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Resonance with respect to the Fučík spectrum *

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Abstract

Let L be a self-adjoint operator on $L^2(\Omega;\mathbb{R})$ with Ω a bounded and open subset of $\mathbb{R}^N$. This article considers the resonance problem with respect to the Fučík spectrum of L, which means that we study equations of the form

$$Lu = \alpha u^+ - \beta u^- + f(\cdot, u),$$

when the homogeneous equation $Lu = \alpha u^+ - \beta u^-$ has non-trivial solutions. Using the computation of degrees that are not necessarily $+1$ or -1, we present results about the existence of solutions. Our results are illustrated with examples and can be seen as generalizations of Landesman-Lazer conditions. Non-existence results are also given.

1 Introduction

In this paper, we continue the study started in [1] about nonlinear equations containing an asymmetric nonlinear term (or "jumping nonlinearity"); i.e., equations of the form

$$Lu = \alpha u^+ - \beta u^- + f \tag{1}$$

or, more generally, of the form

$$Lu = \alpha u^+ - \beta u^- + f(\cdot, u), \tag{2}$$

where $u^+ = \max\{u, 0\}$, and $u^- = \max\{-u, 0\}$. The linear operator L is defined from $\operatorname{dom} L \subset L^2(\Omega;\mathbb{R})$ to $L^2(\Omega;\mathbb{R})$, where Ω is a bounded open subset of $\mathbb{R}^N$. On this operator we shall assume the hypothesis

(H1) The operator L is self-adjoint and has and eigenvalue λ^* such that $\dim\ker(L - \lambda^* I) = n < \infty$.

For studying (2), we assume that L has a compact resolvent and that f is (globally) bounded by an L^2-function. The precise hypotheses about $f(t, u)$ will be stated in Section 4. Also we assume that the pair (α, β) is "not too far" from (λ^*, λ^*), in the sense of the following hypothesis, in which $\sigma(L)$ denotes the spectrum of L.

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(H2) The values α and β are in a closed interval I such that $I \cap \sigma(L) = \{\lambda^*\}$.

The set of points at which the homogeneous equation

$$Lu = \alpha u^+ - \beta u^- \tag{3}$$

has nontrivial solutions is called the Fučík or Dancer-Fučík spectrum, and is denoted by $\Sigma(L)$. Dancer [2] and Fučík [5] recognized the importance of this set in the study of semi-linear boundary value problems. Under hypotheses which include (H1) and (H2) as a particular case, we have studied the structure of the Fučík spectrum within $I \times I$, [1]. We also showed the existence of solutions for (1) and (2) when (α, β) does not belong to the Fučík spectrum and when $f(t, u)$ has sublinear growth in u as $|u| \to \infty$.

In the present work, we investigate the case when (α, β) does belong to the Fučík spectrum. This situation can be considered as a situation of resonance and is divided into two different cases.

In Section 3, we obtain, for fixed (α, β) in $\Sigma(L) \cap (I \times I)$, existence results for (1) under some regularity assumption. Roughly, the assumption means that the set $\{\varphi \mid L\varphi = \alpha\varphi^+ - \beta\varphi^-\} \setminus \{0\}$ is locally a manifold whose dimension may differ on each component, but its tangent space at φ is exactly $\ker(L - \alpha\chi(\varphi^+)I - \beta\chi(\varphi^-)I)$. Here, $\chi(\varphi^+)$ is the characteristic function of the set $\{t \in \Omega \mid \varphi(t) > 0\}$, and $\chi(\varphi^-)$ is defined similarly. Under this regularity assumption, a Landesman-Lazer type condition is obtained for showing the existence of at least one solution to (1). The regularity assumption is satisfied in particular when all solutions of (3) are positive multiples of a finite number of particular solutions, and

$$\dim \ker(L - \alpha\chi(\varphi^+)I - \beta\chi(\varphi^-)I) = 1$$

for any such nontrivial solution. The regularity assumption is also satisfied when

(H3) For each $x \in \ker(L - \lambda^* I)$ there exists a solution φ_x of (3), such that $P\varphi_x = x$, where P denotes the orthogonal projection onto $\ker(L - \lambda^* I)$.

However, under (H3), better existence results, still based on topological degree arguments, can be obtained, which also apply to equation (2). This is the object of Section 4. It corresponds to a Fučík spectrum in $I \times I$ reduced to a curve. The existence of solutions is established if a certain degree is not equal to zero. Consider the particular case where f has limits $f_+(t)$ and $f_-(t)$ for $u \to \pm\infty$. Then, it will be shown that the degree in question can be computed from the indices of the critical values of the function

$$\Psi : S^{n-1} \to \mathbb{R}, \quad x \mapsto \langle f_+, \varphi_x^+ \rangle - \langle f_-, \varphi_x^- \rangle,$$

where S^{n-1} is the unit sphere in $\ker(L - \lambda^* I)$ and $\langle \cdot , \cdot \rangle$ denotes the scalar product in $L^2(\Omega; \mathbb{R})$ (the norm will be denoted by $\|\cdot\|$). When $\dim \ker(L - \lambda^* I) = 2$, the existence result takes a particularly simple form. Let $\{v^{(1)}, v^{(2)}\}$ denote an orthonormal basis of $\ker(L - \lambda^* I)$, and let

$$z_\theta = \cos\theta\, v^{(1)} + \sin\theta\, v^{(2)}\,.$$

We will abbreviate notation by using φ_θ for φ_{z_θ}. Under appropriate hypotheses on L, α, β (see Corollary 3 and Theorem 3), if the function $\Psi : \theta \mapsto \langle f_+, \varphi_\theta^+ \rangle - \langle f_-, \varphi_\theta^- \rangle$ has only simple zeros, and if the number of zeros in $[0, 2\pi)$ is not 2, equation (2) has at least one solution.

The results obtained in Section 4 can be seen as generalizations of the Landesman-Lazer conditions. Those conditions (see [13] for a survey) are familiar in the context of resonance with respect to the "usual spectrum" (i.e., when $\alpha = \beta$) and correspond to a function Ψ of constant sign. It must be noticed that, even when $\alpha = \beta$, the conditions presented in Theorem 3 and some of its corollaries provide a generalization of the classical Landesman-Lazer conditions. The existence conditions of Section 4 have been inspired by results of [4] concerning the periodic boundary-value problem for the second order equation

$$u'' + \alpha u^+ - \beta u^- = f(t)\,.$$

Section 5 concludes our study with a non-existence result for equation (1). Now we present a few results needed later.

2 Preliminary results

Let f_+ and f_- belong to $L^2(\Omega;\mathbb{R})$, with $f_+ - f_- \in L^\infty(\Omega;\mathbb{R})$. For $\varepsilon \geq 0$, define

$$f_\varepsilon(t,u) = \frac{1}{2}\,[f_+(t) + f_-(t)] + \frac{1}{\pi}\,[f_+(t) - f_-(t)]\arctan(\varepsilon u)\,. \tag{4}$$

Clearly, $f_\varepsilon(t,u)$ is Lipschitz continuous in u, with Lipschitz constant $\frac{\varepsilon}{\pi}\,|f_+(t) - f_-(t)|$. To f_ε, we associate the mapping

$$N_\varepsilon : L^2(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R}) : u \mapsto \alpha u^+ - \beta u^- + f_\varepsilon(\cdot, u)\,. \tag{5}$$

It is easy to show that $N_\varepsilon - \frac{1}{2}(\alpha+\beta)I$ is Lipschitz continuous with Lipschitz constant $\frac{1}{2}|\alpha-\beta| + \frac{\varepsilon}{\pi}\|f_+ - f_-\|_{L^\infty}$. The following lemma is a slight generalization of Lemma 1 in [1], to which we refer for the proof.

