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EXISTENCE OF SOLUTIONS FOR A SUBLINEAR SYSTEM OF ELLIPTIC EQUATIONS

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ABSTRACT. We study the existence of non-trivial non-negative solutions for the system

$$\begin{aligned} -\Delta u &= |x|^a v^p \\ \Delta v &= |x|^b u^q \,, \end{aligned}$$

where p and q are positive constants with $pq < 1$, and the domain is the unit ball of $\mathbb{R}^N$ $(N > 2)$ except for the center zero. We look for pairs of functions that satisfy the above system and Dirichlet boundary conditions set to zero. Our results also apply to some super-linear systems.

1. INTRODUCTION

The purpose of this paper is to study the existence of non-trivial non-negative solutions to the Dirichlet problem

$$\begin{aligned} -\Delta u &= |x|^a v^p \quad \text{in } B', \\ \Delta v &= |x|^b u^q \quad \text{in } B', \\ u &= v = 0 \quad \text{on } \partial B \,, \end{aligned} \tag{1.1}$$

where $p > 0$, $q > 0$, $pq < 1$, B is the unit ball of $\mathbb{R}^N$ $(N > 2)$ centered at 0, and $B' = B \setminus \{0\}$.

By a non-negative solution of (1.1) we mean a pair of functions u, v in $C^2(B')$ such that $u \geq 0$, $v \geq 0$, and (u, v) satisfies (1.1). Note that u is super-harmonic whereas v is sub-harmonic in B'.

In [4], we proved the existence of solutions for (1.1) in the super-linear case, $pq > 1$. Bidaut-Veron and Grillot [3] studied the behavior of solutions near zero and the non-existence of non-negative solutions without boundary conditions.

A non-negative non-trivial solution (u, v) is said to be *singular* at zero (or just singular) if

$$\limsup_{x \to 0} (u(x) + v(x)) = +\infty \,.$$

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Note that since v is sub-harmonic it must be singular at zero, and thus any non-trivial non-negative solution to (1.1) is singular at zero.

Let

$$L := \limsup_{x\to 0} |x|^{N-2}(u(x)+v(x)).$$

If $0 < L < +\infty$, we say that (u,v) has a *fundamental* singularity. If $L = +\infty$, we call this singularity a *strong* singularity.

The existence of singular non-negative solutions of fundamental type for systems more general than (1.1) was proved in [5]. Recall that for

$$\begin{gathered} -\Delta u = u^q, \quad u > 0 \quad \text{in } B', \\ u = 0 \quad \text{on } \partial B, \end{gathered} \tag{1.2}$$

solutions that are singular and non-negative exist if

$$q < \frac{N+2}{N-2}.$$

In such a case, the solution u with a singularity at zero satisfies

$$0 \le \limsup_{x\to 0} |x|^{N-2}u(x) < +\infty,$$

and thus the singularity is of fundamental type. See Lions [8], Ni and Sacks [9], Lin [7] and the references therein.

Brézis and Veron [2] showed that for $q \ge N/(N-2)$ solutions of

$$\Delta u = |u|^{q-1}u \quad \text{in } B' \tag{1.3}$$

are bounded near zero. For $q < N/(N-2)$, Veron [10] proved the existence of non-negative singular solutions of (1.3) with either a strong or a fundamental singularity at zero.

Next, we state our main result for Problem (1.1).

Theorem 1.1. *Let $p > 0$, $q > 0$ and $pq < 1$. Then there exists a non-trivial non-negative solution to* (1.1) *if and only if*

$$p < \frac{N+a}{N-2} \ \text{ and } \ N + a + \beta p > 0, \tag{1.4}$$

where

$$\beta := b + 2 - (N-2)q\,. \tag{1.5}$$

Moreover, if (a,b,p,q) satisfies (1.4)*, then for any $c > 0$, there exists a non-negative solution (u,v) such that*

$$\lim_{x\to 0} |x|^{N-2}u(x) = c\,.$$

If in addition

$$q \ge \frac{N+b}{N-2}\,,$$

the above solution has a singularity of strong type at zero.

In Section 2, we shall prove the existence of singular non-negative solutions for a system more general than (1.1); see Theorems 2.1 and 2.3 below. As for (1.3), under additional assumptions for (1.1), we find both fundamental and strong types of singularities. In Section 3, we prove Theorem 1.1, and give some applications of our result for bi-harmonic equations.

