

TWO SYMMETRY PROBLEMS IN POTENTIAL THEORY

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ABSTRACT. We consider two overdetermined boundary value problems as variants on J. Serrin's 1971 classical results and prove in both cases that the domains must be Euclidean balls.

Assume throughout that $\Omega \subseteq \mathbb{R}^N$ is a bounded domain whose boundary is of class C^2 and contains the origin strictly in its interior. Let ν be the outer unit normal to $\partial\Omega$. Summation over repeated indices is in effect. In 1971, James Serrin [1] proved the following classical result.

Theorem 1. *Suppose there exists a function $u \in C^2(\bar{\Omega})$ satisfying the elliptic differential equation*

$$a(u, |p|)\Delta u + h(u, |p|)u_i u_j u_{ij} = f(u, |p|) \quad \text{in } \Omega$$

where a, f and h, p_i, p_j are continuously differentiable functions of u and p (here $p = (u_1, \dots, u_n)$ denotes the gradient vector of u). Suppose also that $u > 0$ in Ω and that u satisfies the boundary conditions

$$u = 0, \quad \frac{\partial u}{\partial \nu} = \text{constant} \quad \text{on } \partial\Omega.$$

Then Ω must be a ball and u is radially symmetric.

The method of proof combines the Maximum Principles and the device (which goes back to A. D. Alexandroff: *every embedded surface in \mathbb{R}^N with constant mean curvature must be a sphere*) of moving planes to a critical position and then showing that the solution is symmetric about the limiting plane. In a subsequent article, H. F. Weinberger [3] gave a simplified proof for the special case of the Poisson differential equation, $\Delta u = -1$.

Our aim at present is to introduce some variants on Serrin's result and arrive at the same symmetry conclusions by employing elementary arguments. The next statement involves radial dependence on the boundary conditions.

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Proposition 1. *Suppose there exists a solution $u \in C^2(\bar{\Omega})$ to the overdetermined problem:*

$$\begin{aligned} \Delta u &= -1 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} &= -cr && \text{on } \partial\Omega; \end{aligned} \tag{1}$$

where $r = \sqrt{x_1^2 + \dots + x_N^2}$ and c is a constant. Then Ω is an N -dimensional ball.

Before turning to the details we like to discuss some physical motivations for the problem itself. Consider a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross-sectional form Ω . If we fix rectangular coordinates in space with the z -axis directed along the pipe, it is well known that the flow velocity u is then a function of x, y alone satisfying the Poisson differential equation (for $N = 2$) $\Delta u = -A$ in Ω , where A is a constant related to the viscosity of the fluid and to the rate of change of pressure per unit length along the pipe. Supplementary to the differential equation one has the adherence condition $u = 0$ on $\partial\Omega$.

Finally, the tangential stress per unit area on the pipe wall is given by the quantity $\mu \frac{\partial u}{\partial \nu}$, where μ is the viscosity. Our proposition then states that the ratio of the tangential stress on the pipe wall to its radial distance is the same at all points of the wall if and only if the pipe has a circular cross section.

Exactly the same differential equation and boundary condition arise in the linear theory of torsion of a solid straight bar of cross-section Ω ; see [2] pp.109-119. In light of this, Proposition 1 states that, when a solid straight bar is subject to torsion, the ratio of the magnitude of the resulting traction, which occurs at the surface of the bar, to the radial distance of the surface is independent of position if and only if the bar has a circular cross-section.

Lemma 1. *Under the hypothesis of Proposition 1, the following holds*

$$\int_{\Omega} u dx = c^2 \int_{\Omega} r^2 dx. \tag{2}$$

Proof. Let us introduce the auxiliary function $h = 2u - x_i u_i$. Then clearly $\Delta h = 0$. Green's identities

$$\int_{\Omega} (h \Delta u - u \Delta h) dx = \int_{\partial\Omega} \left(h \frac{\partial u}{\partial \nu} - u \frac{\partial h}{\partial \nu} \right) d\sigma,$$