Lemma 1 *Let L, α, β satisfy (H1) and (H2), and f_ε be as above. Then, provided that ε is small enough, for any $x \in \ker(L - \lambda^* I)$, the problem*

$$Lu = \lambda^* x + (I-P)[\alpha u^+ - \beta u^- + f_\varepsilon(\cdot,u)], \tag{6}$$

$$Pu = x \tag{7}$$

has a unique solution $u_x = u_x(f_\varepsilon, \alpha, \beta)$. Moreover, this solution is Lipschitzian with respect to x, α, β. More precisely, for x, y in a bounded set $B_1 \subset \ker(L - \lambda^* I)$, there exists a constant $K > 0$ such that

$$\|u_x(f_\varepsilon,\alpha,\beta) - u_y(f_\varepsilon,\alpha',\beta')\| \leq K\,(\|x-y\| + |\alpha-\alpha'| + |\beta-\beta'|)\,.$$

Let $u_x = u_x(f_\varepsilon, \alpha, \beta)$ be the solution of (6), (7). Define

$$c(x, f_\varepsilon, \alpha, \beta) \quad = \quad -P[\alpha u_x^+ - \beta u_x^- + f_\varepsilon(\cdot, u_x)] + \lambda^* x,$$

so that u_x satisfies

$$Lu_x = \alpha u_x^+ - \beta u_x^- + f_\varepsilon(\cdot, u_x) + c(x, f_\varepsilon, \alpha, \beta). \tag{8}$$

It is clear that

$$Lu = \alpha u^+ - \beta u^- + f_\varepsilon(\cdot, u) \tag{9}$$

admits a solution if and only if there exists $x \in \ker(L - \lambda^* I)$ such that $c(x, f_\varepsilon, \alpha, \beta) = 0$. Therefore, (9) can be reduced to a problem in a finite-dimensional space. In relation with equation (3), we will write $c_0(x, \alpha, \beta)$ for $c(x, 0, \alpha, \beta)$. Hence, we have

$$c_0(x, \alpha, \beta) = -P[\alpha u_x^+ - \beta u_x^-] + \lambda^* x,$$

where $u_x = u_x(0, \alpha, \beta)$, given by Lemma 1, is the unique solution of

$$\begin{gathered} Lu = \lambda^* x + (I - P)[\alpha u^+ - \beta u^-], \\ Pu = x\,. \end{gathered}$$

To the function c_0, we associate the function

$$h_0 : \ker(L - \lambda^* I) \times I \times I : (x, \alpha, \beta) \mapsto \langle c_0(x, \alpha, \beta), x\rangle\,;$$

some of its properties, established in [1], are recalled below.

Lemma 2 *Let (H1) and (H2) hold. Then, the function h_0 admits partial derivatives with respect to $\alpha, \beta \in I$, is differentiable with respect to $x \in \ker(L - \lambda^* I)$,*

$$\frac{\partial}{\partial\alpha} h_0(x, \alpha, \beta) = -\|u_x^+\|^2, \quad \frac{\partial}{\partial\beta} h_0(x, \alpha, \beta) = -\|u_x^-\|^2,$$

and

$$\nabla_x h_0(x, \alpha, \beta) = 2c_0(x, \alpha, \beta). \tag{10}$$

Based on the function h_0, the following theorem, which has been proved in [1] and in [11], provides a characterization of the points of the Fučík spectrum, within the square $I \times I$.

Theorem 1 *Let (H1) and (H2) hold. Then, (α, β) in $I \times I$ belongs to the Fučík spectrum of L if and only if 0 is a critical value of the function*

$$h_0(\cdot, \alpha, \beta) : x \mapsto \langle c_0(x, \alpha, \beta), x\rangle,$$

with the critical value being reached at some point $x \neq 0$.

The Fučík spectrum contains in particular the sets

$$F^- = \{(\alpha, \beta) \in I \times I \mid \min_{x \in \ker(L - \lambda^* I),\, \|x\| = 1} \langle c_0(x, \alpha, \beta), x\rangle = 0\}$$

and

$$F^+ = \{(\alpha, \beta) \in I \times I \mid \max_{x \in \ker(L - \lambda^* I),\, \|x\| = 1} \langle c_0(x, \alpha, \beta), x\rangle = 0\}.$$

3 A first existence result at resonance

Let (α, β) be fixed in $\Sigma(L) \cap I \times I$. We will obtain a Landesman-Lazer type existence condition for equation (1) under a regularity assumption. Let $\mathcal{S}$ denote the set of all solutions of equation (3), and assume that

(H3') For each $\varphi \in \mathcal{S}$, $\varphi \neq 0$, there exists $\delta > 0$ such that, if $\tilde{\varphi} \in \mathcal{S}$ and $\|\varphi - \tilde{\varphi}\| \leq \delta$, then $\dim\ker(L - \alpha\chi(\varphi^+)I - \beta\chi(\varphi^-)I) = \dim\ker(L - \alpha\chi(\tilde{\varphi}^+)I - \beta\chi(\tilde{\varphi}^-)I)$. Moreover, for each $\psi \in \ker(L - \alpha\chi(\varphi^+)I - \beta\chi(\varphi^-)I)$, $\operatorname{dist}(\varphi + \varepsilon\psi, \mathcal{S}) = o(\varepsilon)$ for $\varepsilon \to 0$.

If it happens that $\mathcal{S}$ is locally a manifold whose tangent space at φ is exactly $\ker(L - \alpha\chi(\varphi^+)I - \beta\chi(\varphi^-)I)$, then (H3') is clearly satisfied. This will be the case when all solutions of (3) are positive multiples of a finite number of particular solutions and

$$\dim\ker(L - \alpha\chi(\varphi^+)I - \beta\chi(\varphi^-)I) = 1,$$

for any such nontrivial solution φ. The condition (H3') also holds when (H3) is satisfied; but, in this last case, better existence results can be obtained, as shown in Section 4.

For the sake of simplicity, in this section we will consider only equation (1) with a function f independent of the unknown function u. We will also need the following (weak) hypotheses

(H4) For some $p > 2$, $\operatorname{dom} L$, equipped with the graph norm, is continuously injected into $L^p(\Omega; \mathbb{R})$

(H5) For any nontrivial solution φ of the homogeneous equation (3),

$$\operatorname{meas}\{t \in \Omega \mid \varphi(t) = 0\} = 0.$$

Theorem 2 *Assume that (α, β) belongs to the Fučík spectrum, that (H1), (H2), (H3'), (H4), and (H5) hold, that the homogeneous equation (3) has no solution of constant sign, and that for any nontrivial solution φ of (3), we have $\langle f, \varphi \rangle > 0$. Moreover, assume that, for $\varepsilon > 0$ small enough, $(\alpha + \varepsilon, \beta + \varepsilon)$ does not belong to the Fučík spectrum and the degree, with respect to open bounded sets containing 0, of the mapping $c_0(\cdot, \alpha + \varepsilon, \beta + \varepsilon) : \ker(L - \lambda^* I) \to \ker(L - \lambda^* I)$, is different from 0. Then, equation (1) has at least one solution.*

The sign condition can be reversed to $\langle f, \varphi \rangle < 0$, taking then $\varepsilon < 0$ in the condition on the degree.