2. Existence results for general systems

In this section, we prove the existence of singular non-negative radially symmetric solutions to

$$\begin{gathered}-(r^{N-1}u'(r))' = r^{N-1}f(r,v(r)) \quad \text{in } (0,1),\\ (r^{N-1}v'(r))' = r^{N-1}g(r,u(r)) \quad \text{in } (0,1),\\ u(1)=v(1)=0,\end{gathered} \tag{2.1}$$

without sub-linear type conditions. In particular the results in this section apply to (1.1) with $pq \neq 1$. When $pq < 1$, our results are optimal as stated in Theorem 1.1. When $pq > 1$, our results extend some results in [4] to the inequality case.

Throughout this section we will assume that f and g are non-negative continuous functions from $(0,1)\times\mathbb{R}^+$ to $\mathbb{R}$ and satisfying

$$0 \le f(r,s) \le f_1(r,s), \quad 0 \le g(r,s) \le g_1(r,s), \tag{2.2}$$

where f_1 and g_1 are continuous functions that are non-decreasing as functions of s.

Set $u_0(r) := r^{2-N}-1$, and fixed positive values α and d, define

$$v_\alpha(r) := du_0(r) + \int_r^1 s^{1-N}\int_s^1 t^{N-1}g_1(t,\alpha t^{2-N})\,dt\,ds\,. \tag{2.3}$$

To state the main result of this section, we assume that

$$\Lambda_\alpha := (N-2)^{-1}\int_0^1 t^{N-1}f_1(t,v_\alpha(t))dt < \infty\,. \tag{H1}$$

Theorem 2.1. *Assume that f and g are two non-negative continuous functions satisfying* (2.2)*. Assume that there exists $\alpha > 0$ such that (H1) is satisfied and $\Lambda_\alpha < \alpha$. Then there exist infinitely many positive solutions to* (2.1)*. Moreover, for any $c \in [0, \alpha-\Lambda_\alpha)$ there exists a solution (u,v) such that*

$$\lim_{r\to 0^+} r^{N-2}u(r) = c\,. \tag{2.4}$$

Proof. Let c be such that $0 \le c < \alpha - \Lambda_\alpha$. Consider the the system of integrals

$$\begin{aligned}u(r) &= cu_0(r) + \int_r^1 s^{1-N}\int_0^s t^{N-1}f(t,v(t))\,dt\,ds\,,\\ v(r) &= du_0(r) + \int_r^1 s^{1-N}\int_s^1 t^{N-1}g(t,u(t))\,dt\,ds\,.\end{aligned} \tag{2.5}$$

Define the operator $T=(T_1,T_2)$, where

$$\begin{aligned}T_1(u,v)(r) &= cu_0(r) + \int_r^1 s^{1-N}\int_0^s t^{N-1}f(t,v(t))\,dt\,ds\,,\\ T_2(u,v)(r) &= du_0(r) + \int_r^1 s^{1-N}\int_s^1 t^{N-1}g(t,u(t))\,dt\,ds\,.\end{aligned} \tag{2.6}$$

Then a non-negative fixed point (u,v) of the operator T is is a non-negative solution to (2.1). To apply the Schauder fixed point Theorem to T, we do the following three steps. First construct an invariant set $\mathcal{M}$ under T. Second transform the set $\mathcal{M}$ into a set $\mathcal{A}$, and thus the operator T into an operator W. Third prove the continuity and compactness of W on $\mathcal{A}$.

Step 1. Let $\mathcal{M}$ be a subset of $(C(0,1])^2$ defined by

$$\mathcal{M} := \{(u,v) : 0 \le u(r) \le \alpha r^{2-N}, \quad 0 \le v(r) \le v_\alpha(r)\}. \tag{2.7}$$

Next, we show that $T(\mathcal{M}) \subset \mathcal{M}$. Let $(u,v) \in \mathcal{M}$, and thus $v(r) \le v_\alpha(r)$. Therefore, from the definition of T_1 and (2.2) we have

$$\begin{aligned} T_1(u,v)(r) \le & cu_0(r) + \int_r^1 s^{1-N} \int_0^s t^{N-1} f_1(t, v_\alpha(t)) dt ds \\ \le & cu_0(r) + (N-2)\Lambda_\alpha \int_r^1 s^{1-N} ds \\ \le & \alpha r^{2-N}, \end{aligned}$$

where we used the choice of c. Now, we show that $T_2(u,v)(r) \le v_\alpha(r)$. Since $(u,v) \in \mathcal{M}$, and from the definition of v_α given by (2.3)

$$T_2(u,v)(r) \le du_0(r) + \int_r^1 s^{1-N} \int_s^1 t^{N-1} g_1(t, \alpha t^{2-N})\, dt\, ds = v_\alpha(r).$$

