Electronic Journal of Differential Equations, Vol. **1999**(1999), No. 25, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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A necessary and sufficient condition for the diagonalization of multi-dimensional quasilinear systems *

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Abstract

In this paper, the author obtains a necessary and sufficient condition on the diagonalization of multi-dimensional quasilinear systems of first order, and gives some physical applications.

1 Introduction

Consider the following multi-dimensional quasilinear system

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{m} A_i(u)\frac{\partial u}{\partial x_i}=0\,, \tag{1.1}$$

where $u=(u_1,\ldots,u_n)^T$ is the unknown vector function, $A_i(u)=\left(a^i_{jk}(u)\right)$ is an $n\times n$ matrix with suitable smooth elements $a^i_{jk}(u)$ $(i=1,\ldots,m;\ j,k=1,\ldots,n)$, m and n are two integers ≥ 1.

We say system (1.1) is *diagonalizable*, if there exists a smooth transformation $w=(w_1(u),\cdots,w_n(u))^T$ with non-vanishing Jacobian such that (1.1) can be equivalently rewritten as the following coupled system

$$\frac{\partial w_j}{\partial t}+\sum_{i=1}^{n}\lambda^i_j(w)\frac{\partial w_j}{\partial x_i}=0 \quad (j=1,\ldots,n), \tag{1.2}$$

where $\lambda^i_j(w)$ $(i=1,\ldots,m;\ j=1,\ldots,n)$ are smooth functions of w.

Such functions $w_j=w_j(u)$ $(j=1,\ldots,n)$ are called *strict Riemann invariants*, and $\lambda^i_j(w)$ is called *the x_i-directional speed of w_j* $(i=1,\ldots,m;\ j=1,\ldots,n)$.

The diagonal quasilinear system (1.2) is special and important system possessing many good properties. For instance, it is easier to find exact solutions

* *1991 Mathematics Subject Classifications:* 35F20, 35L40.
Key words and phrases: Quasilinear systems, diagonalization, strict Riemann invariants, conservation laws.

Submitted July 1, 1999. Published July 29, 1999.

and to establish the existence and uniqueness theory of solutions, etc. ([CH], [Ho], [L1]). However, not all quasilinear systems are diagonalizable. Hence, it is interesting and significant to discuss the following two kinds of problems

(I) Under which conditions is system (1.1) diagonalizable? If it is diagonalizable, what are the Riemann invariants?

(II) Under which conditions is system (1.1) not diagonalizable?

In this paper, we consider these problems for the multi-dimensional quasilinear system (1.1) and give a necessary and sufficient condition for its diagonalization. Moreover, in order to illustrate that our criterion is effective, we give some applications.

In the case of one space dimension, it is well known that every quasilinear hyperbolic system with two partial differential equations and with two unknown functions is always diagonalizable ([CH], [L1]). For general quasilinear systems in one space dimension, this result is obtained by making use of Nijenhuis tensor N^i_{jk} of the matrix $A_1(u)$:

$$N^i_{jk} = \sum_{l=1}^{n} \left[a^1_{lj} \frac{\partial a^1_{ik}}{\partial u_j} - a^1_{lk} \frac{\partial a^1_{ij}}{\partial u_l} - a^1_{il} \left(\frac{\partial a^1_{lk}}{\partial u_j} - \frac{\partial a^1_{lj}}{\partial u_k} \right) \right] \quad (i,j,k = 1,\dots,n)\,, \tag{1.3}$$

and introducing the tensor

$$T^i_{jk} = \sum_{p,q=1}^{n} \left(N^i_{pq} a^1_{pj} a^1_{qk} - N^p_{jq} a^1_{ip} a^1_{qk} - N^p_{qk} a^1_{ip} a^1_{qj} + N^p_{jk} a^1_{iq} a^1_{qp} \right), \tag{1.4}$$

where $i,j,k = 1,\dots,n$. Haantjes [Ha] proved that (1.1) with $m = 1$ is diagonalizable if and only if

$$T^i_{jk} \equiv 0 \quad (i,j,k = 1,\dots,n). \tag{1.5}$$

Applications of this criterion were discussed in [Fe], [FT] and [T]. The advantage of the criterion is that we do not need to calculate eigenvalues and eigenvectors of matrix $A_1(u)$. However, it seems to me that it is not easy to generalize the criterion to multi-dimensional system. Moreover, we do not know how to solve the Riemann invariants. Employing the eigenvalues and the eigenvectors of matrix $A_1(u)$, Serre [S] considered this problem and gave another necessary and sufficient condition. Assuming for simplicity that the Spectrum $(A_1(u))$ consists of n distinct real values, he proved that (1.1) (in which $m = 1$) is diagonalizable if and only if the Frobenius conditions $l_i\{r_j, r_k\} = 0$ $(i,j,k = 1,\dots,n; j,k \neq i)$ hold, where l_i and r_i denote the left and right eigenvectors related to the eigenvalue $\lambda_i(u)$ of $A_1(u)$, and $\{\cdot,\cdot\}$ is the Poisson bracket of vector fields in u-space. By enlarging the system, Garabedian observed that every quasilinear system in one space dimension can be diagonalized and hence solved locally by Picard iteration ([G.Page 100]), although the diagonalizable system may be inhomogeneous.

