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On the tidal motion around the earth complicated by the circular geometry of the ocean's shape *

Ranis N. Ibragimov

Abstract

We study the Cauchy-Poisson free boundary problem on the stationary motion of a perfect incompressible fluid circulating around the Earth. The main goal is to find the inverse conformal mapping of the unknown free boundary in the hodograph plane onto some fixed boundary in the physical domain. The approximate solution to the problem is obtained as an application of this method. We also study the behaviour of tidal waves around the Earth. It is shown that on a positively curved bottom the problem admits two different high order systems of shallow water equations, while the classical problem for the flat bottom admits only one system.

1 Introduction

We study the Cauchy-Poisson problem on the stationary motion of a perfect fluid which has a free boundary and has a solid bottom represented by a circle with a sufficiently large radius. We have shown in [4] that such problem can be associated with a two dimensional model to an oceanic motion around the Earth since we consider strictly longitudinal flow. Since the problem is a free boundary problem, the analysis is rather difficult.

The permanent water waves have been considered in a great number of papers. However, most researchers are concerned with fluid motion which is infinitely deep and extends infinitely both rightward and leftward. See Crapper [1], Stoker [9] or Stokes [10] for the history. Such problems are usually called Stoke problem (if the surface tension is neglected) and Wilton's problem (if the surface tension is taken into account).

We consider water waves for which ratio of depth of fluid above the circular bottom to the radius of the circle is small (shallow water). In these the disturbance to the water does not penetrate unchanged to the bottom and the effective inertia of the water is therefore reduced.

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Our primary concern is to find the conformal mapping (for the Stoke problem) of the unknown free boundary onto fixed one. The resulting Dirichlet problem can be solved numerically using Okamoto method [3]. The more detailed structure of the bifurcation of solutions for the related problem was numerically computed by Fujita at al. [3]. The existence of nontrivial solutions for the analogous reduced Dirichlet problem can be found in Okamoto [7], Ibragimov [4] as well as in classical literature (see e.g., [8] or [9]).

The higher order shallow water equations in the non-stationary case are derived in this paper. It is shown that the present problem admits two different systems of shallow water equations while the classical problem for the flat bottom admits only one system (see [2]).

We note that papers [3], [6] and [7] are concerned with fluid whose surface tension is taken into account. In fact, the surface tension plays the role of a "regulator" of the problem which simplifies the analysis substantially. Furthermore the nature of the problem requires the surface tension to be neglected. Thus, the present paper represents more systematic approach to the problem.

The present paper aims to investigate the problem by using a conformal mapping which distinguishes our paper from [3], [4], [6] and [7].

2 Basic Equations

The analysis of this problem is performed in the following notation: R is the radius of the circle, r is a distance from the origin, θ is a polar angle, h_0 is the undisturbed level of the liquid above the circle and $h = h(\theta)$ is the level of the disturbance of the free boundary above the circle. For the sake of simplicity we assume that the pressure is constant on the free boundary. The stream function $\psi = \psi(r, \theta)$ defines the velocity vector, i.e.,

$$v^r = -\frac{1}{r}\psi_\theta, \quad v^\theta = \psi_r\,.$$

Hence irrotational motion of an ideal incompressible fluid of the constant pressure in the homogeneous gravity field g =const is described by the stream function ψ in the domain

$$\Omega_h = \{(r,\theta) : 0 \le \theta \le 2\pi, R \le r \le R + h_0 + h(\theta)\}$$

which is bounded by the bottom $\Gamma_R = \{(r,\theta) : r = R, \theta \in [0, 2\pi]\}$ and the free boundary with equation $\Gamma_h = \{(r,\theta) : r = R + h_0 + h(\theta), \theta \in [0, 2\pi]\}$. Note that ψ is a harmonic function in Ω_h, since we assumed that the flow is irrotational. More specifically, we assume that the fluid is incompressible and inviscid and that the flow is stationary. Then the problem is to find the function $h(\theta)$ and the stationary, irrotational flow beneath the free boundary $r = R + h_0 + h(\theta)$ given by the stream function ψ which satisfy the following differential equations