Step 2. Let $\varepsilon > 0$, and let $\vartheta \in C^1((0,1)) \cap C([0,1])$ be a non-negative function such that

$$\vartheta(r) := \begin{cases} 0 & \text{if } r = 0, \\ v_\alpha^{-1-\varepsilon}(r) & \text{if } r \in (0, 1/2), \\ 1 & \text{if } r \in (3/4, 1]. \end{cases}$$

Since $v_\alpha(r) \ge dr^{2-N}$ near zero, the continuity of ϑ at zero follows.

Let $\mathcal{A}$ be the subset of $(C[0,1])^2$ defined by

$$\mathcal{A} = \{(y,z) : 0 \le y(r) \le \alpha r^\varepsilon, \ 0 \le z(r) \le \vartheta(r) v_\alpha(r)\}.$$

Define in $\mathcal{A}$ the operator

$$W(y,z)(r) = (W_1(y,z)(r), W_2(y,z)(r)), \tag{2.8}$$

where

$$\begin{aligned} W_1(y,z)(r) &= r^{N-2+\varepsilon} T_1\left(r^{2-N-\varepsilon} y(r), \vartheta^{-1}(r) z(r)\right), \\ W_2(y,z)(r) &= \vartheta(r) T_2\left(r^{2-N-\varepsilon} y(r), \vartheta^{-1}(r) z(r)\right), \end{aligned} \tag{2.9}$$

and thus

$$\begin{aligned} W_1(y,z)(r) &= r^{N-2+\varepsilon}\Big(cu_0(r) + \int_r^1 s^{1-N} \int_0^s t^{N-1} f\left(t, \vartheta^{-1}(t) z(t)\right)\, dt\, ds\Big), \\ W_2(y,z)(r) &= \vartheta(r)\Big(du_0(r) + \int_r^1 s^{1-N} \int_s^1 t^{N-1} g\left(t, t^{2-N-\varepsilon} y(t)\right)\, dt\, ds\Big). \end{aligned} \tag{2.10}$$

By (2.8) and (2.9) we have that (y,z) is a fixed point of W if and only if $(u,v) = (r^{2-N-\varepsilon} y, \vartheta^{-1} z)$ is a fixed point of T. Moreover, from Step 1 we have that $W(\mathcal{A}) \subset \mathcal{A}$. Furthermore, $\mathcal{A}$ is a closed convex bounded subset of $(C[0,1])^2$. In order to show existence of a fixed point, via Schauder fixed point theorem, to W in $\mathcal{A}$ it is enough to prove that W is a continuous and compact operator, which is done in the next step.

Step 3. First, we show that $W(\mathcal{A})$ is a relatively compact subset of $(C[0,1])^2$. Since $W(\mathcal{A})$ is bounded, by Ascoli-Arzela theorem, it is enough to prove that $W(\mathcal{A})$ is an equicontinuos subset of $(C[0,1])^2$. This can be done by proving the existence of two functions $\psi, \varphi \in L^1(0,1)$ and a positive constant C such that for any $r \in [0,1]$,

$$\left|\frac{d}{dr}W_1(y,z)(r)\right| \le C\psi(r) \tag{2.11}$$

and

$$\left|\frac{d}{dr}W_2(y,z)(r)\right| \le C\varphi(r)\,. \tag{2.12}$$

From (2.10) and with $\,' = d/dr$ we have

$$\begin{aligned} W'_1(y,z)(r) =&(N-2+\varepsilon)r^{-1}W_1(y,z)(r) \\ &- c(N-2)r^{\varepsilon-1} - r^{\varepsilon-1}\int_0^r t^{N-1}f\left(t, \vartheta^{-1}(t)z(t)\right)\,dt\,. \end{aligned}$$