For nonlinear systems in several space dimensions, only a few results are known. Dafermos [D] and Lopes Filho & Nussenzweig Lopes [LN] examined the system of two partial differential equations in several space variables, and obtained some results on Riemann invariants, characteristic structure and the stability of admissible L^∞ solution. These results were started by an observation in [Ra] that only systems with commuting matrices can possess BV estimates.

This paper aims to generalize Serre's result to quasilinear system that might be multi-dimensional, that is to say, to give a necessary and sufficient condition on diagonalization of (1.1).

The main results of this paper are presented and proved in Section 2. Section 3 is devoted to the applications of the main results to some physical systems such as the system of gas dynamics, the system of multicomponent chromatography, the system of electrophoresis and the quasilinear hyperbolic system of conservation laws with rotational invariance.

2 Main results

Lemma 2.1 *We can derive from (1.1) a partial differential equation in diagonal form*

$$\frac{\partial R(u)}{\partial t}+\sum_{i=1}^{m}\lambda_i(u)\frac{\partial R(u)}{\partial x_i}=0 \quad (\nabla_u R(u)\neq 0), \tag{2.1}$$

where $\lambda_i(u)$ $(i=1,\dots,m)$ *are smooth functions, if and only if there exists a smooth row vector* $l=l(u)$ $(l(u)\neq 0)$ *such that* $l(u)$ *is a common left eigenvector of* $A_i(u)$ $(i=1,\dots,m)$*, the corresponding eigenvalue is* $\lambda_i(u)$*, and it holds that*

$$L(u)\wedge dL(u)=0, \tag{2.2}$$

where

$$L(u)=l(u)du. \tag{2.3}$$

Such a smooth function $R=R(u)$ is called *one strict Riemann invariant.* Obviously, system (1.1) is diagonalizable if and only if there exist n independent strict Riemann invariants.

Proof of Lemma 2.1 **Necessity**: Suppose that there exists a smooth function $R=R(u)$ $(\nabla_u R\neq 0)$ such that (2.1) holds. Thus, from (2.1) we have

$$\nabla_u R(u)\left(\frac{\partial u}{\partial t}+\sum_{i=1}^{m}\lambda_i(u)\frac{\partial u}{\partial x_i}\right)=0. \tag{2.4}$$

On the other hand, multiplying (1.1) by $\nabla_u R(u)$ from the left gives

$$\nabla_u R(u)\left(\frac{\partial u}{\partial t}+\sum_{i=1}^{m}A_i(u)\frac{\partial u}{\partial x_i}\right)=0. \tag{2.5}$$

The combination of (2.4) and (2.5) yields

$$\nabla_u R(u) A_i(u) = \lambda_i(u) \nabla_u R(u) \quad (i = 1, \ldots, m). \tag{2.6}$$

(2.6) shows that $\nabla_u R(u)$ is a common left eigenvector of $A_i(u)$ $(i = 1, \ldots, m)$ and the corresponding eigenvalue is $\lambda_i(u)$. Taking $l(u) = \nabla_u R(u)$, we get (2.2) immediately.

Sufficiency: Suppose that $l(u)$ is a common left eigenvector of $A_i(u)$ $(i = 1, \ldots, m)$ and $\lambda_i(u)$ is the corresponding real eigenvalue. Multiplying (1.1) by $l(u)$ from the left gives

$$l(u) \left(\frac{\partial u}{\partial t} + \sum_{i=1}^{m} \lambda_i(u) \frac{\partial u}{\partial x_i} \right) = 0. \tag{2.7}$$

On the other hand, (2.2) implies that the equation $L(u) = 0$ is completely integrable. Hence, by the well-known Frobenius Theorem, there exists a smooth function $R = R(u)$ $(\nabla_u R(u) \neq 0)$ such that $l(u) /\!\!/ \nabla_u R(u)$. Thus, from (2.7) we get (2.1) immediately. This finishes the proof. $\square$