The boundary conditions on u , and the harmonicity of h lead to

$$\int_{\Omega} h dx = c \int_{\partial\Omega} r h d\sigma \tag{3}$$

Now we compute the left and right hand sides of (3) individually. Applying the divergence theorem, we get

$$\begin{aligned} \int_{\Omega} h dx &= \int_{\Omega} (2 + N)u dx - \int_{\Omega} \operatorname{div}(xu) dx \\ &= \int_{\Omega} (2 + N)u dx - \int_{\partial\Omega} ux \cdot \nu d\sigma \\ &= \int_{\Omega} (2 + N)u dx, \quad \text{for } u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{4}$$

Again since u vanishes on the boundary $\partial\Omega$ (therefore $\nu = \pm \frac{\nabla u}{\|\nabla u\|}$) and $r_i = \frac{x_i}{r}$, we gather that (note: the argument here could have proceeded using the so-called Pohozaev's identity, but we drop it so as not to use any "heavy gun")

$$\begin{aligned}
 c \int_{\partial\Omega} r h d\sigma &= -c \int_{\partial\Omega} r x_i u_i d\sigma = -c \int_{\partial\Omega} r^2 \frac{\partial u}{\partial r} d\sigma \\
 &= -c \int_{\partial\Omega} r^2 \frac{\partial u}{\partial \nu} \frac{\partial r}{\partial \nu} d\sigma = c^2 \int_{\partial\Omega} r^3 \frac{\partial r}{\partial \nu} d\sigma \\
 &= \frac{c^2}{4} \int_{\partial\Omega} \frac{\partial(r^4)}{\partial \nu} d\sigma = \frac{c^2}{4} \int_{\Omega} \Delta(r^4) dx \\
 &= c^2(N+2) \int_{\Omega} r^2 dx.
 \end{aligned} \tag{5}$$

Combining (3), (4) and (5) proves the Lemma. \square

Proof of Proposition 1. Consider the functional $\Phi = u_i u_i - c^2 r^2$. Then,

$$\begin{aligned}
 \Delta\Phi &= 2u_{ij}u_{ij} + 2u_i\Delta u_i - 2c^2N \\
 &= 2u_{ij}u_{ij} - 2c^2N \quad \text{since } \Delta u \text{ is a constant} \\
 &= 2 \sum_{i,j} \left(u_{ij} + \frac{\delta_{ij}}{N} \right)^2 + \frac{2}{N} - 2c^2N \\
 &\geq \frac{2}{N} - 2c^2N.
 \end{aligned} \tag{6}$$

For a moment assume that $cN \leq 1$. Then we have,

$$\Delta\Phi \geq 0 \quad \text{in } \Omega. \tag{7}$$

Note that $\nu = \pm \nabla u / \|\nabla u\|$, since u vanishes on the boundary. Thus on $\partial\Omega$ we have

$$\Phi = \left(\frac{\partial u}{\partial \nu} \right)^2 - c^2 r^2 = 0. \tag{8}$$

Applying Green's identities and replacing the boundary conditions results in

$$\begin{aligned}
 \int_{\Omega} \Phi dx &= \int_{\Omega} \nabla u \cdot \nabla u - c^2 \int_{\Omega} r^2 dx \\
 &= \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} d\sigma - \int_{\Omega} u \Delta u dx - c^2 \int_{\Omega} r^2 dx \\
 &= \int_{\Omega} u dx - c^2 \int_{\Omega} r^2 dx.
 \end{aligned} \tag{9}$$

By Lemma 1 above, equation (9) yields

$$\int_{\Omega} \Phi = 0.$$

Standard Maximum Principles together with the properties (7)-(9) of Φ imply that $\Phi \equiv 0$ in Ω . This in particular forces equality in (6), i.e.

$$u_{ij} + \frac{\delta_{ij}}{N} \equiv 0.$$

Consequently, u takes the radial form

$$u = a - \frac{r^2}{2N}.$$

Again, since u vanishes on the boundary $\partial\Omega$, we obtain that Ω is indeed a ball, as asserted. \square

Remarks.

- (1) It is not hard to see that the C^2 -smoothness assumptions, on u , made in the preceding Proposition (even Proposition 2, below) could be relaxed except the proofs would then get rather technical.
- (2) The assumption $cN \leq 1$ in the proof of Proposition 1 is not essential. To see this, notice in fact that due to the conditions (1) on u coupled with the divergence theorem verify that

$$\begin{aligned} |\Omega| &= - \int_{\Omega} \Delta u dx = - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma = c \int_{\partial\Omega} r d\sigma \\ &\geq cN \int_{\partial\Omega} \frac{r dA}{N} = cN |\Omega| \quad \text{from geometry.} \end{aligned}$$

Therefore, $cN \leq 1$.