Proof. We will use the homotopy

$$c(x, f, \alpha + \varepsilon s, \beta + \varepsilon s) = 0\,, \tag{11}$$

where $s \in [0, 1]$. By the last hypothesis, for η small enough, the degree, with respect to the unit ball centered at 0, of the mapping $x \mapsto c(x, \eta f, \alpha + \varepsilon, \beta + \varepsilon)$ is different from 0 and, since

$$c(rx, rf, \alpha', \beta') = r\, c(x, f, \alpha', \beta') \text{ for } r > 0 \text{ and for any } \alpha', \beta' \in I, \tag{12}$$

the same is true for the degree of $c(x,f,\alpha+\varepsilon,\beta+\varepsilon)$, with respect to sufficiently large balls centered at 0. Using the invariance of the degree with respect to a homotopy, the theorem is then proved if we can find a priori bounds for the solutions of (11). By contradiction, assume that there exists sequences $\{s_n\},\{x_n\}$, with $s_n\in[0,1]$, $\{x_n\}\subset\ker(L-\lambda^*I)$, $\|x_n\|\to\infty$, such that

$$c(x_n,f,\alpha+\varepsilon s_n,\beta+\varepsilon s_n)=0\,. \tag{13}$$

Denoting by u_n the function associated to x_n by Lemma 1 ($u_n=u_{x_n}(f,\alpha,\beta)$), we can equivalently write

$$Lu_n=(\alpha+\varepsilon s_n)u_n^+-(\beta+\varepsilon s_n)u_n^-+f\,. \tag{14}$$

Let $y_n=x_n/\|x_n\|$; using (12) again, we have

$$c\left(y_n,\frac{f}{\|x_n\|},\alpha+\varepsilon s_n,\beta+\varepsilon s_n\right)=0\,. \tag{15}$$

Passing, if necessary, to subsequences, we can assume that $\{y_n\},\{s_n\}$, converge; let their respective limits be denoted by y^*,s^*. Going to the limit in (15), we obtain, c being continuous,

$$c_0(y^*,\alpha+\varepsilon s^*,\beta+\varepsilon s^*)=0\,. \tag{16}$$

Since $\|y^*\|=1$, this implies that $(\alpha+\varepsilon s^*,\beta+\varepsilon s^*)$ belongs to the Fučík spectrum, from which follows, by the last hypothesis of the theorem, that $s^*=0$. Consequently, the function $u_{y^*}(0,\alpha,\beta)$, associated to y^* by Lemma 1, is a solution of (3). In the sequel, we will write φ for $u_{y*}(0,\alpha,\beta)$. Let $v_n=u_n/\|x_n\|$, and $\varphi_n\in\mathcal{S}$ be such that

$$\|\varphi_n-v_n\|=\operatorname{dist}(v_n,\mathcal{S}).$$

Hence, both $\{v_n\}$ and $\{\varphi_n\}$ converge to φ. By (14), the equation verified by v_n can be written under the form

$$\begin{aligned} Lv_n \;=\;& \alpha\chi(\varphi_n^+)v_n+\beta\chi(\varphi_n^-)v_n\\ &+\varepsilon s_n\chi(\varphi_n^+)v_n+(\alpha+\varepsilon s_n)[\chi(v_n^+)-\chi(\varphi_n^+)]v_n\\ &+\varepsilon s_n\chi(\varphi_n^-)v_n+(\beta+\varepsilon s_n)[\chi(v_n^-)-\chi(\varphi_n^-)]v_n+\frac{f}{\|x_n\|}\,. \end{aligned} \tag{17}$$

Since $\varphi_n\in\ker(L-\alpha\chi(\varphi_n^+)I-\beta\chi(\varphi_n^-)I)$, equation (17) has a solution only if

$$\begin{aligned} &\varepsilon s_n\langle\chi(\varphi_n^+)v_n,\varphi_n\rangle+(\alpha+\varepsilon s_n)\langle[\chi(v_n^+)-\chi(\varphi_n^+)]v_n,\varphi_n\rangle \qquad (18)\\ &+\varepsilon s_n\langle\chi(\varphi_n^-)v_n,\varphi_n\rangle+(\beta+\varepsilon s_n)\langle[\chi(v_n^-)-\chi(\varphi_n^-)]v_n,\varphi_n\rangle+\frac{\langle f,\varphi_n\rangle}{\|x_n\|} \;=\; 0\,. \end{aligned}$$

Let w_n denote the orthogonal projection of v_n onto the space $K_n:=\ker(L-\alpha\chi(\varphi_n^+)I-\beta\chi(\varphi_n^-)I)$. If $w_n\neq\varphi_n$, then by assumption (H3'),

$$\operatorname{dist}(v_n,\mathcal{S})\le\operatorname{dist}(v_n,\varphi_n+\varepsilon(w_n-\varphi_n))+o(\varepsilon)=\operatorname{dist}(v_n,\varphi_n)-O(\varepsilon)+o(\varepsilon),$$

contradicting the definition of φ_n. Hence, $\varphi_n = w_n$ and

$$L\varphi_n = \alpha\chi(\varphi_n^+)\varphi_n + \beta\chi(\varphi_n^-)\varphi_n. \tag{19}$$

By the arguments used to prove Lemma 1, it can be shown that, under hypotheses (H1) and (H2), the operator $[L - \alpha\chi(\varphi_n^+)I - \beta\chi(\varphi_n^-)I]_{\varphi_n^\perp} : K_n^\perp \to K_n^\perp$ is invertible. Using the norm of its inverse and the fact that $\operatorname{dom} L$, equipped with the graph norm, is continuously injected into $L^p(\Omega;\mathbb{R})$, we can show, subtracting (19) from (17) and using the fact that $\varphi_n - v_n \in K_n^\perp$, that

$$\begin{aligned} &\|v_n - \varphi_n\|_{L^p} \\ &\quad\leq\ C_n\Big[\varepsilon s_n + \frac{\|f\|}{\|x_n\|} + \|[\chi(v_n^+) - \chi(\varphi_n^+)]v_n\| + \|[\chi(v_n^-) - \chi(\varphi_n^-)]v_n\|\Big]\,, \end{aligned} \tag{20}$$

for some constant C_n. By assumption (H3'), and the semi-continuity of separated parts of the spectrum (see Theorem 4.3.16 in Kato [8]), $C := \sup_{n\in\mathbb{N}} C_n < +\infty$. But, for n large,

$$\|[\chi(v_n^+) - \chi(\varphi_n^+)]v_n\|^2 = \int_{v_n\varphi_n<0} v_n^2 \leq \int_{v_n\varphi_n<0} (v_n - \varphi_n)^2\,. \tag{21}$$

Using Hölder's inequality and the fact that $\operatorname{dom} L \subset L^p(\Omega;\mathbb{R})$ for some $p > 2$, we can write

$$\int_{v_n\varphi_n<0} (v_n - \varphi_n)^2 \leq \|v_n - \varphi_n\|_{L^p}^2 \operatorname{meas}\big\{t\in\Omega \mid v_n(t)\varphi_n(t) < 0\big\}^{1-2/p}. \tag{22}$$

As both $\{v_n\}$, $\{\varphi_n\}$ converge to φ in $L^2(\Omega;\mathbb{R})$, we have, by hypothesis (H5),

$$\operatorname{meas}\{v_n(t)\varphi_n(t) < 0\} \leq \operatorname{meas}\{v_n(t)\varphi(t) < 0\} + \operatorname{meas}\{\varphi_n(t)\varphi(t) < 0\} \overset{n\to\infty}{\to} 0\,.$$

Consequently, by (21) and (22), $\|(\chi(v_n^+) - \chi(\varphi_n^+))v_n\| = o(\|v_n - \varphi_n\|_{L^p})$ for $n \to \infty$. A similar result holds for $\|(\chi(v_n^-) - \chi(\varphi_n^-))v_n\|$. Hence, for n large, the last two terms in (20) can be combined with the left hand side to yield an inequality of the type

$$\|v_n - \varphi_n\|_{L^p} \leq C'\big[\varepsilon s_n + \frac{\|f\|}{\|x_n\|}\big]\,, \tag{23}$$

for some $C' > C$. Taking (21), (22) into account, we then see that, in (18), the terms $(\alpha + \varepsilon s_n)\langle[\chi(v_n^+) - \chi(\varphi_n^+)]v_n, \varphi_n\rangle$ and $(\beta + \varepsilon s_n)\langle[\chi(v_n^-) - \chi(\varphi_n^-)]v_n, \varphi_n\rangle$ can be neglected with respect to the sum of the other terms, which have the same (positive) sign for n large. The equality (18) is thus seen to lead to a contradiction. $\diamondsuit$

When the function $S^{n-1} \subset \ker(L - \lambda^* I) \to \mathbb{R} : x \mapsto h_0(x, \alpha, \beta)$ is negative, except at one point $x = x^*$, the result of Theorem 2 is fairly classical. The point (α, β) then lies on the set (in general, a curve) denoted by F^+ in Section 2, and the condition on the degree is automatically satisfied. For instance, when dim $\ker(L-\lambda^* I) = 1$, the conclusions can be seen as a particular case of results of Gallouët and Kavian [7]. However, Theorem 2 can also be used when λ^* is an eigenvalue of multiplicity greater than 1, and when the point (α, β) belongs to a Fučík curve that lies between the sets F^- and F^+. Such types of spectra have been considered in [1]. The application of Theorem 2 to that situation is illustrated by the following example.