Remark 2.1 *The proof of the sufficiency in Lemma 2.1 provides a method to obtain the strict Riemann invariant. The first integral of a common left eigenvector of $A_i(u)$ $(i = 1, \ldots, n)$ is a strict Riemann invariant.*

By Lemma 2.1, we have

Theorem 2.1 *System (1.1) is diagonalizable if and only if there exist n independent row vectors $l_j(u) = (l_{j1}(u), \cdots, l_{jn}(u))$ $(j = 1, \ldots, n)$ such that each $l_j(u)$ is a common left eigenvector of $A_i(u)$ $(i = 1, \ldots, m)$ and*

$$L_j(u) \wedge dL_j(u) = 0, \tag{2.8}$$

where

$$L_j(u) = l_j(u) du. \tag{2.9}$$

Corollary 2.1 *If system (1.1) is diagonalizable, then*

$$A_i(u) A_{i'}(u) = A_{i'}(u) A_i(u) \quad (i, i' = 1, \ldots, m). \tag{2.10}$$

Proof. By Theorem 2.1, there exist n independent row vectors $l_j(u)$ $(j = 1, \ldots, n)$ such that

$$\begin{aligned} l_j(u) A_i(u) &= \lambda_j^i(u) l_j(u), \\ l_j(u) A_{i'}(u) &= \lambda_j^{i'}(u) l_j(u), \end{aligned} \tag{2.11}$$

where $i, i' = 1, \ldots, m$; $j = 1, \ldots, n$, and $\lambda_j^i(u)$ is the eigenvalue of $A_i(u)$ corresponding to the common left eigenvector $l_j(u)$.

Multiplying the first (resp. second) equation of (2.11) by $A_{i'}$ (resp. A_i) from the right gives

$$l_j(u) A_i(u) A_{i'}(u) = l_j(u) A_{i'}(u) A_i(u) \quad (j = 1, \ldots, n). \tag{2.12}$$

Noting the fact that $l_j(u)$ $(j = 1, \ldots, n)$ are independent, from (2.12) we get (2.10) immediately. This completes the proof. $\square$

Remark 2.2 *Diagonal system of partial differential equations is hyperbolic. But when $m > 1$, it is not strictly hyperbolic in the sense of Majda ([M]). In fact, if all $A_i(u)$ $(i = 1, \dots, m)$ are diagonal matrices, then for any given u on the domain under consideration, there is at least one unit vector $\omega = (\omega_1, \cdots, \omega_m)$ such that the matrix $\sum_{i=1}^n \omega_i A_i(u)$ does not have n distinct real eigenvalues. Not only that, it can not have any non-degenerate wave cones whatsoever ([D], [LN]). This is different from non-diagonal systems ([L2]).*

In what follows, we discuss three special but important cases.

Case I. Hyperbolic systems with at least one-directional strict hyperbolicity

For simplicity of statement, we introduce

Definition 2.1 *We say an $n \times n$ matrix $A(u)$ is hyperbolic, if $A(u)$ has n real eigenvalues and is diagonalizable for any given u on the domain under consideration; $A(u)$ is strictly hyperbolic, if $A(u)$ has n distinct real eigenvalues.*

Lemma 2.2 *Suppose that matrix $A(u)$ is strictly hyperbolic and $B(u)$ is hyperbolic. Then there exist n independent row vectors $l_j(u) = (l_{j1}(u), \cdots, l_{jn}(u))$ $(j = 1, \dots, n)$ such that each $l_j(u)$ is a common left eigenvector of $A(u)$ and $B(u)$, if and only if*

$$AB = BA. \tag{2.13}$$

Proof. The proof of the necessity is the same as that of Corollary 2.1, moreover, we do not require that $A(u)$ is strictly hyperbolic.

It remains to prove the sufficiency.

Let λ_j^A $(j = 1, \dots, n)$ be the n distinct real eigenvalues of $A(u)$. Without loss of generality, we may suppose that

$$\lambda_1^A(u) < \cdots < \lambda_n^A(u). \tag{2.14}$$

Moreover, let $l_j^A(u)$ be the left eigenvector of $A(u)$ corresponding to $\lambda_j^A(u)$ $(j = 1, \dots, n)$ and introduce

$$\mathcal{L}^A = \begin{pmatrix} l_1^A(u) \\ \vdots \\ l_n^A(u) \end{pmatrix}.$$

Thus, we have

$$\mathcal{L}^A A \left(\mathcal{L}^A\right)^{-1} = \operatorname{diag}\left(\lambda_1^A(u), \cdots, \lambda_n^A(u)\right). \tag{2.15}$$