$$\Delta\psi = 0(\text{in } \Omega_h), \qquad \psi = 0(\text{on } \Gamma_R), \qquad \psi = a \ (\text{on } \Gamma_h), \tag{1}$$

$$|\nabla\psi|^2 + 2gh = \text{constant} \quad (\text{ on } \Gamma_h), \tag{2}$$

$$\tfrac{1}{2}\int_0^{2\pi} (R + h_0 + h(\theta))^2\, d\theta = \pi(R + h_0)^2, \tag{3}$$

where the constant a denotes the flow rate. Equations (1)-(3) represent the free boundary Cauchy-Poisson problem in which the boundary Γ_h is unknown as well as the stream function.

3 The Inverse Transforms Principle

Constant flow

The exact solution

$$h \equiv 0 \quad \text{and} \quad \psi = \psi_0 = a \log r \tag{4}$$

of (1)-(3) corresponds to the constant flow with undisturbed free boundary. The trivial solution (4) represents a flow whose streamlines are concentric circles with the common center at the origin.

The following non-dimensional quantities are introduced:

$$r = R + h_0 r', \quad h = h_0 h', \quad \psi = a\psi', \quad \epsilon = \frac{h_0}{R}, \quad \mathcal{F} = \frac{h_0\sqrt{g h_0}}{a},$$

where $\mathcal{F}$ is a Froude number and R is used as a vertical scale. We consider ϵ as the small parameter of the problem. After dropping the prime, Equations (1)-(3) are written by ψ', h', and (r', θ) as follows

$$\Delta_{(\epsilon)}\psi = 0 \quad (\text{in } \Omega_h), \tag{5}$$

$$\psi = 0 (\text{on } \Gamma_R), \tag{6}$$

$$\psi = 1 (\text{on } \Gamma_h), \tag{7}$$

$$\left|\nabla_{(\epsilon)}\psi\right|^2 + 2\mathcal{F}^{-2}h = \text{ constant} \quad (\text{on } \Gamma_h), \tag{8}$$

$$\tfrac{1}{2}\int_0^{2\pi}(1+\epsilon+\epsilon h(\theta))^2 d\theta = \pi(1+\epsilon)^2 \quad (\text{on } \Gamma_h). \tag{9}$$

Here the Laplace and gradient operators are given by

$$\Delta_{(\epsilon)} = (\epsilon\partial_\theta)^2 + [(1+\epsilon r)\partial_r]^2, \quad \nabla_{(\epsilon)} = \Big(\frac{\epsilon\partial_\theta}{(1+\epsilon r)}, \partial_r\Big),$$

where the subscripts imply the differentiation.

We further consider the complex potential $\omega(\zeta) = \phi + i\psi$ where $\zeta = (1+\epsilon r)e^{i\theta}$ is the independent complex variable and $\phi(\zeta)$ is the velocity potential which is characterized by the analyticity of $\phi + i\psi$, i.e.,

$$\phi_r = \frac{\epsilon\psi_\theta}{(1+\epsilon r)}, \quad \frac{\epsilon\phi_\theta}{(1+\epsilon r)} = -\psi_r.$$

We note that the complex velocity $d\omega/d\zeta$ is a single-valued analytic function of ζ, although ω is not single-valued. In fact, when we turn around the bottom $r = 1$ once, ϕ increases by

$$-\int_0^{2\pi} \psi_r(1,\theta)\, d\theta$$

which has a positive sign by the maximum principle (Hopf's lemma). Hence, if we remove the width of annulus region $\theta = 0$, $r \in [1, 1+\epsilon]$, then at every point (r, θ), $\omega(\zeta)$ is single-valued analytic function which maps the rectangular (in the $\omega(\alpha)$-hodograph plane) domain with ϕ in $[0, -2\pi/\log(1+\epsilon)]$ and ψ in $[0, 1]$ as coordinates onto the annulus