Example 1. Fučík curves for the following boundary value problem have been studied in [1].

$$u^{(4)} + (m^2+n^2)u'' = \alpha u^+ - \beta u^-, \tag{24}$$

$$u(0) = u(\pi) = 0, u''(0) = u''(\pi) = 0\,. \tag{25}$$

We choose here $m = 6$ and $n = 13$. The value $\lambda^* = m^2 n^2 = 6084$ is an eigenvalue of multiplicity 2 for the operator

$$L : \operatorname{dom} L \subset L^2((0,\pi);\mathbb{R}) \to L^2((0,\pi);\mathbb{R}) : u \mapsto u^{(4)} + (m^2+n^2)u''\,.$$

We take $\operatorname{dom} L = H^4((0,\pi);\mathbb{R})$, the real Sobolev space of order 4. The eigenspace associated with λ^* is spanned by the functions $\sin mx \sin nx$. It has been shown in [1] that four Fučík curves pass through the point (λ^*, λ^*); their respective slopes at that point are $-0.7232, -6/7, -7/6, -1.3828$. We will assume that (α, β) lies on the curve of slope $-7/6$. We expect hypothesis (H5) of Theorem 2 to hold, although the verification appears difficult. On the other hand, the condition on the degree is seen to be verified (at least for $|\beta-\alpha|$ small). Indeed, the degree, with respect to open bounded sets containing 0, of $x \mapsto c_0(x, \alpha+\varepsilon, \beta+\varepsilon)$ is equal to -1, for ε small (see [1]). Assuming that f satisfies the condition in Theorem 2 (only one function φ needs to be considered here), we conclude that equation (1) has at least one solution, provided the condition (H5) is indeed satisfied.

4 Fučík spectrum reduced to a curve

The Fučík spectrum turns out to be particularly simple when (H3) holds. In this case, it results from Lemma 2 that the sets F^- and F^+ defined in Section 2 coincide and no other point of the Fučík spectrum is contained in $I \times I$. The condition (H3) appears in [1] and, under a different form, in [3]. It is shown there that the condition is satisfied for periodic boundary value problems for ordinary differential equations, when the operator L is autonomous (see [1] or [3] for a precise statement).

Let $f(t,u)$ satisfy the Carathéodory conditions; i.e., $f(\cdot, u)$ is measurable for all $u \in \mathbb{R}$, $f(t, \cdot)$ is continuous for a.e. $t \in \Omega$. Moreover, assume that there exists $K \in L^2(\Omega;\mathbb{R})$, such that

$$|f(t,u)| \le K(t), \text{ for all } u \in \mathbb{R}\,.$$

Let

$$g_{\pm}(t) = \liminf_{u\to\pm\infty} f(t,u), G_{\pm}(t) = \limsup_{u\to\pm\infty} f(t,u).$$

Assume that $G^{+} - g_{-} \in L^{\infty}(\Omega;\mathbb{R})$. For $\varepsilon \geq 0$, and with an arbitrary choice of f_+, f_- satisfying

$$g_-(t) \leq f_-(t) \leq G_-(t),\ g_+(t) \leq f_+(t) \leq G_+(t), \tag{26}$$

we define f_ε as in Section 2 by

$$f_\varepsilon(t,u) = \frac{1}{2}\,[f_+(t) + f_-(t)] + \frac{1}{\pi}\,[f_+(t) - f_-(t)]\arctan(\varepsilon u)\,. \tag{27}$$

To f, we associate the mapping

$$N : L^2(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R}) : u \mapsto \alpha u^+ - \beta u^- + f(\cdot,u)\ ; \tag{28}$$

N_ε is defined as before by

$$N_\varepsilon : L^2(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R}) : u \mapsto \alpha u^+ - \beta u^- + f_\varepsilon(\cdot,u). \tag{29}$$

Under (H3), existence results will be obtained, based on the computation of the coincidence degree $D((L,N),B_r)$, with respect to balls $B_r \subset L^2(\Omega;\mathbb{R})$, centered at 0 and of large radius r (see [12] for a definition of the coincidence degree). Under conditions to be given below, the computation of that degree will be shown to reduce to the computation of the Brouwer's degree of the mapping

$$c_{f_+,f_-} : \ker(L-\lambda^* I) \to \ker(L-\lambda^* I) : x \mapsto \nabla_x\left[\langle f_+,\varphi_x^+\rangle - \langle f_-,\varphi_x^-\rangle\right], \tag{30}$$

with respect to the unit ball in $\ker(L-\lambda^* I)$ (since the function c_{f_+,f_-} is not likely to be continuous at 0, the degree is to be understood as the degree of a continuous extension to the unit ball, of a restriction of c_{f_+,f_-} to the unit sphere). Notice that the mapping c_{f_+,f_-} is homogeneous of degree 0, so that the Brouwer's degree of c_{f_+,f_-} is the same with respect to all balls centered at 0. The precise statement of the relation between the two degrees is given below in Theorem 3. It can be seen as an extension, for the resonance with respect to the Fučík spectrum, of a result of Krasnosel'skii [10] for the "classical" resonance (the case $\alpha = \beta$).

When (H3) is satisfied, it results from Lemma 1 that, with $(\alpha,\beta) \in \Sigma(L) \cap (I \times I)$, there exists a *unique* solution φ_x of (3) such that $P\varphi_x = x$. We will use a weakened version of assumption (H4):

(H4') For some $p > 2$, $\varphi_x \in L^p(\Omega;\mathbb{R})$ for every $x \in \ker(L-\lambda^* I)$ and the mapping $\ker(L-\lambda^* I) \to L^p(\Omega;\mathbb{R}) : x \mapsto \varphi_x$ is Lipschitz continuous.

This will allow to consider non-elliptic operators as the wave operator (see Example 2).

Theorem 3 *Assume that hypotheses (H1), (H2), (H3), (h4'), and (H5) hold. Also assume that L has a compact resolvent and that $f(t,u)$ satisfies the hypotheses given above. Let N and c_{f_+,f_-} be defined by (28) and (30), respectively. Assuming $c_{f_+,f_-}(x) \neq 0$ for any f_+, f_- satisfying (26) and any $x \neq 0$, the coincidence degree $D((L,N),B_r)$, with respect to sufficiently large balls B_r is equal, up to the sign, to the Brouwer's degree*

$$d_B(c_{f_+,f_-}, B_1 \cap \ker(L-\lambda^* I), 0).$$

Of course, the proof needs to show that the above degree is independent of f_+, f_- satisfying (26).

Some preliminary steps are needed for the proof of Theorem 3. We start with a lemma stating that the mapping $x \to \varphi_x$ has a strong Fréchet derivative, denoted $\varphi'_x : \ker(L-\lambda^* I) \to L^2(\Omega;\mathbb{R})$, i.e.

$$\|\varphi_y - \varphi_z - \varphi'_x(y-z)\| = o(\|y-z\|) \text{ for } y \to x, z \to x.$$

Lemma 3 *Let L, α, β satisfy hypotheses $(H_1), (H_2), (H_3), (H_4'), (H_5)$. Then, the function $x \mapsto \varphi_x$ admits a strong Fréchet derivative $\varphi'_x : \ker(L-\lambda^* I) \to L^2(\Omega;\mathbb{R})$ at $x \neq 0$, $w_h = \varphi'_x h$ being the unique solution of*

$$Lw_h = (I-P)[\alpha\chi(\varphi_x^+) + \beta\chi(\varphi_x^-)]w_h + \lambda^* h, \tag{31}$$

$$Pw_h = h\,. \tag{32}$$

Moreover, w_h is also a solution of

$$Lw_h = [\alpha\chi(\varphi_x^+) + \beta\chi(\varphi_x^-)]w_h \tag{33}$$

and, for fixed h, the function $x \mapsto \varphi'_x h$ is continuous with respect to x.