Thus, using invariance property of W in $\mathcal{A}$ and the definition of Λ_α we obtain

$$\left|\frac{d}{dr}W_1(y,z)(r)\right| \le \Big((N-2)(\alpha+c+\Lambda_\alpha)+\varepsilon\alpha\Big)r^{\varepsilon-1}.$$

Hence, W_1 satisfies (2.11) with $\psi(r) = r^{\varepsilon-1}$. Similarly, by (2.10) we obtain

$$\begin{aligned} W'_2(y,z)(r) =&\frac{\vartheta'(r)}{\vartheta(r)}W_2(y,z)(r) - d(N-2)r^{1-N}\vartheta(r) \\ &- \vartheta(r)r^{1-N}\int_r^1 t^{N-1}g(t, t^{2-N-\varepsilon}y(t))dt\,. \end{aligned}$$

Using again the invariance property of W in $\mathcal{A}$ we obtain

$$|W'_2(y,z)(r)| \le |\vartheta'(r)|v_\alpha(r) + \vartheta(r)r^{1-N}\big((N-2)d + \int_r^1 t^{N-1}g_1(t,\alpha t^{2-N})dt\big),$$

and by definition (2.3) of v_α we have

$$|W'_2(y,z)(r)| \le |\vartheta'(r)|v_\alpha(r) + \vartheta(r)|v'_\alpha(r)| = \varphi(r).$$

The function $\varphi \in L^1(0,1)$, since it is bounded for $r > 1/2$ and for r near zero

$$\varphi(r) = -(2+\varepsilon)v'_\alpha(r)v_\alpha^{-1-\varepsilon}(r).$$

Finally, we prove the continuity of W in $\mathcal{A}$. Let (y_n, w_n) be any sequence converging on $\mathcal{A}$ to (y,w) and let $r \in [0,1]$ be fixed. From the definition of W given by (2.10) and the continuity of $u \mapsto f(t,u)$, $u \mapsto g(t,u)$, uniform convergence of (y_n, w_n) to (y,w) and the Lebesgue dominated convergence theorem we easily deduce that

$$\lim_{n\to\infty} W(y_n, w_n)(r) = W(y,w)(r) \tag{2.13}$$

for all $r \in [0,1]$. Moreover, since $\mathcal{A}$ is closed and $W(\mathcal{A})$ is equicontinuous, then $\{W(y_n,w_n) : n \ge 1\} \cup \{W(y,w)\}$ is an equicontinuous family. Thus from pointwise convergence (2.13) we obtain the uniform convergence, that is, $W(y_n,w_n)$ converges to $W(y,w)$ uniformly. Therefore W is a continuous operator.

Thus by Schauder fixed point theorem, there exists $(u,v) \in \mathcal{M}$ satisfying

$$\begin{aligned} u(r) &= cu_0(r) + \int_r^1 s^{1-N}\int_0^s t^{N-1}f(t,v(t))dtds, \\ v(r) &= du_0(r) + \int_r^1 s^{1-N}\int_s^1 t^{N-1}g(t,u(t))\,dt\,ds\,. \end{aligned}$$

Hence there exists a positive solution to (2.1).

The behavior of u at zero is a consequence of L'Hôpital rule.

$$\begin{aligned}\lim_{r\to 0^+} r^{N-2}u(r) &= c + \lim_{r\to 0^+} \frac{\int_r^1 s^{1-N}\int_0^s t^{N-1}f(t,v(t))\,dt\,ds}{r^{2-N}},\\ &= c + \lim_{r\to 0^+} \frac{r^{1-N}\int_0^r t^{N-1}f(t,v(t))dt}{(N-2)r^{1-N}},\\ &= c + \frac{1}{N-2}\lim_{r\to 0^+}\int_0^r t^{N-1}f(t,v(t))dt,\\ &= c.\end{aligned}$$

□

As a consequence of the construction of non-negative solutions given in the above theorem, we have the following result about existence of positive solutions with a strong singularity.

Corollary 2.2. *Assume that the hypotheses in Theorem 2.1 hold and $g(r,s)$ is non decreasing in s. Then,*

(i) *If*

$$\int_0^1 t^{N-1}g(t,\alpha t^{2-N})dt = +\infty \quad \textit{for any } \alpha > 0, \tag{2.14}$$

there exists a non-negative solution (u,v) to (2.1) with a strong singularity.