$$\Gamma_h^0 = \{(r, \theta) : 1 < r < 1+\epsilon, \theta \in [0, 2\pi]\}.$$

We represent the constant flow (4) by

$$\omega(\zeta) = \phi + i\psi = \frac{i\log(1+\epsilon r) - \theta}{\log(1+\epsilon)}, \tag{10}$$

where $r = \xi_0(\psi)$ and $\theta = \eta_0(\phi)$ transform the rectangular domain in the hodograph plane onto Γ_h^0. Consequently, each conformal mapping by the function $\omega(\zeta)$ between hodograph and physical planes represents an irrotational flow in the physical ζ plane. Furthermore, (10) implies that

$$\eta_0(\phi) = -\phi\log(1+\epsilon) \quad \text{and} \quad \xi_0(\psi) = \epsilon^{-1}([1+\epsilon]^\psi - 1). \tag{11}$$

Reduction on the boundary

Now a two-dimensional infinitesimal disturbance ξ' and η' is superimposed on ξ_0 and η_0. Then the resulting transform components are

$$\xi = \xi_0 + \xi', \quad \eta = \eta_0 + \eta'.$$

The perturbed quantities ξ' and η' are assumed to be small quantities so that the nontrivial solution is close to the trivial one.

Then with (11) the inverse transform can be combined in form

$$\log\zeta = \psi\log(1+\epsilon) - i\phi\log(1+\epsilon) + \log(1 + \frac{\epsilon\xi'}{1+\epsilon\xi_0}) + i\eta',$$

since $\log(1+\epsilon\xi_0) = \psi\log(1+\epsilon)$. Consequently, the polar angle θ and radius r are given by equations

$$\theta = -\phi\log(1+\epsilon) + \eta', \quad r = \epsilon^{-1}[(1+\epsilon)^\psi(1 + \frac{\epsilon\xi'}{1+\epsilon\xi_0}) - 1].$$

Since the motion is irrotational, we can decrease the dimension of the problem by one. In other words, we introduce the boundary value for the function ξ and reduce the basic equations to the quantities which arise from the condition on the free boundary $\psi = 1$. To this end we introduce the regular function $f(\omega) = \alpha + i\beta$ such that

$$(\alpha, \beta) = \frac{1}{(\lambda^2 + (\eta')2)_\phi}(\lambda_\phi, -\eta'_\phi).$$

Then the nonlinear boundary condition (8) can be reduced to the differential equation of the conservation form for $f(\omega)$ by virtue of the following

Theorem 1 *Let the function $\lambda(\phi,\psi)$ defined by*

$$1+\frac{\epsilon\xi'}{1+\epsilon\xi_0}=e^{\lambda(\phi,\psi)} \tag{12}$$

be differentiable. We introduce the derivative operator $\partial_{\mathbf{n}}$ along the normal to $\psi=1$ by $\partial_{\mathbf{n}}\mu=\lambda_\psi|_{\psi=1}$, where $\mu(\phi)=\lambda(\phi,1)$. Then on the free boundary, the following three relations hold:

$$\beta_\phi=-\partial_{\mathbf{n}}\alpha=-\partial_\phi\partial_{\mathbf{n}}\left\{(\tau+\frac{1+\epsilon^2}{2})e^{2\mu}\right\}, \tag{13}$$

where $\tau=(b-\mu\mathcal{F}^{-2})(1+\epsilon)^2$, and $b=const$ is the Bernoulli constant.