Proof. Notice first that, by the same arguments as for Lemma 1, the system (31), (32) defines a unique function w_h; moreover, it is clear that w_h is a bounded linear function of $h \in \ker(L-\lambda^* I)$. Let us prove that w_h is indeed the strong Fréchet derivative of $x \mapsto \varphi_x$. By definition, we have

$$\begin{aligned} L\varphi_x &= \alpha\varphi_x^+ - \beta\varphi_x^-\ ,\ P\varphi_x = x, \\ L\varphi_y &= \alpha\varphi_y^+ - \beta\varphi_y^-\ ,\ P\varphi_y = y. \end{aligned}$$

With $h = y-z$, and $w_h = w_{y-z}$ defined by (31), (32), we get, by subtraction,

$$\begin{aligned} L(\varphi_y - \varphi_z - w_{y-z}) &= \alpha(I-P)[\varphi_y^+ - \varphi_z^+ - \chi(\varphi_x^+)(\varphi_y - \varphi_z)] \\ &\quad +\alpha(I-P)\chi(\varphi_x^+)(\varphi_y - \varphi_z - w_{y-z}) \\ &\quad -\beta(I-P)[\varphi_y^- - \varphi_z^- - \chi(\varphi_x^-)(\varphi_y - \varphi_z)] \\ &\quad -\beta(I-P)\chi(\varphi_x^-)(\varphi_y - \varphi_z - w_{y-z}), \end{aligned} \tag{34}$$

taking into account that $P(\alpha\varphi_y^+ - \beta\varphi_y^-) = \lambda^* y$, $P(\alpha\varphi_z^+ - \beta\varphi_z^-) = \lambda^* z$. On the other hand, we have

$$P(\varphi_y - \varphi_z - w_{y-z}) = 0. \tag{35}$$

By hypotheses (H4') and (H5), the mappings $u \mapsto u^+$ and $u \mapsto u^-$ have strong Fréchet derivatives at φ_x, as functions from $L^p(\Omega;\mathbb{R})$ to $L^2(\Omega;\mathbb{R})$ (see [1]), their respective derivatives being the mappings $h \mapsto \chi(\varphi_x^+)h$ and $h \mapsto -\chi(\varphi_x^-)h$. This means that

$$\frac{\|u^+ - v^+ - \chi(\varphi_x^+)(u-v)\|}{\|u-v\|_{L^p}} \to 0 \text{ for } u \stackrel{L^p}{\to} \varphi_x, v \stackrel{L^p}{\to} \varphi_x\,.$$

But, the mapping $\ker(L-\lambda^* I) \to L^p(\Omega;\mathbb{R}) : x \mapsto \varphi_x$ is Lipschitz continuous; consequently,

$$\frac{\|\varphi_y^+ - \varphi_z^+ - \chi(\varphi_x^+)(\varphi_y - \varphi_z)\|}{\|y-z\|} \to 0 \text{ for } y \to x, z \to x\,. \tag{36}$$

A similar result holds for the negative parts. On the other hand, it follows from (34), (35), using arguments similar to those of Lemma 1, that, for some $C_1 > 0$,

$$\begin{aligned} \|\varphi_y - \varphi_z - w_{y-z}\| \quad &\le \quad C_1\left[\|\varphi_y^+ - \varphi_z^+ - \chi(\varphi_x^+)(\varphi_y - \varphi_z)\|\right. \\ &\qquad \left. + \|\varphi_y^- - \varphi_z^- - \chi(\varphi_x^-)(\varphi_y - \varphi_z)\|\right] \end{aligned}$$

which, combined with (36), shows that the mapping $h \mapsto w_h$ is the strong Fréchet derivative of $x \mapsto \varphi_x$.

On the other hand, φ_x being a solution of (3), we have

$$P[\alpha\varphi_x^+ - \beta\varphi_x^-] = \lambda^* x, \text{ for all } x \in \ker(L-\lambda^* I). \tag{37}$$

Since $\chi(\varphi_x^+), -\chi(\varphi_x^-)$ are the Fréchet derivatives, at φ_x, of $u \mapsto u^+$ and $u \mapsto u^-$ respectively, we see, by differentiating (37), that

$$P[\alpha\chi(\varphi_x^+)\varphi_x' h + \beta\chi(\varphi_x^-)\varphi_x' h] = \lambda^* h,$$

which derives (33) from (32).

It remains to show that $\varphi_x' h$ is continuous with respect to x. From (31), (32), we have

$$\begin{aligned} L(\varphi_x' h - \varphi_y' h) \quad &= \quad (I-P)[\alpha\chi(\varphi_x^+) + \beta\chi(\varphi_x^-)(\varphi_x' h - \varphi_y' h)] \\ &\qquad + \alpha(I-P)[\chi(\varphi_x^+) - \chi(\varphi_y^+)]\varphi_y' h \\ &\qquad + \beta(I-P)[\chi(\varphi_x^-) - \chi(\varphi_y^-)]\varphi_y' h, \\ P(\varphi_x' h - \varphi_y' h) \quad &= \quad 0. \end{aligned}$$

Using again arguments like in Lemma 1, we obtain, for some constant C_2,

$$\|\varphi_x' h - \varphi_y' h\| \le C_2[\|\chi(\varphi_x^+) - \chi(\varphi_y^+)\| + \|\chi(\varphi_x^-) - \chi(\varphi_y^-)\|]\,.$$

The conclusion then follows from hypothesis (H5). $\diamondsuit$

Lemma 4 *Let L, α, β satisfy hypotheses $(H_1), (H_2), (H_3), (H_4')$ and (H5). Let $\{x_n\} \subset \ker(L-\lambda^* I)$ be such that $\|x_n\| \to \infty, x_n/\|x_n\| \to x^*$. Then, with f_ε defined by (27) (assuming ε fixed, but small enough),*

$$\lim_{n\to\infty} c(x_n, f_\varepsilon, \alpha, \beta) = -c_{f_+,f_-}(x^*). \tag{38}$$

Proof. With the notations of Section 2, we have

$$Lu_x = \alpha u_x^+ - \beta u_x^- + f_\varepsilon(\cdot, u_x) + c(x, f_\varepsilon, \alpha, \beta),$$

and, by the previous lemma, $L(\varphi_x' h) = [\alpha\chi(\varphi_x^+) + \beta\chi(\varphi_x^-)](\varphi_x' h)$. It then follows from the self-adjointness of L, that

$$\begin{aligned} 0 &= \langle f_\varepsilon(\cdot, u_x), \varphi_x' h\rangle + \langle c(x, f_\varepsilon, \alpha, \beta), \varphi_x' h\rangle \\ &\quad + (\alpha - \beta)\langle u_x^+ \chi(\varphi_x^-), \varphi_x' h\rangle + (\beta - \alpha)\langle u_x^- \chi(\varphi_x^+), \varphi_x' h\rangle. \end{aligned} \tag{39}$$

To estimate the last two terms in the above formula, we will use inequalities like

$$\begin{aligned} \langle u_x^+ \chi(\varphi_x^-), \varphi_x' h\rangle &\leq \Big(\int_{u_x > 0, \varphi_x < 0} u_x^2\Big)^{1/2} \Big(\int_{u_x > 0, \varphi_x < 0} (\varphi_x' h)^2\Big)^{1/2} \\ &\leq \|u_x - \varphi_x\| \Big(\int_{u_x > 0, \varphi_x < 0} (\varphi_x' h)^2\Big)^{1/2}. \end{aligned} \tag{40}$$

It is easy to show that $\|u_x - \varphi_x\|$ is bounded, independently of x. Hence, we will have

$$\lim_{n\to\infty} \langle u_{x_n}^+ \chi(\varphi_{x_n}^-), \varphi_{x_n}' h\rangle = 0, \tag{41}$$

if we can show that

$$\lim_{n\to\infty} \int_{u_{x_n} > 0, \varphi_{x_n} < 0} (\varphi_{x_n}' h)^2 = 0. \tag{42}$$