(ii) *If*

$$\int_0^1 t^{N-1}g(t,\alpha t^{2-N})dt < +\infty \quad \textit{for any } \alpha > 0,$$

any non-negative non-trivial radially symmetric solution has fundamental singularity.

Proof. Assume first that (2.14) is satisfied. Let (u,v) be a solution to (2.1) constructed in Theorem 2.1 with $c>0$. Thus,

$$v(r) = du_0(r) + \int_r^1 s^{1-N}\int_s^1 t^{N-1}g(t,u(t))\,dt\,ds\,.$$

By a generalize version of L'Hôpital rule (see Proposition 7.1 in [6]) we have

$$\begin{aligned}\liminf_{r\to 0^+} r^{N-2}v(r) &\ge \liminf_{r\to 0^+} \frac{\int_r^1 s^{1-N}\int_s^1 t^{N-1}g(t,u(t))\,dt\,ds}{r^{2-N}},\\ &\ge \liminf_{r\to 0^+} \frac{r^{1-N}\int_r^1 t^{N-1}g(t,u(t))dt}{(N-2)r^{1-N}},\\ &= \frac{1}{N-2}\lim_{r\to 0^+}\int_r^1 t^{N-1}g(t,u(t))dt = +\infty,\end{aligned}$$

where the last equality holds by (2.14) and since $\lim_{r\to 0^+} r^{N-2}u(r) = c$.

Assume now, that $\int_0^1 t^{N-1}g(t,\alpha t^{2-N})dt < +\infty$, and let (u,v) be a non-negative solution to (2.1). Since $-r^{N-1}u'(r)$ is non decreasing, we easily obtain that $u(r) \le \alpha r^{2-N}$, where $\alpha = -u'(1)/(N-2)$. Moreover, from the second in (2.1) v satisfies

$$v(r) = du_0(r) + \int_r^1 s^{1-N}\int_s^1 t^{N-1}g(t,u(t))\,dt\,ds$$

and thus

$$v(r) \leq \Big(d + (N-2)^{-1} \int_0^1 t^{N-1} g(t, \alpha t^{2-N}) dt\Big) u_0(r),$$

and the conclusion follows. $\square$

Next, we will show a general existence result of fundamental singular solutions which is included in [5], Theorem 4.3. We give an idea of the proof for the sake of completeness .

For this purpose, let $\alpha > 0$ and let

$$u_\alpha(r) := \int_r^1 s^{1-N} \int_0^s t^{N-1} f_1(t, \alpha t^{2-N})\, dt\, ds\,.$$

Theorem 2.3. *Assume that f and g are two non negative functions satisfying* (2.2). *Assume that*

$$\int_0^1 t^{N-1} f_1(t, \alpha t^{2-N}) dt < \infty,$$

and

$$\lambda_\alpha := \frac{1}{N-2} \int_0^1 t^{N-1} g_1(t, u_\alpha(t)) dt < \infty.$$

Moreover, suppose that for some $\alpha > 0$, we have $\lambda_\alpha < \alpha$. Then, for any $d \in (\lambda_\alpha, \alpha]$, there exists a non-negative solution (u,v) to (2.1) *such that*

$$\lim_{r \to 0^+} r^{N-2} (u,v)(r) = (0,d).$$

Proof. The proof of this result is similar to the proof of Theorem 2.1. Let $d \in (\lambda_\alpha, \alpha]$, and let $F = (F_1, F_2)$ be given by

$$\begin{aligned} F_1(u,v)(r) &= \int_r^1 s^{1-N} \int_0^s t^{N-1} f(t, v(t)) dt, \\ F_2(u,v)(r) &= d u_0(r) - \int_r^1 s^{1-N} \int_0^s t^{N-1} g(t, u(t)) dt. \end{aligned}$$

Define $\mathcal{N}$ as the subset of $C((0,1])^2$ such that

$$\mathcal{N} := \{(u,v) \mid 0 \leq u \leq u_\alpha,\ 0 \leq v \leq \alpha u_0\}.$$

Under the assumptions of the theorem, it is not difficult to prove that $F(\mathcal{N}) \subset \mathcal{N}$. The rest of the proof follows the ideas of Theorem 2.1. $\square$

Next, we will apply Theorem 2.1 to problem (2.1) with

$$0 \leq f(r,s) \leq r^a s^p, \quad 0 \leq g(r,s) \leq r^b s^q. \tag{2.15}$$

Theorem 2.4. *Let $p > 0$ and $q > 0$, with $pq \neq 1$ and suppose that (a,b,p,q) satisfies* (1.4). *Assume that f and g are two non negative functions satisfying* (2.15). *Then, there exist $c_0 > 0$ such that for any $c \in [0, c_0)$ there exists (u,v) a non-negative singular solution to* (2.1) *such that*

$$\lim_{r \to 0^+} r^{N-2} u(r) = c.$$

Moreover, if $pq < 1$ then $c_0 = +\infty$.