Proof. Since the velocity potential ω defined by (10) is analytic function, ξ and η are single-valued and they satisfy the Cauchy-Riemann equations

$$\frac{\epsilon}{1+\epsilon\xi}\xi'_\phi=\eta'_\psi,\quad \frac{\epsilon}{1+\epsilon\xi}\xi'_\psi-\frac{\epsilon^2(1+\epsilon\xi)^{-1}}{(1+\epsilon\xi_0)}\xi_{0_\psi}=-\eta'_\phi$$

which can be simplified as

$$\lambda_\phi=\eta'_\psi,\qquad \lambda_\psi=-\eta'_\phi. \tag{14}$$

From (11), (14) and the representation $\zeta=(1+\epsilon\xi')e^{i\eta'}$, it follows that

$$\left|\frac{d\zeta}{d\phi}\right|^2=\epsilon^2(\epsilon^{-1}+1)^2e^{2\lambda}\lambda_\phi^2+(1+\epsilon\xi)^2\eta'^2_\phi \tag{15}$$

since $\xi_0=1$ on the free boundary.

By virtue of the (14) and (15) and presentation $\eta_\phi=-\log(1+\epsilon)-\partial_{\mathbf{n}}\mu$, the Bernoulli equation (8) takes the form

$$\tau[e^{2\lambda}\lambda_\phi^2+\frac{1}{(1+\epsilon)^2}((1+\epsilon\xi')^2(\log(1+\epsilon)+\partial_{\mathbf{n}}\mu)^2)]-\frac{1}{2}=0$$

which can be transformed to the conservation law i.e.,

$$\alpha(\phi,1)=\frac{\partial}{\partial\phi}(\tau+\frac{(1+\epsilon^2)}{2})e^{2\mu}. \tag{16}$$

Thus, the first equation in (13) holds due to analyticity of function $f(\omega)$ and the second equation in (13) follows from the definition of the normal derivative operator $\partial_{\mathbf{n}}$. Finally, the last equation is the consequence of the changing of the order of differentiation $\partial_\phi\partial_{\mathbf{n}}\mu=\partial_{\mathbf{n}}\mu_\phi$. ♢

Note that function $\mu(\phi)$ is being found by analyticity of $f(\omega)$ and thus transformation $\xi(\phi,\psi)$ is determined by definition (12) as

$$\xi'=(\frac{1}{\epsilon}+\xi_0(\psi))(e^{\lambda(\phi,\psi)}-1).$$

Solution to the Dirichlet Problem in a fixed domain

In view of Theorem 1, it follows from the definition for function $\alpha(\phi,1)$ and (16) that integrating (13) over ϕ along $\psi=1$ leads to the following equation on the free boundary:

$$4\tau e^{2\mu}[\log(1+\epsilon)+\partial_{\mathbf{n}}\mu]+\partial_{\mathbf{n}}([2\tau+1+\epsilon^2]e^{2\mu})-\epsilon\delta_0=0\,,$$

where δ_0 is the constant of integrating which represent the horizontal impulse flow.

Finally, simplifying the last equation we arrive to the Dirichlet problem in the fixed domain

$$\lambda_{\phi\phi}+\lambda_{\psi\psi}=0 \quad (0<\psi<1), \tag{17}$$

$$\lambda(\phi,0)=0, \quad \lambda(\phi,1)=\mu, \tag{18}$$

$$\begin{aligned}\mu\partial_{\mathbf{n}}\mu\mathcal{F}^{-2}-(b+\frac{1}{4}[1-\mathcal{F}^{-2}]-\frac{\epsilon}{2})\partial_{\mathbf{n}}\mu&\\ +\frac{\log(1+\epsilon)}{2}(\mu\mathcal{F}^{-2}-b)+\frac{\delta e^{-2\mu}}{(1+\epsilon)^2}\quad&=\quad 0\,,\end{aligned} \tag{19}$$

where we denote $\delta=\frac{\delta_0}{8}$.