- (3) Using the same argument and replacing the second boundary condition by $\frac{\partial u}{\partial \nu} = -cr^\alpha$, where $\alpha \geq 1$ (note: this already implies that $cN \text{diam}(\Omega)^{\alpha-1} \leq 1$), $d = \text{diam}(\Omega)$ and

$$0 < c \leq \frac{2^{\alpha-1}}{d^{\alpha-1} \sqrt{\alpha N(N + 2\alpha - 2)}}$$

we still get the conclusion of Proposition 1.

- (4) Even more generally, if $\frac{\partial u}{\partial \nu} = -f(r)$ where $g(r) = f^2(r)$ satisfies the conditions

$$\begin{aligned} \int_{\Omega} (rg' - 2g) dx &\geq 0, \\ g'' + \frac{N-1}{r} g' - \frac{2}{N} g &\leq 0 \end{aligned}$$

then Proposition 1 holds. Moreover, it turns out that either $f \equiv c_1$ or $f = c_2 r$ for some positive constants c_1 and c_2 . One of the implications of which is that under these general suppositions, there can only be two possible forms for the boundary derivative: either $\partial u / \partial \nu = -c_1$ as in Theorem 1 of Serrin, or $\partial u / \partial \nu = -c_2 r$ as in Proposition 1 of this article!

Proposition 2. *Suppose there exists a solution $u \in C^2(\bar{\Omega})$ to following overdetermined boundary-value problem*

$$\begin{aligned} \Delta u &= -r^\alpha \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \\ x_i u_i + c_0 r^{2+\alpha} + c_1 &= 0 \quad \text{on } \partial\Omega; \end{aligned} \tag{10}$$

where α , c_0 and c_1 are constants. If $\beta := (\alpha + 2)(c_0(\alpha + N) - 1)$ is not a negative integer, then Ω is an N -dimensional ball.

Proof. Introduce the functional $V = x_i u_i + c_0 r^{\alpha+2} + c_1 + \beta u$. Then by direct computation we obtain that V is harmonic. As can easily be seen, the boundary conditions make $V \equiv 0$ on $\partial\Omega$. Thus,

$$\begin{aligned} \Delta V &= 0 \quad \text{in } \Omega \\ V &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Classical Maximum Principles show that $V \equiv 0$ inside Ω , too. Hence, it follows that

$$x_i u_i + c_0 r^{\alpha+2} + c_1 + \beta u = 0 \quad \text{in } \bar{\Omega}. \tag{11}$$

Rewriting equation (11) one obtains

$$\frac{\partial}{\partial r} \left(r^\beta (u - u(0)) + \frac{c_0 r^{\beta+\alpha+2}}{\beta + \alpha + 2} \right) = r^{\beta-1} \left(\beta (u - u(0)) + r \frac{\partial u}{\partial r} + c_0 r^{\alpha+2} \right) = 0.$$

This in turn implies that

$$r^\beta (u - u(0)) + \frac{c_0 r^{\beta+\alpha+2}}{\beta + \alpha + 2} \equiv G(\Theta)$$

for angular variables Θ . Hence,

$$u - u(0) = -c_0 \frac{r^{\alpha+2}}{\beta + \alpha + 2} + r^{-\beta} G(\Theta) \tag{12}$$

Now, if $\beta \geq 0$ then u cannot be regular at the origin unless $G(\Theta) \equiv 0$.

If $\beta < 0$ is not a negative integer, then once more we must have $G(\Theta) \equiv 0$ for u must satisfy $\Delta u = -r^\alpha$ and $r^{-\beta} G(\Theta)$ is not harmonic with this value of β . Therefore, the solution u is radial and

$$u = u(0) - c_0 \frac{r^{\alpha+2}}{\beta + \alpha + 2}.$$

After using (10) and the vanishing of u on $\partial\Omega$, this proves that Ω is a ball thereby completing the proof of the proposition.

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