeating the arguments of Lemma 3.1 we have $\phi_s(t) \leq \tilde{y}_s(t)$. If $-\tilde{\Phi}_s$ is non-negative, then $-\phi_s$ is non-negative and $|\phi_s(t)| \leq |\tilde{y}_s(t)|$. Since the initial function is a difference of two non-negative functions, we easily have $|\phi_s(t)| \leq 2|\tilde{y}_s(t)|$. Now the required result is due to the previous lemma. Q.E.D.

It is simple to check that according to (2.1), problem (2.3), (1.2) has a solution ϕ represented by

$$\phi(t) = \int_{-\infty}^{\infty} \tilde{\phi}_s(t) dE_s$$

where $\tilde{\phi}_s(t)$ is a solution to problem (3.6), (4.6). Due to the previous lemma and Lemma 4.1 we have

$$\|\phi\|_{L^2(R_+)} \leq c_1(\|\Phi(0)\|_H + \|B_0\Phi\|_{L^2[-h,0]}). \tag{4.8}$$

Proof of Theorem 2.1 Due to (2.5) and (4.8), Lemma 4.2 and condition (2.2) yield

$$\begin{aligned} \|u\|_{L^2(R_+)} &\leq c_1(\|\Phi(0)\|_H + \|B_0\Phi\|_{L^2[-h,0]}) + \theta_\mu\|F\|_{L^2(R_+)} \\ &\leq c_1(\|B_0\Phi\|_{L^2[-h,0]} + \|\Phi(0)\|_H) + \theta_\mu(q\|u\|_{L^2(R_+)} + l_0). \end{aligned}$$

Now condition (2.5) implies the required result. Q.E.D.

5 Absolute stability and the generalized Aizerman-Myshkis hypothesis

Definition 5.1 *The zero solution of equation (1.1) is said to be absolutely stable in the class of nonlinearities satisfying the condition*

$$\|F(., u(.))\|_{L^2(R_+)} \leq q\|u\|_{L^2(R_h)} \tag{5.1}$$

if there exists a positive constant c_1 independent of the specific form of function F (but dependent on q), such that the inequality

$$\|u\|_{L^2(R_+)} \leq c_1(\|\Phi(0)\|_H + \|B_0\Phi\|_{L^2[-h,0]})$$

holds for any solution u of (1.1) with the initial condition (1.2).

Let a, b, c be an $n \times n$-matrix, a column-matrix and a row-matrix, respectively. In 1949 M. A. Aizerman conjectured the following hypothesis: for the absolute stability of the zero solution of the equation $\dot{x} = A_0x + bf(cx)$ in the class of nonlinearities $f : \mathbf{R}^1 \to \mathbf{R}^1$, satisfying $0 \leq f(s)/s \leq q$ ($q = const > 0, s \in \mathbf{R}^1$, $s \neq 0$) it is necessary and sufficient that the linear equation $\dot{x} = ax + q_1bcx$ be asymptotically stable for any $q_1 \in [0, q]$ [2]. These hypothesis caused the great interest among the specialists. Counterexamples were set up that demonstrated it was not, in general, true, (see [15], [18], and references therein). Therefore, the following problem arose: to find the class of systems that satisfy Aizerman's hypothesis. The author showed that any system satisfies Aizerman hypothesis if its impulse function is non-negative [5]. The similar result was proved for multivariable systems and distributed ones, cf. [6]. On the other hand, A.D.

Myshkis [14] pointed out at the importance of consideration of the generalized Aizerman problem for retarded systems. The problem pointed by A.D. Myshkis was considered in [8] and [9]. In the present paper we will consider the following generalization of the Aizerman problem:

Put

$$A_0 = \int_{-\infty}^{\infty} a_s^- dE_s.$$

Problem 1: *To separate a class of equations (1.1), such that the asymptotic stability of the linear equations*

$$\dot{u} + A_0 u + B_0 \int_0^h u(t-\tau)d\mu(\tau) = \tilde{q}u \tag{5.2}$$

with some $\tilde{q} \in [0, q]$ provides the absolute stability of equation (1.1) in the class of nonlinearities (5.1).

Theorem 2.1 with $l_0 = 0$ implies

Theorem 5.2 *Let conditions (2.1), (2.6) and (2.7) hold. In addition, let the Green function to equation (2.3) be positive definite. Then the zero solution to equation (1.1) is absolutely stable in the class of nonlinearities (5.1).*

Let us check that Theorem 5.2 separates a class of nonlinearities satisfying Problem 1. To this end we will show that, if the Green function is positive definite, the stability of equation (5.2) with $\tilde{q} = q$ implies condition (2.6). Indeed, let v be a solution of (5.2). Then

$$v(t) = \int_0^{\infty} v_s(t)dP_s g,$$

where $v_s(t)$ is the Eg-coordinate function of $v(t)$. According to (2.1) and (5.2) v_s satisfies the equation

$$\dot{v}_s(t) + a_s^- v_s(t) + b_s^- \int_0^h v_s(t-\tau)d\mu = qv(t).$$

Since equation (5.2) is assumed to be asymptotically stable, the roots of the function

$$\lambda + a_s^- + b_s^- \int_0^h e^{-\lambda\tau}d\mu - q$$

are in the open left half-plane. So

$$-q + i\omega + a_s^- + b_s^- \int_0^h e^{-i\omega\tau}d\mu \neq 0$$

for all $\omega \in \mathbb{R}$. Hence, with $\omega = 0$ we get (2.6), as claimed.