On the other hand, by (2.13) we have

$$\left(\mathcal{L}^A A \left(\mathcal{L}^A\right)^{-1}\right)\left(\mathcal{L}^A B \left(\mathcal{L}^A\right)^{-1}\right) = \left(\mathcal{L}^A B \left(\mathcal{L}^A\right)^{-1}\right)\left(\mathcal{L}^A A \left(\mathcal{L}^A\right)^{-1}\right). \tag{2.16}$$

Noting (2.14) and (2.15), from (2.16) we see that $\mathcal{L}^A B \left(\mathcal{L}^A\right)^{-1}$ is a diagonal matrix. This finishes the proof. $\square$

Therefore, by Theorem 2.1 we have

Corollary 2.2 *Suppose that $A_i(u)$ $(i=1,\dots,m-1)$ are hyperbolic and $A_m(u)$ is strictly hyperbolic. Then system (1.1) is diagonalizable if and only if*

$$A_i(u)A_m(u) = A_m(u)A_i(u) \quad (i=1,\dots,m-1) \tag{2.17}$$

and

$$L_j^{A_m}(u) \wedge dL_j^{A_m}(u) = 0 \quad (j=1,\dots,n), \tag{2.18}$$

where $L_j^{A_m}(u) = l_j^{A_m}(u)du$, in which $l_j^{A_m}(u)$ is a left eigenvector of $A_m(u)$ corresponding to $\lambda_j^{A_m}(u)$.

Such a system is called *the hyperbolic system with at least one-directional strict hyperbolicity.*

Case II. Symmetric systems

The following Lemma is well known.

Lemma 2.3 *Suppose that $A(u)$ and $B(u)$ are $n\times n$ real symmetric matrices. Then the conclusion of Lemma 2.2 is still valid.*

Hence, by Theorem 2.1 we get

Corollary 2.3 *Suppose that $A_i(u)$ $(i=1,\dots,m)$ are $n\times n$ real symmetric matrices. Then system (1.1) is diagonalizable if and only if*

$$A_i(u)A_{i'}(u) = A_{i'}(u)A_i(u) \quad (i,i'=1,\dots,m) \tag{2.19}$$

and (2.8) holds, where $L_j(u) = l_j(u)du$, in which $l_j(u)$ stands for a common left eigenvector of $A_i(u)$ $(i=1,\dots,m)$ corresponding to $\lambda_j^i(u)$.

Case III. Systems with constant multiplicity eigenvalues

Now we turn to consider the case that $A_i(u)$ $(i=1,\dots,m)$ have constant multiplicity eigenvalues. Without loss of generality, we suppose that

$$\lambda^i(u) \stackrel{\triangle}{=} \lambda_1^i(u) \equiv \cdots \equiv \lambda_{p_i}^i(u) < \lambda_{p_i+1}^i(u) < \cdots < \lambda_n^i(u) \quad (i=1,\dots,m), \tag{2.20}$$

where p_i is an integer >1.

Suppose that there exist n independent smooth row vectors $l_j(u) = (l_{j1}(u),\cdots,l_{jn}(u))$ $(j=1,\dots,n)$ such that $A_i(u)$ $(i=1,\dots,m)$ can be diagonalized simultaneously, namely,

$$\mathcal{L}(u)A_i(u)\mathcal{L}^{-1}(u) = \operatorname{diag}\left(\lambda_1^i(u),\cdots,\lambda_n^i(u)\right) \quad (i=1,\dots,m), \tag{2.21}$$

where $\mathcal{L}(u) = (l_{jk}(u))$ is an $n\times n$ matrix and $\lambda_j^i(u)$ is the eigenvalue of $A_i(u)$ corresponding to the left eigenvector $l_j(u)$.

We have

Theorem 2.2 *Under the hypotheses (2.20)-(2.21), system (1.1) is diagonalizable if and only if*

$$L_1(u) \wedge \cdots \wedge L_p(u) \wedge dL_\alpha(u) = 0 \quad (\alpha = 1, \cdots, p) \tag{2.22}$$

and

$$L_\beta(u) \wedge dL_\beta(u) = 0 \quad (\beta = p+1, \ldots, n), \tag{2.23}$$

where $p = \min_{i=1,\ldots,m}\{p_i\}$ *and* $L_j(u) = l_j(u)du \ \ (j = 1, \ldots, n)$.