Proof. Let

$$f_1(r,s) = r^a s^p, \quad g_1(r,s) = r^b s^q.$$

The function v_α defined by (2.3) is given now by

$$v_\alpha(r) := du_0(r) + \alpha^q \int_r^1 s^{1-N} \int_s^1 t^{N-1+b-(N-2)q}\, dt\, ds\,. \tag{2.16}$$

Next, we show that (H1) is equivalent to

$$N + a + \min\{\beta, 2-N\}p > 0,$$

where β is defined by (1.5). Let

$$w_1(r) := \int_r^1 s^{1-N} \int_s^1 t^{N-1+b-(N-2)q}\, dt\, ds\,. \tag{2.17}$$

Thus, by setting $\rho := \beta + N - 2$,

$$w_1(r) = \begin{cases} \frac{u_0(r)}{\rho(N-2)} + \frac{r^\beta - 1}{\rho\beta} & \text{if } \beta \neq 0 \text{ and } \rho \neq 0, \\ \frac{u_0(r)}{(N-2)^2} + \frac{\log(r)}{N-2} & \text{if } \beta = 0, \\ \int_r^1 s^{1-N} |\log(s)| ds & \text{if } \rho = 0. \end{cases}$$

Moreover, if $\rho = 0$,

$$\lim_{r\to 0^+} r^{N-2} |\log(r)|^{-1} w_1(r) = (N-2)^{-1}.$$

Now, the proof of the equivalence to (H1) follows easily.

To prove the existence of a non-negative solution it is sufficient to find d and α positive constants such that

$$\Lambda_\alpha = (N-2)^{-1} \int_0^1 t^{N-1+a} v_\alpha^p(t) dt < \alpha\,. \tag{2.18}$$

Since $v_\alpha = du_0 + \alpha^q w_1$, where w_1 is given by (2.17), and using the inequality $(x+y)^p \le C(x^p + y^p)$, for any non-negative numbers x and y, and where $C = \max\{1, 2^{p-1}\}$, we see that (2.18) is satisfied if

$$Ad^p + B\alpha^{pq} < (N-2)\alpha\,, \tag{2.19}$$

where

$$A := \int_0^1 t^{N-1+a} u_0^p(t) dt, \quad B := \int_0^1 t^{N-1+a} w_1^p(t)\, dt\,.$$

If we choose, for instance, d such that $Ad^p = B\alpha^{pq}$, (2.19) is satisfied for any α such that

$$2B\alpha^{pq-1} < N-2\,.$$

Moreover, by Theorem 2.1 there exists (u,v) non-negative singular solution such that $\lim_{r\to 0^+} r^{N-2} u(r) = c$, for any $c \in [0, \alpha - \Lambda_\alpha)$, and thus if $pq < 1$ and since $\alpha - \Lambda_\alpha$ tends to infinity as α does, the existence in the sub-linear case is for any $c > 0$. $\square$

The following result is an application of Theorem 2.3 to problem (2.1).

Theorem 2.5. *Assume that f and g are two non-negative functions satisfying* (2.15)*, with $p > 0$ and $q > 0$ and $pq \neq 1$. Also assume that (a, b, p, q) satisfies*

$$p < (N + a)/(N - 2) \quad \text{and} \quad N + b + \mu q > 0,$$

where $\mu := \min\{a + 2 - (N - 2)p, 0\}$. Then, there exist $d_0 \geq 0$ and $d_1 > 0$, with $d_0 < d_1$, such that for any $d \in (d_0, d_1)$ there exists (u, v) a non-negative singular solution to (2.1) *such that*

$$\lim_{r\to 0^+} r^{N-2}(u, v)(r) = (0, d).$$

Moreover, if $pq < 1$, $d_1 = +\infty$ and if $pq > 1$, $d_0 = 0$.