With $\|x_n\| \to \infty$, $x_n/\|x_n\| \to x^*$, we have $(u_{x_n} - \varphi_{x_n})/\|x_n\| \to 0$, $\varphi_{x_n}/\|x_n\| \to \varphi_{x^*}$, and using (H5), it follows that

$$\lim_{n\to\infty} \operatorname{meas}\{t \in \Omega \mid u_{x_n}(t)\varphi_{x_n}(t) < 0\} = 0,$$

which implies

$$\lim_{n\to\infty} \int_{u_{x_n} > 0, \varphi_{x_n} < 0} (\varphi_{x^*}' h)^2 = 0\,.$$

The limit (42) then follows from the continuity of $x \mapsto \varphi_x'$. Using (41) and a similar result for $\langle u_{x_n}^- \chi(\varphi_{x_n}^+), \varphi_{x_n}' h\rangle$, we deduce from (39) that

$$\lim_{n\to\infty} \big[\langle f_\varepsilon(\cdot, u_{x_n}), \varphi_{x_n}' h\rangle + \langle c(x_n, f_\varepsilon, \alpha, \beta), \varphi_{x_n}' h\rangle\big] = 0. \tag{43}$$

But, $f_\varepsilon(t, u_{x_n})$, converges pointwise to $f_+(t)\chi(\varphi_{x^*}^+) + f_-(t)\chi(\varphi_{x^*}^-)$, and, by Lebesgue dominated convergence theorem, convergence is also true in the L^2-norm. On the other hand,

$$\begin{aligned} \langle c(x, f_\varepsilon, \alpha, \beta), \varphi_x' h\rangle &= \langle c(x, f_\varepsilon, \alpha, \beta), P(\varphi_x' h)\rangle \\ &= \langle c(x, f_\varepsilon, \alpha, \beta), h)\rangle. \end{aligned}$$

Putting together those observations, we conclude that

$$\lim_{n\to\infty} \langle c(x_n, f_\varepsilon, \alpha, \beta), h\rangle = -\langle f_+(t)\chi(\varphi_{x^*}^+) + f_-(t)\chi(\varphi_{x^*}^-), \varphi_{x^*}' h\rangle$$

from which (38) follows. ♢

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let us choose arbitrarily f_+, f_- satisfying (26); with that choice of f_+, f_-, let $f_\varepsilon, N_\varepsilon$ be defined as above by (27), (29) respectively. We will first show that, for r sufficiently large, the following equality holds

$$D((L,N),B_r) = D((L,N_\varepsilon),B_r), \tag{44}$$

between the coincidence degree involving the mappings N and N_ε. By the invariance property of the degree with respect to homotopies, that result is established if we can find an a priori bound for the solutions of

$$Lu = sNu + (1-s)N_\varepsilon u, \quad s \in [0,1]. \tag{45}$$

By contradiction, assume that there exist sequences $\{u_n\}, \{s_n\}$, with $\|u_n\| \to \infty$, $s_n \in [0,1]$, such that

$$Lu_n = s_n N u_n + (1-s_n)N_\varepsilon u_n.$$

Let $x_n = Pu_n$; extracting if necessary a subsequence, we can assume that $x_n/\|x_n\|$ converges to some $x^* \in \ker(L-\lambda^* I)$ and s_n converges to some $s^* \in [0,1]$. It is easy to prove that $\|x_n\| \to \infty$ and that $u_n/\|x_n\| \to \varphi_{x^*}$, using the fact that f and f_ε are bounded by an L^2-function. Since, by Lemma 1,

$$Lu_n = N_\varepsilon u_n + c(x_n, f_\varepsilon, \alpha, \beta), \tag{46}$$

we get, subtracting (46) from (45),

$$0 = s_n(Nu_n - N_\varepsilon u_n) - c(x_n, f_\varepsilon, \alpha, \beta),$$

i.e.

$$0 = s_n[f(\cdot, u_n) - f_\varepsilon(\cdot, u_n)] - c(x_n, f_\varepsilon, \alpha, \beta).$$

Multiplying the last equality by $\varphi'_{x_n} h$, for an arbitrary $h \in \ker(L-\lambda^* I)$, we get, using (43) and Lemma 4,

$$s^* \lim_{n\to\infty} \langle f(\cdot,u_n), \varphi'_{x_n} h\rangle + (1-s^*)\langle f_+\chi(\varphi^+_{x^*}) + f_-\chi(\varphi^-_{x^*}), \varphi'_{x^*} h\rangle = 0\,. \tag{47}$$

On the other hand, passing if necessary to a further subsequence, we can assume that $f(\cdot, u_n)$ converges weakly to some function

$$\tilde f_+\chi(\varphi^+_{x^*}) + \tilde f_-\chi(\varphi^-_{x^*}),$$

with

$$\begin{aligned}
\liminf_{u\to-\infty} f(t,u) \le \quad & \tilde f_-(t) \quad \le \limsup_{u\to-\infty} f(t,u), \\
\liminf_{u\to+\infty} f(t,u) \le \quad & \tilde f_+(t) \quad \le \limsup_{u\to+\infty} f(t,u).
\end{aligned}$$

It then follows from (47) that

$$s^*\langle \tilde f_+\chi(\varphi^+_{x^*}) + \tilde f_-\chi(\varphi^-_{x^*}), \varphi'_{x^*} h\rangle + (1-s^*)\langle f_+\chi(\varphi^+_{x^*}) + f_-\chi(\varphi^-_{x^*}), \varphi'_{x^*} h\rangle = 0\,. \tag{48}$$

Let

$$f^* = (s^* \tilde{f}_+ + (1-s^*)f_+)\chi(\varphi^+_{x^*}) + (s^* \tilde{f}_- + (1-s^*)f_-)\chi(\varphi^-_{x^*})\,;$$

The equation (48) can also be written

$$\nabla_x \left[\langle (f^*_+, \varphi^+_{x^*} \rangle - \langle (f^*_-, \varphi^-_{x^*} \rangle] \right\rangle = 0$$

or

$$c_{f^*_+, f^*_-}(x^*) = 0.$$

But, this contradicts the hypotheses and proves (44).

It remains to show that, for r large,

$$D((L, N_\varepsilon), B_r) = \pm d_B(c_{f_+, f_-}, B_1 \cap \ker(L - \lambda^* I), 0).$$

Denoting by $K : \operatorname{Range}(L - \lambda^* I) \to \operatorname{Range}(L - \lambda^* I)$ the right inverse of $L - \lambda^* I$ with respect to P, and using a Lyapunov-Schmidt decomposition, the problem $Lu = N_\varepsilon u$ is transformed into

$$u = Pu + K(I - P)N_\varepsilon + c(x_n, f_\varepsilon, \alpha, \beta).$$

By the homotopy

$$u = Pu + sK(I - P)N_\varepsilon + c(x_n, f_\varepsilon, \alpha, \beta),$$

it is seen that the Leray-Schauder degree, with respect to large balls of

$$u \mapsto u - Pu - K(I - P)N_\varepsilon - c(x_n, f_\varepsilon, \alpha, \beta)$$

is the same as that of

$$u \mapsto u - Pu - c(x_n, f_\varepsilon, \alpha, \beta).$$

By a product formula, this last degree is the same, for r large, as the Brouwer degree $d_B(c(\cdot, f_\varepsilon, \alpha, \beta), B_r \cap \ker(L - \lambda^* I), 0)$. The conclusion then follows from (38). $\diamondsuit$

Working as in [1] (see Theorem 10), the degree $d_B(c_{f_+, f_-}, B_1 \cap \ker(L - \lambda^* I), 0)$ can be computed from the indices of critical points, with negative critical values, of the function

$$\Psi : x \mapsto \langle f_+, \varphi^+_x \rangle - \langle f_-, \varphi^-_x \rangle,$$

restricted to the unit sphere S^{n-1} in $\ker(L - \lambda^* I)$. More precisely, the following corollary holds.