Now the problem (17)-(19) is reduced to finding of one function $\mu(\phi)$ since if function μ is known then function $\lambda(\phi,\psi)$ is defined as the solution of the mixed problem for the Laplace equation (17) and the boundary conditions (18). In particular, $\lambda_\psi|_{\psi=1}$ can be considered as the result of action of the operator $\partial_{\mathbf{n}}$ on function μ. Namely, we represent $\lambda(\phi,\psi)$ by the Fourier series (see for example Ovsjannikov [8] or Stoker [9]). Then the dependence between $n-th$ Fourier coefficients of functions λ, μ and $\partial_{\mathbf{n}}\mu$ is given by

$$[\lambda(\phi,\psi)]_n=\frac{\sinh n\psi}{\sinh n}\mu_n, \quad [\mu(\phi)]_n=\mu_n \quad [\partial_{\mathbf{n}}\mu(\phi)]_n=\mu_n\cot n\,, \tag{20}$$

in which $\mu(\phi)=\mu_n e^{in\phi}$ (summation is assumed). Thus problem (17)-(19) is written in terms of $[\mu(\phi)]_n$ only. Since $[\partial_{\mathbf{n}}\mu]_n$ are given by (20), we can represent the disturbance $\mu(\phi)$ by the expansion in series with respect to parameter ϵ (see also Ovsjannikov [8]). Consequently we apply the stretching transformation and expansion

$$(\mu,\partial_{\mathbf{n}}\mu,)=\sum_0^\infty \epsilon^i\left\{\epsilon(\overline{\mu}_i,\epsilon(\overline{\partial}_{\mathbf{n}}\overline{\mu})_i)\right\} \quad (i=0,\infty) \tag{21}$$

which is a characteristic of a shallow water. Substitution of representation (21) into (19) and elimination mod ϵ^3 (neglecting of the terms with ϵ^m, $m\geqslant 4$) yields the approximate solution of the form $[\mu_i]_n=[\mu_i(b,\mathcal{F})]_n$ as follows:

$$\begin{aligned}&\delta-\frac{b}{2}++\epsilon\Big\{-2\delta+\frac{b}{4}+\frac{\mathcal{F}^{-2}}{2}\mu_1-\frac{1}{4}[1-\mathcal{F}^{-2}+4b](\partial_{\mathbf{n}}\mu)_1-2\delta\mu_1\Big\}\\ &+\epsilon^2\Big\{3\delta-\frac{b}{6}-\frac{\mathcal{F}^{-2}}{4}\mu_1+\frac{\mathcal{F}^{-2}}{4}\mu_2+\frac{1}{2}(\partial_{\mathbf{n}}\mu)_1-\frac{1}{4}[1-\mathcal{F}^{-2}+4b](\partial_{\mathbf{n}}\mu)_2\end{aligned}$$

$$
\begin{aligned}
&+4\delta\mu_1 + 2\delta\mu_1^2 - 2\delta\mu_2 + \mu_1(\partial_{\mathbf{n}}\mu)_1\big\} + \epsilon^3\big\{-4\delta + \frac{b}{8} + \frac{\mathcal{F}^{-2}}{6}\mu_1 - \frac{\mathcal{F}^{-2}}{4}\mu_2 \\
&+\frac{\mathcal{F}^{-2}}{2}\mu_3 + \frac{1}{2}(\partial_{\mathbf{n}}\mu)_2 - \frac{1}{4}[1-\mathcal{F}^{-2}+4b](\partial_{\mathbf{n}}\mu)_3 - 6\delta\mu_1 - 4\delta\mu_1^2 - \frac{4}{3}\delta\mu_1^3 \\
&+4\delta\mu_2 - 2\delta\mu_3 + 4\delta\mu_1\mu_2 + \mu_1(\partial_{\mathbf{n}}\mu)_2 + (\partial_{\mathbf{n}}\mu)_1\mu_2\big\} + o(\epsilon^4). \qquad (22)
\end{aligned}
$$

Thus, in view of (20), Expression (22) represents the recurrent system of algebraic equations for determination of all $[\mu_i]_n$, where the horizontal impulse flow has asymptotic $\delta = \frac{b}{2}$.