Remark 2.1. In [4] we proved existence of solutions for (1.1) in the super-linear case, that is when $pq > 1$. In the super-linear case, Theorem 2.4 and Theorem 2.5 do not give the optimal region of the values (a, b, p, q) of existence of non-negative solutions to (1.1). However, we will show later that for the sub-linear case Theorem 2.4 is optimal, see Theorem 1.1.

As a consequence of Theorem 2.4, Corollary 2.2 and Theorem 2.5 we have the following.

Corollary 2.6. *Let $p > 0$ and $q > 0$, and consider*

$$\begin{aligned} -\Delta u &= v^p \quad \text{in } B', \\ \Delta v &= u^q \quad \text{in } B', \\ u = v &= 0 \quad \text{on } \partial B, \end{aligned} \tag{2.20}$$

with $q \geq N/(N - 2)$ and $N + \big(2 - (N - 2)q\big)p > 0$. Then, there exist fundamental and strongly singular non-negative solutions of (2.20).

3. Proof of main theorem 1.1

In this section we prove our main result and we give some applications to biharmonic equations.

Proof of Theorem 1.1. Let $p > 0$, $q > 0$ and $pq < 1$. Assume that (a, b, p, q) satisfies (1.4). The existence of a non-negative singular solution to (1.1) follows from Theorem 2.4.

Assume that (a, b, p, q) does not satisfies (1.4), with $p > 0$, $q > 0$ and $pq < 1$ and let (u, v) be a non-negative solution to (1.1). We will show that (u, v) must be the trivial solution.

First, when $N + a + \beta p \leq 0$, the conclusion follows form Theorem 1.2 in [3].

Now, assume that $p \geq (N+a)/(N-2)$. Since u is a non-negative super-harmonic function, from Theorem 1 in [1], we obtain that

$$|x|^a v^p \in L^1_{loc}(B).$$

Moreover, since v is sub-harmonic there exists a non-negative constant c (possible $c = \infty$) such that $\lim_{r\to 0^+} r^{N-2}\overline{v}(r) = c$, where $\overline{v}$ is the spherical average of v. Assume first that $c = 0$. Let $w(s) := s\overline{v}(r)$ with $s = r^{N-2}$. We easily obtain that w is a convex function satisfying $w(0) = w(1) = 0$, and thus $v = 0$. On the other hand, if $c \neq 0$, we have for some positive constant C such that for all r near 0,

$$\overline{v}(r) \geq Cr^{2-N}\,. \tag{3.1}$$

By Remark 3.1 in [3] and (3.1) we deduce that $\overline{v^p}(r) \geq C\overline{v}^p(r) \geq Cr^{(2-N)p}$, and thus

$$\infty > \int_{B_\epsilon(0)} |x|^a v^p(x)dx \geq C\int_0^\epsilon r^{a+N-1}\overline{v}^p(r)dr \geq C\int_0^\epsilon r^{a+N-1-p(N-2)}dr,$$

contradicting $p \geq \frac{N+a}{N-2}$.

The last assertion on the theorem follows from Corollary 2.2. □

The following two results are applications of Theorem 1.1 to the bi-harmonic equation.

Corollary 3.1. *Let $N > 2$, and let $0 < q < 1$. Then there exist positive solutions of*

$$\begin{gathered} \Delta^2 u + |x|^b u^q = 0 \quad \text{in } B_1'(0), \\ u = \Delta u = 0 \quad \text{on } \partial B_1(0), \end{gathered} \tag{3.2}$$

such that $-\Delta u \geq 0$, if and only if

$$q < \frac{N+b+2}{N-2}.$$

Corollary 3.2. *Let $N > 2$, and $0 < q < 1$. Then there exist positive solutions of* (3.2) *such that $\Delta u \geq 0$, if and only if*

$$q < \frac{N+b}{N-2}.$$

As a consequence of the results in [4] and the two corollaries above, we obtain the following.

Corollary 3.3. *Let $N > 2$, and $0 < q \neq 1$. Then*

(i) *There exist positive solutions of* (3.2) *with $b = 0$ such that $-\Delta u \geq 0$, if and only if*

$$(N-4)q < N.$$

(ii) *There exist positive solutions of* (3.2) *with $b = 0$ such that $\Delta u \geq 0$, if and only if*

$$q < \frac{N}{N-2}.$$

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