Corollary 1 *Under the hypotheses of Theorem 3, the coincidence degree $D((L, N), B_r)$, with respect to large balls B_r, is, up to the sign, given by*

$$1 - \sum \operatorname*{ind}_x \nabla_x \Psi_{|S_{n-1}}(x)\,, \tag{49}$$

where the summation is taken over $x \in S^{n-1}$ such that $\nabla_x(\Psi_{|S^{n-1}})(x) = 0$ and $\Psi_{|S_{n-1}}(x) < 0$. As a consequence, if the sum of the indices in (49) is different from 1, problem (2) has at least one solution.

When $\dim\ker(L-\lambda^* I) = 1$, letting $\ker(L-\lambda^* I) = \mathbb{R}x^*$, the result must be interpreted as follows: if $\Psi(x^*)\Psi(-x^*) > 0$, then the degree is equal to ± 1; whereas, when $\Psi(x^*)\Psi(-x^*) < 0$, the degree is equal to 0.

The following result is an immediate consequence of the above corollary.

Corollary 2 *Under the hypotheses of Theorem 3, assume that, with g_+, G_- defined by (26),*

$$\langle g_+, \varphi_x^+\rangle - \langle G_-, \varphi_x^-\rangle > 0, \text{ for any } x \in \ker(L-\lambda^* I), x \neq 0. \tag{50}$$

Then, problem (2) has at least one solution.

The condition (50), in which the sign could be reversed (with g_+, G_- replaced respectively by G_+, g_-), is a condition of Landesman-Lazer type. It has been obtained by Gallouët and Kavian [7] for the case $\dim\ker(L-\lambda^* I) = 1$. When f does not depend on u, it amounts to the condition appearing in Theorem 2, i.e. $\langle f, \varphi\rangle > 0$, for any nontrivial solution φ of (3).

Actually, when $\dim\ker(L-\lambda^* I) = 1$, existence results can be obtained by the above approach only when Ψ always takes the same sign (for any f_+, f_- satisfying (26) and any $x \in \ker(L-\lambda^* I), x \neq 0$), a situation which corresponds to Landesman-Lazer conditions. This is no longer true for higher dimensional kernels. It can be seen in particular when $\dim\ker(L-\lambda^* I) = 2$, in which case the result of Corollary 1 takes a very simple form. To state it, we let $\{v^{(1)}, v^{(2)}\}$ be an orthonormal basis of $\ker(L-\lambda^* I)$, define z_θ by

$$z_\theta = \cos\theta\, v^{(1)} + \sin\theta\, v^{(2)}\,,$$

and use the abbreviation φ_θ for φ_{z_θ}.

Corollary 3 *Let the hypotheses of Theorem 3 be satisfied, with $\dim\ker(L-\lambda^* I) = 2$. Assume moreover that, for any f_+, f_- satisfying (26), the function $\Psi : \theta \mapsto \langle f_+, \varphi_\theta^+\rangle - \langle f_-, \varphi_\theta^-\rangle$ has only simple zeros, the number of zeros in $[0, 2\pi)$ being different from 2. Then, equation (2) has at least one solution.*

For equation (1), the arguments can be simplified with respect to the treatment above. Indeed, the problem can then be immediately reduced to a computation of degree in $\ker(L-\lambda^* I)$, by a Lyapunov-Schmidt decomposition. In fact, (1) has a solution if and only if there exists $x \in \ker(L-\lambda^* I)$ such that $c(x, f, \alpha, \beta) = 0$ (for equation (2), the difference is that this reduction is not possible in general). It is not necessary to compute a coincidence degree for the pair (L, N), and, hence, the compactness hypothesis can be removed, leading to the following result.

Theorem 4 *Assume that hypotheses (H1), (H2), (H3), (H4'), and (H5) hold, and that f is a given element of $L^2(\Omega;\mathbb{R})$. For $x \in \ker(L-\lambda^* I)$, let $\Psi(x) = \langle f, \varphi_x\rangle$. Then, if*

$$\sum_x \operatorname{ind} \nabla_x \Psi_{|S_{n-1}}(x) \neq 1,$$

where the summation is taken over $x \in S^{n-1}$ such that $\nabla_x(\Psi_{|S^{n-1}})(x) = 0$ and $\Psi_{|S_{n-1}}(x) < 0$, then equation (1) has at least one solution.

Again, in the case $\dim\ker(L-\lambda^* I) = 2$, a particularly simple existence condition can be written.

Corollary 4 *Let the hypotheses of Theorem 4 be satisfied, with* $\dim\ker(L-\lambda^* I) = 2$. *Assume moreover that the function* $\Psi : \theta \mapsto \langle f, \varphi_\theta\rangle$ *has only simple zeros, the number of zeros in* $[0, 2\pi)$ *being different from* 2. *Then, equation (1) has at least one solution.*

The results of [4], concerning the $2\pi-$periodic boundary-value problem for the equation

$$u'' + \alpha u^+ - \beta u^- = f(t), \tag{51}$$

where f is $2\pi-$periodic, can be seen as an application of the above corollary (at least for the points (α, β) in the Fučík spectrum "not too far" from the diagonal in the (α, β)-plane). Assuming that

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{n},$$

for some integer n, the function φ_θ can be defined by $\varphi_\theta(t) = \varphi(t+\theta)$, φ being a solution of $u'' + \alpha u^+ - \beta u^- = 0$. We can choose

$$\varphi(t) = \begin{cases} \frac{1}{\sqrt{\alpha}} \sin(\sqrt{\alpha} t) & (t \in [0, \frac{\pi}{\sqrt{\alpha}}]), \\ -\frac{1}{\sqrt{\beta}} \sin(\sqrt{\beta}(t - \frac{\pi}{\sqrt{\alpha}})) & (t \in [\frac{\pi}{\sqrt{\alpha}}, \frac{2\pi}{n}]). \end{cases}$$

Computing

$$\Psi(\theta) = \int_0^{2\pi} f(t)\varphi(t+\theta)\, dt,$$

we can assert that (51) has a solution, provided that Ψ has only simple zeros, the number of zeros in $[0, 2\pi)$ being different from 2.

Another application of Corollary 4 is given below.

Example 2. The above result can be applied to the problem

$$u_{tt} - u_{xx} = \alpha u^+ - \beta u^- + f\,, \tag{52}$$

$$u(0,t) = u(\pi, t) = 0\,, \text{ for all } t \in [0, 2\pi], \tag{53}$$

$$u(x,t) = u(x, t+2\pi), \text{ for all } x \in [0,\pi], t \in \mathbb{R}\,, \tag{54}$$

where the function $f : [0,\pi] \times \mathbb{R} \to \mathbb{R} : (x,t) \mapsto f(x,t)$ is 2π-periodic in t, and belongs to $L^2(\Omega; \mathbb{R})$, where $\Omega = (0,\pi) \times (0, 2\pi)$. We define

$$\varphi_{lm}(x,t) = \begin{cases} \frac{\sqrt{2}}{\pi} \sin(lx)\sin(mt), & l \in \mathbb{N}, m \in \mathbb{N}\,, \\ \frac{1}{\pi} \sin(lx), & l \in \mathbb{N}, m = 0\,, \\ \frac{\sqrt{2}}{\pi} \sin(lx)\cos(mt), & l \in \mathbb{N}, -m \in \mathbb{N}\,, \end{cases}$$

The set $\{\varphi_{lm}, l \in \mathbb{N}, m \in \mathbb{Z}\}$ forms an orthonormal basis in $L^2(\Omega;\mathbb{R})$ and each $u \in L^2(\Omega;\mathbb{R})$ admits a representation

$$u = \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} u_{lm}\varphi_{lm},$$

with $u_{lm} = \langle u, \varphi_{lm}\rangle$. The abstract realization of the wave operator $u_{tt} - u_{xx}$, with the periodic-Dirichlet boundary conditions (53), (54), is the linear operator $L : \operatorname{dom} L \subset L^2(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R})$, defined by

$$Lu = \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} (l^2 - m^2) u_{lm}\varphi_{lm},$$

where

$$\operatorname{dom} L = \left\{ u \in L^2(\Omega;\mathbb{R}) \mid \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} |l^2 - m^2|^2 \, |u_{lm}|^2 < \infty \right\}.$$