The shape of the free boundary $h(\theta)$ can be determined numerically using Okmaoto's method [3]. The existence of exact solution (ψ, h) can be established analytically by the Fixed Point Theorem (see, for example, Okamoto & Shoji [7] or Ibragimov [4]).

4 Behavior of Tides waves

Existence of stationary waves

The main concern of this part is the evolution of tides around the Earth in time t. In order to bring out the essential parameters of the problem, the dimensional fundamental equations, (1)-(2), are written in non-stationary case as follows:

$$
\begin{gathered}
\Delta(\psi) = 0 \quad (\text{in } \Omega_h), \\
\psi_\theta = 0 \quad (\text{on } \Gamma_R), \\
rh_t + \psi_\theta + \psi_r h_\theta = 0 \qquad (\text{on } \Gamma_h), \\
-h_\theta\psi_{t\theta} + r^2\psi_{tr} + \tfrac{r}{2}\big(\tfrac{\psi_\theta^2}{r^2} + \psi_r^2\big)_\theta + rgh_\theta = 0 \quad (\text{on } \Gamma_h),
\end{gathered}
$$

where $\Delta = (\partial_{\theta\theta} + r^2\partial_{rr} + r\partial_r)$.

The perturbed quantities h' and ψ' are interrelated as

$$
h = h', \quad \psi = -\frac{\gamma}{2\pi}\log r + \psi',
$$

where γ is the intensity of the vortex. For the small disturbances we obtain the linear problem in the domain $D_0 = \{(r,\theta) : R \leqslant r \leqslant R + h_0, 0 \leqslant \theta \leqslant 2\pi\}$ as follows

$$
\Delta(\psi') = 0 \quad (\text{in } D_0), \tag{23}
$$

$$
\psi'_\theta = 0 \quad (\text{on } \Gamma_R), \tag{24}
$$

$$
h'_\theta + \tfrac{\psi'_\theta}{r} - \tfrac{\gamma h'_\theta}{2\pi r^2} = 0 \quad (\text{on } \Gamma_{h_0}), \tag{25}
$$

$$
r^2\psi'_{tr} - \tfrac{\gamma}{2\pi}\psi'_{r\theta} + rgh'_\theta = 0 \quad (\text{on } \Gamma_{h_0}). \tag{26}
$$

Since Equations (23)-(26) are linear, the method of superposition is applicable (see also Friedrichs [2]). Hence it is sufficient to look for periodic solutions of the form

$$
(h', \psi') = (H, \Psi(r))\exp\{i(k\theta - wt)\} \tag{27}
$$

in which the wave number k is a given real quantity and eigenvalues w give the different modes of the tide's wave propagation. Substitution of representation (27) into (23)-(26) leads to the expression

$$\Psi(r) = c(r^k - R^{2k}r^{-k})$$

and to the equations

$$\begin{gathered} H(w + \frac{k\gamma}{2\pi r^2}) - \frac{kc}{r}(r^k - R^{2k}r^{-k}) = 0, \\ \frac{kg}{cr}H - (w + \frac{k\gamma}{2\pi r^2})(kr^{k-1} + kR^{2k}r^{-k-1}) = 0 \end{gathered}$$

in which c is a constant of integrating. Consequently, the determinantal equation for the longitudinal tide wave is as follows

$$w = \pm\sqrt{\frac{\frac{kg}{(R+h_0)}\left[(R+h_0)^{k-1} - R^{2k}(R+h_0)^{-k-1}\right]}{(R+h_0)^{k-1} + R^{2k}(R+h_0)^{-k-1}}} - \frac{k\gamma}{2\pi(R+h_0)^2} \tag{28}$$

Thus surface tide waves (on the constant flow) are dispersive with two different modes of propagation. Simplification of relation (28) shows that the tide wave is propagated with a speed

$$a_0 = \frac{w}{k} = \pm\sqrt{gh_0}\sqrt{\frac{\tanh[k\ln(1+\epsilon)]}{kRh_0(1+\epsilon)}} - \frac{\gamma}{2\pi R^2(1+\epsilon)^2}.$$