6 Input-output stability

Let us consider the equation

$$\dot{u} + A(t)u + B(t)\int_0^h u(t-\tau)d\mu(\tau) = \Psi(t, u(.), \zeta(.)) \ \ (t \geq 0) \tag{6.1}$$

where $\zeta : R_+ \to H$ is a given function (input), $\Psi : R_+ \times L^2(R_h, H) \times L^2(R_+, H) \to H$ is *a causal nonlinearity* in the sense that

$$\Psi(t, u_1(.), \zeta) = \Psi(t, u_2(.), \zeta) \text{ if } u_1(\tau) = u_2(\tau)$$

for all

$$\tau \leq t, \zeta \in L^2(R_+, H) \text{ and } u_1, u_2 \in L^2(R_h, H).$$

Definition 6.1 *We will say that equation (6.1) is L^2-input-output stable, if for any $\epsilon > 0$, there is a $\delta > 0$, such that $\|\zeta\|_{L^2(R_+)} \leq \delta$ implies $\|u\|_{L^2(R_+)} \leq \epsilon$ for any solution u of (6.1) under the zero initial condition $u(t) = 0$ $(t \leq 0)$.*

Theorem 6.2 *Let the conditions (2.1), (2.6), (2.7) and*

$$\|\Psi(., v(.), \zeta)\|_{L^2(R_+)} \leq q\|v\|_{L^2(R_h)} + \mu\|\zeta\|_{L^2(R_+)} + l_0$$

$$(v \in L^2(R_h, H), \zeta \in L^2(R_+, H), \mu = const > 0), \tag{6.2}$$

hold. In addition, let the Green function to equation (2.3) be positive definite. Then equation (6.1) is L^2-input-output stable. Moreover, there is a constant $c_1 > 0$, such that

$$\|u\|_{L^2(R_+)} \leq c_1(\|\zeta\|_{L^2(R_+)} + l_0)$$

for any solution u of (6.1) with the zero initial condition.

Indeed, condition (6.2) implies inequality (2.2) with $\|\zeta\|_{L^2(R_+)} + l_0$ instead of l_0. Now the result is due to Theorem 2.1.

7 Example

First note that condition (5.1) holds, in particular, if $F(t, u(.)) = F_0(u(t), u(t-h))$ where $F_0 : H^2 \to H$ is a function satisfying

$$\|F(z_1, z_2)\|_H \leq q_0\|z_0\|_H + q_1\|z_1\|_H \ \ (q_0, q_1 = const;\ z_0, z_1 \in H). \tag{7.1}$$

Indeed, in this case

$$\|F_0(u(t), u(t-h))\|_{L^2(R_+)} \leq q_0\|u\|_{L^2(R_+)} + q_1\|u(t-h)\|_{L^2(R_+)} \leq$$

$$(q_0 + q_1)\|u\|_{L^2(R_h)}. \tag{7.2}$$

So condition (3.1) holds with $q = q_0 + q_1$.

Let us consider the equation

$$\frac{\partial u(t,x)}{\partial t} = g(t)\frac{\partial^2 u(t,x)}{\partial x^2} - c(t)\frac{\partial^2 u(t-h,x)}{\partial x^2} + F_1(x, u(t,x), u(t-h,x))$$
$$(0 < x < 1,\ t \geq 0) \tag{7.3}$$

where $g(t), c(t)$ are continuous bounded positive functions, F_1 is a scalar continuous function defined on $[0,1] \times \mathbb{R}^2$ and having the property

$$|F_1(x, z_0, z_1)| \leq q_0|z_0| + q_1|z_1| \ \ (x \in [0,1], z_0, z_1 \in \mathbb{R}). \tag{7.4}$$

According to (7.2), condition (7.4) implies (5.1) with $q = q_0 + q_1$.

Take some selfadjoint boundary conditions. Let S be the operator defined by $Su = -d^2u/dx^2$ $(0 < x < 1)$ with the taken boundary conditions and the positive eigenvalues λ_k $(k = 1, 2, ...)$ numerated in the increasing order with the multiplicities taken into account. Then (7.1) can be written as (1.1) with $A(t) = g(t)S, B(t) = -c(t)S$ and

$$a_s(t) \equiv a_k(t) = g(t)\lambda_k, b_s(t) \equiv b_k(t) = -c(t)\lambda_k.$$

So

$$a_s^- \equiv a_k^- = \lambda_k \inf_{t\geq 0} g(t);\ b_s^- \equiv b_k^- = -\lambda_k \sup_{t\geq 0} c(t).$$

Then condition (2.6) takes the form

$$\lambda_1(\inf_{t\geq 0} g(t) - \sup_{t\geq 0} c(t)) > q. \tag{7.5}$$

Condition (2.7) is fulfilled, provided for a $\beta < 1$,

$$\beta \inf_{t\geq 0} g(t) \geq \sup_{t\geq 0} c(t). \tag{7.6}$$

Condition (2.9) is always holds. Due to Theorem 2.1, under conditions (7.5)-(7.6), any solution to equation (7.3) is in $L^2(R_+, H)$, provided condition (7.4) holds. Moreover, due to Theorem 5.2, the zero solution to equation (7.3) is absolutely stable in the class of nonlinearities (7.4).

In particular, if we take the Dirichlet boundary conditions $u(t,0) = u(t,1) = 0$ $(t \geq 0)$, then $\lambda_k = \pi^2 k^2$ $(k = 1, 2, ...)$.

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