It is well known that L is densely defined, self-adjoint, closed, that $\operatorname{Im} L = (\ker L)^{\perp}$, and that L has a pure point spectrum $\sigma(L)$ made of eigenvalues:

$$\sigma(L) = \{\lambda_{lm} = l^2 - m^2, l \in \mathbb{N}, m \in \mathbb{Z}\}.$$

It is not hard to see that each eigenvalue $\lambda_{lm} \neq 0$ has finite multiplicity, 0 is of infinite multiplicity, and $\sigma(L)$ can be written in the following form:

$$\sigma(L) = \mathbb{Z} \setminus \{-1, -4, 4m + 2, m \in \mathbb{Z}\}.$$

In order to apply Corollary 4 to problem (52), (53), (54), we take, for instance, $\lambda^* = -3$, and first observe that hypothesis (H_1) is satisfied and that $\lambda^* = -3$ is of multiplicity 2. The eigenvalues closest to -3 are -5 and 0, so that (H_2) is satisfied with $I = [-5 + \varepsilon, -\varepsilon]$, ε being positive. The hypothesis (H_3) follows from the fact that, if $u(x, t)$ is a solution of

$$u_{tt} - u_{xx} = \alpha u^+ - \beta u^-, \tag{55}$$

with the periodic-Dirichlet boundary value conditions (53), (54), the same is true for $u(x, t + \theta)$, for any $\theta \in [0, 2\pi]$ (see [1] for a similar argument applied to ordinary differential equations). Actually, it is easy to check that the Fučík spectrum for that problem contains the curve

$$\frac{1}{\sqrt{1-\alpha}} + \frac{1}{\sqrt{1-\beta}} = 1,$$

passing trough the point $(-3, -3)$, corresponding to the eigenvalue chosen here. The corresponding solutions of (55), (53), (54) can be written explicitly under the form

$$\varphi_\theta(x, t) = \sin x \; v(t + \theta/2), \; \theta \in [0, 2\pi],$$

v being a nontrivial π-periodic solution of

$$v'' + (1-\alpha)v^+ - (1-\beta)v^- = 0.$$

Thus, assumptions (H4'), (H5) are also satisfied.

Applying Corollary 4, we then obtain the following existence result.

Corollary 5 *Let α, β be in $I = [-5+\varepsilon, -\varepsilon]$. Assume that the function*

$$\Psi : \theta \mapsto \int_0^{2\pi} \int_0^{\pi} \sin x \, f(x,t) \; v(t+\theta) \, dx \, dt$$

has only simple zeros, the number of zeros in $[0, 2\pi)$ being denoted by $2z$. Then, if $z \neq 1$, problem (52), (53), (54) has at least one solution.

In particular, if $f(x,t) = G(x)H(t)$, the number $2z$ is equal to the number of zeros, in $[0, 2\pi)$, of the function

$$\theta \mapsto \int_0^{2\pi} H(t) v(t+\theta) \, dt \; ,$$

provided that $\int_0^{\pi} \sin x \, G(x) \, dx \neq 0$.

5 A non-existence result.

The non-existence result presented below complements the existence results of the previous sections. It is inspired from a result written by Lazer and McKenna [9] for the 2π-periodic boundary value problem for the differential equation

$$u'' + \alpha u^+ - \beta u^- = \cos(nx),$$

with n an integer and

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{\beta}} = \frac{2}{n}, (n-1)^2 < \alpha, \beta < (n+1)^2.$$

Theorem 5 *Let L, α, β satisfy hypotheses (H1), (H2), (H3). Then, if*

$$\|Pf\| > \frac{|\beta - \alpha|}{2d - |\beta - \alpha|} \|(I-P)f\|, \tag{56}$$

where $d = \operatorname{dist}((\alpha+\beta)/2, \sigma(L) \setminus \{\lambda^*\})$, the equation

$$Lu = \alpha u^+ - \beta u^- + f$$

has no solution.

Proof. By contradiction, assume that u is a solution. By hypothesis (H3), it is possible to find $\tilde{u}\,\mathrm{dom}\,L$ such that

$$L\tilde{u} = \alpha\tilde{u}^+ - \beta\tilde{u}^-, P\tilde{u} = Pu.$$

Since $u - \tilde{u} \in [\ker(L - \lambda^* I)]^\perp$, letting $\mu = (\alpha + \beta)/2$, we can write

$$(L - \mu I)(u - \tilde{u}) = (I - P)[(\alpha - \mu)(u^+ - \tilde{u}^+) - (\beta - \mu)(u^- - \tilde{u}^-) + f].$$

But, the function $u \mapsto (\alpha - \mu)u^+ - (\beta - \mu)u^-$ is Lipschitzian with Lipschitz constant $|\beta - \alpha|/2$, so that,

$$\|\tilde{L}_\mu^{-1}\|^{-1}\|u - \tilde{u}\| \le \|(I - P)f\| + \frac{|\beta - \alpha|}{2}\,\|u - \tilde{u}\|,$$

with $\tilde{L}_\mu = (L - \mu I)_{|[\ker(L-\lambda^* I)]^\perp}$. Since $\|\tilde{L}_\mu^{-1}\|^{-1} = \mathrm{dist}(\mu, \sigma(L_{|[\ker(L-\lambda^* I)]^\perp})) = d$ and since $d > |\beta - \alpha|/2$, we obtain the relation

$$\|u - \tilde{u}\| \le \frac{\|(I - P)f\|}{d - |\beta - \alpha|/2}. \tag{57}$$

On the other hand, projecting the equations

$$Lu = \alpha u^+ - \beta u^+ + f \quad \text{and} \quad L\tilde{u} = \alpha\tilde{u}^+ - \beta\tilde{u}^+$$

on $\ker(L - \lambda^* I)$, we get

$$\begin{aligned} P[(\alpha - \lambda^*)u^+ - (\beta - \lambda^*)u^-] &= -Pf, \\ P[(\alpha - \lambda^*)\tilde{u}^+ - (\beta - \lambda^*)\tilde{u}^-] &= 0, \end{aligned}$$

or

$$\begin{aligned} P[(\alpha - \mu)u^+ - (\beta - \mu)u^-] &= -Pf + (\lambda^* - \mu)Pu, \\ P[(\alpha - \mu)\tilde{u}^+ - (\beta - \mu)\tilde{u}^-] &= (\lambda^* - \mu)P\tilde{u}, \end{aligned}$$

from which we deduce by subtraction, using again the Lipschitz constant for $u \mapsto (\alpha - \mu)u^+ - (\beta - \mu)u^-$,

$$\|Pf\| \le \frac{|\beta - \alpha|}{2}\|u - \tilde{u}\|.$$

Combining this with (57) leads to

$$\|Pf\| \le \frac{|\beta - \alpha|/2}{d - |\beta - \alpha|/2}\|(I - P)f\|,$$

in contradiction with hypothesis (56). $\diamondsuit$

Notice that, under the hypothesis of Theorem 5, if $\dim(\ker(L-\lambda^* I)) = 2$, the confrontation with Theorem 4 implies that the function $\theta \mapsto \langle f, \varphi_\theta\rangle$ necessarily has two zeros in $[0, 2\pi)$, if its zeros are simple.

Example 2'. Coming back to the periodic-Dirichlet boundary value problem

$$\begin{gathered} u_{tt} - u_{xx} = \alpha u^+ - \beta u^- + f\,, \\ u(0,t) = u(\pi,t) = 0\,, \text{for all } t \in [0,2\pi], \\ u(x,t) = u(x,t+2\pi), \text{ for all } x \in [0,\pi], t \in \mathbb{R}\,, \end{gathered}$$

we see that all the conditions of Theorem 5 are fulfilled with $\lambda^* = -3$, if $(I - P)f = 0, f \neq 0$, i.e. if $f \in \ker(L - \lambda^* I)\setminus\{0\}$. Hence, the boundary value problem written above has no solution if $f(x,t) = \sin x \, \sin 2t$ and if $\alpha, \beta \in [-5+\varepsilon, -\varepsilon]$, for some $\varepsilon > 0$.

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