Hence the condition of the existence of stationary tide waves ($a_0 = 0$) is

$$|\gamma| \leqslant 2\pi R\epsilon^{-\frac{1}{2}}(1+\epsilon)^2\sqrt{gh_0}.$$

Splitting phenomena for shallow water equations

We suppose that the parameter ϵ is infinitesimally small. So we consider R as the natural physical scale. Note that kinematic condition can be written as the mass balance equation. Namely,

$$h_t + (R+h)^{-1}\partial_\theta \int_R^{R+h} v^\theta dr = 0 \tag{29}$$

since the radial velocity component is given by

$$v^r = -r^{-1}\int_R^{R+h} v^\theta_\theta \, dr\,.$$

Hence the mass balance equation (29) takes the form

$$rh_t + \partial_\theta(uh) = 0 \quad (\text{on } \Gamma_h),$$

where the average velocity $u(\theta, t)$ is defined by relation

$$u(\theta,t) = h^{-1} \int_R^{R+h} v^\theta(r,\theta,t)\, dr\,.$$

To go further, it is better to introduce an nondimensionalization here. We put

$$t = \frac{R}{U} t', \quad \psi = h_0 U \psi', \quad u = U u',$$

where U is a unit of velocity. Hereafter index prime will be omitted. Then the impulse equation is written as

$$-\frac{\epsilon^2 h_\theta \psi_{t\theta}}{(1+\epsilon h)^2} + \psi_{tr} + \frac{1}{2(1+\epsilon h)} \partial_\theta \Big(\frac{\epsilon^2 \psi_\theta^2}{(1+\epsilon h)^2} + \psi_r^2 \Big) + \frac{h_\theta}{(1+\epsilon h)} = 0. \tag{30}$$

We represent the stream function ψ by the Lagrangian expansion (see also Ovsjannikov [8] or Friedrichs [2])

$$\psi = \sum_{i=0}^{\infty} \epsilon^i \psi^{(i)}\,.$$

Then the Laplace equation takes the form $(\operatorname{mod} \epsilon^2)$

$$\begin{aligned} &\psi_{rr}^{(0)} + \epsilon(\psi_{rr}^{(1)} + 2r\psi_{rr}^{(0)} + \psi_r^{(0)}) \\ +&\epsilon^2(\psi_{\theta\theta}^{(0)} + \psi_{rr}^{(2)} + 2r\psi_{rr}^{(1)} + r^2\psi_{rr}^{(0)} + \psi_r^{(1)} + r\psi_r^{(0)}) = 0\,. \end{aligned} \tag{31}$$

Equation (31) represents the recurrent system of differential equations for the determination of $\psi^{(i)}$ as the solution of the Cauchy problem with boundary conditions $\psi(0,\theta,t) = 0$, $\psi(1+h,\theta,t) = uh$ for $\psi^{(0)}$ and zero boundary conditions for $\psi^{(1)}$ and $\psi^{(2)}$. Hence function ψ $(\operatorname{mod} \epsilon^2)$ is as follows:

$$\psi = ur + \epsilon\Big(u\frac{r^2}{2} - uh\frac{r}{2}\Big) + \epsilon^2\Big(-u_{\theta\theta}\frac{r^3}{6} + uh\frac{r^2}{4} + u_{\theta\theta}h^2\frac{r}{6} - uh^2\frac{r}{4}\Big). \tag{32}$$

We use the Tailor expansion

$$(1+\epsilon h)^{-1} = 1 - \epsilon h + (\epsilon h)^2 + \cdots \tag{33}$$

to write (30) as

$$\psi_{tr} + \frac{1}{2}(\epsilon^2\psi_\theta^2 + \psi_r^2)_\theta - \epsilon^2 h_\theta \psi_{t\theta} + (\epsilon h - 1)\Big(\frac{1}{2}\epsilon h(\psi_r^2)_\theta + \epsilon h h\Big)_\theta + h_\theta = 0\,. \tag{34}$$

Multiply (30) by $(1+\epsilon h)^2$ and then use expansion (33). Then (30) becomes

$$\psi_{tr} + \frac{1}{2}(\epsilon^2\psi_\theta^2 + \psi_r^2)_\theta - \epsilon^2 h_\theta \psi_{t\theta} + \epsilon h\Big(\psi_{tr}(2+\epsilon h) + \frac{1}{2}(\psi_r^2)_\theta + h_\theta\Big) + h_\theta = 0\,. \tag{35}$$

If we substitute ψ defined by (32) into (34), we obtain the following equation of the shallow water theory

$$
\begin{aligned}
u_t + uu_\theta + h_\theta + \epsilon(\frac{h}{2}u_t - \frac{u}{2}h_t + u^2 h_\theta - hh_\theta) \\
+\epsilon^2\Big(hh_\theta u_{\theta t} - \frac{h^3}{3}u_{\theta\theta t} + \frac{h^2}{4}u_t + \frac{h}{3}u_{\theta\theta}h_t + hh_\theta u_\theta^2 \\
+h^2 u_\theta u_{\theta\theta} + \frac{3}{4}uu_\theta + hu^2 h_\theta - \frac{1}{3}u_\theta u_{\theta\theta} - \frac{1}{3}uu_{\theta\theta\theta} + h^2 h_\theta\Big) &= 0\,.
\end{aligned}
\tag{36}
$$

Consequently, the substitution of ψ (32) into (35) yields

$$
\begin{aligned}
u_t + uu_\theta + h_\theta + \epsilon\Big(\frac{h}{2}u_t - \frac{u}{2}h_t + \frac{h}{2}uu_t + \frac{u^2}{2}h_\theta + \frac{3}{2}huu_\theta\Big) \\
+\epsilon^2\Big(- hh_\theta u_{\theta t} - \frac{h^2}{3}u_{\theta\theta t} + \frac{9}{4}h^2 u_t + \frac{h}{3}h_t u_{\theta\theta} - uhh_t \\
+hh_\theta u_\theta^2 + h^2 u_\theta u_{\theta\theta} + \frac{3}{4}h^2 uu_\theta + \frac{1}{4}u^2 hh_\theta - \frac{1}{3}u_{\theta\theta\theta} \\
+\frac{3}{4}h^2 u_\theta + \frac{1}{2}uh_\theta + \frac{uh}{2}h_\theta\Big) &= 0\,.
\end{aligned}
\tag{37}
$$

Equatrions (36) and (37) supplied with the kinematic condition

$$
\partial_t(\epsilon h^2 + 2h) + 2\partial_\theta(uh) = 0 \tag{38}
$$

represent two systems of the shallow water equations.

To verify that these two systems are different, we compute their first integrals in the stationary case as follows

$$
\begin{aligned}
h + \epsilon(2c^2 h - \frac{1}{2}h^2) + \epsilon^2(3c^2 h^2 - \frac{1}{3}h^3) = J_1, \\
h + \frac{1}{2}\epsilon c^2 h + \frac{1}{2}\epsilon^2(\frac{17}{4}c^2 h^2 - \frac{1}{2}c^2 h^2 - ch) = J_2,
\end{aligned}
$$

where c, J_1, J_2 are constants of integration. Obviously, $J_1 \neq J_2$.

At first sight, it seems that the problem (23)-(26) does not have a unique solution because of that fact. However, it can be shown that the solution of the problem is invariant with respect to the decomposition of the function which represent the free boundary. Since it is not difficult, the proof is omitted.

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Ranis N. Ibragimov
Department of Mathematics and Statistics
University of Victoria
P.O. Box 3045 STN CSC
Victoria, B.C., V8P 2A1, Canada
e-mail: ranis@nbnet.nb.ca