Proof. **Necessity**: Noting (2.20)-(2.21), from (1.1) we have

$$l_\alpha(u)\left(\frac{\partial u}{\partial t} + \sum_{i=1}^{m} \lambda^i(u)\frac{\partial u}{\partial x_i}\right) = 0 \quad (\alpha = 1, \cdots, p). \tag{2.24}$$

By the fact that (1.1) is diagonalizable, there exists a $p \times p$ smooth invertible matrix $C(u) = (C_{\mu\nu}(u))_{\mu,\nu=1}^{p}$ and smooth functions $w_\alpha = w_\alpha(u)$ $(\alpha = 1, \cdots, p)$ such that

$$\sum_{\mu=1}^{p} C_{\alpha\mu}(u)L_\mu(u) = dw_\alpha(u) \quad (\alpha = 1, \cdots, p). \tag{2.25}$$

Then, it follows from (2.25) that

$$\sum_{\mu=1}^{p} C_{\alpha\mu}(u)dL_\mu(u) + \sum_{\mu=1}^{p} dC_{\alpha\mu}(u) \wedge L_\mu(u) = 0 \quad (\alpha = 1, \cdots, p). \tag{2.26}$$

Thus, we have

$$\sum_{\mu=1}^{p} C_{\alpha\mu}(u)L_1(u) \wedge \cdots \wedge L_p(u) \wedge dL_\mu(u) = 0 \quad (\alpha = 1, \cdots, p). \tag{2.27}$$

Noting the fact that $C(u)$ is invertible, by (2.27) we get (2.22) immediately.

The proof of (2.23) is the same as that of Theorem 2.1.

The sufficiency can be proved in a manner similar to the proof of the sufficiency of Lemma 2.1. For brevity, we omit it here. This completes the proof. $\square$

Remark 2.3 *When $p = n - 1$, (2.22) becomes trivial, namely, (2.22) holds automatically; When $p = n$, (2.22) and (2.23) always hold, so system (1.1) is always diagonalizable in this case.*

Remark 2.4 *When $m = 1$, Theorem 2.1 goes back to a Serre's result in [S].*

3 Applications

In this section, we give the applications of the results presented in Section 2 to some physical systems.

System of gas dynamics

Consider the system of gas dynamics in three space dimensions ([CH])

$$\frac{\partial U}{\partial t}+A(U)\frac{\partial U}{\partial x}+B(U)\frac{\partial U}{\partial y}+C(U)\frac{\partial U}{\partial z}=0, \tag{3.1}$$

where

$$U=\begin{pmatrix}\rho\\ u\\ v\\ w\\ S\end{pmatrix},\quad A(U)=\begin{pmatrix}u & \rho & 0 & 0 & 0\\ \frac{1}{\rho}\frac{\partial p}{\partial \rho} & u & 0 & 0 & \frac{1}{\rho}\frac{\partial p}{\partial S}\\ 0 & 0 & u & 0 & 0\\ 0 & 0 & 0 & u & 0\\ 0 & 0 & 0 & 0 & u\end{pmatrix}, \tag{3.2}$$

$$B(U)=\begin{pmatrix}v & 0 & \rho & 0 & 0\\ 0 & v & 0 & 0 & 0\\ \frac{1}{\rho}\frac{\partial p}{\partial \rho} & 0 & v & 0 & \frac{1}{\rho}\frac{\partial p}{\partial S}\\ 0 & 0 & 0 & v & 0\\ 0 & 0 & 0 & 0 & v\end{pmatrix},\quad C(U)=\begin{pmatrix}w & 0 & 0 & \rho & 0\\ 0 & w & 0 & 0 & 0\\ 0 & 0 & w & 0 & 0\\ \frac{1}{\rho}\frac{\partial p}{\partial \rho} & 0 & 0 & w & \frac{1}{\rho}\frac{\partial p}{\partial S}\\ 0 & 0 & 0 & 0 & w\end{pmatrix},$$

$\rho > 0$ is the density, (u, v, w) is the velocity, S is the entropy, p is the pressure and the state equation is

$$p = p(\rho, S) > 0, \tag{3.3}$$

in which $p(\rho, S)$ satisfies that on each finite domain of $\rho > 0$,

$$\frac{\partial p}{\partial \rho}(\rho, S) > 0. \tag{3.4}$$

By a simple calculation, we observe that there is one and only one independent row vector

$$l_5(U) \stackrel{\triangle}{=} (0,\ 0,\ 0,\ 0,\ 1) \tag{3.5}$$

such that $l_5(U)$ is a non-zero common left eigenvector of $A(U), B(U)$ and $C(U)$, and it holds that

$$L_5(U) \wedge dL_5(U) = 0, \tag{3.6}$$

where $L_5(U) = l_5(U)dU$. Hence, by Lemma 2.1 we obtain

Theorem 3.1 *From system (3.1), one and only one non-trivial partial differential equation in diagonal form can be reduced, namely, the entropy equation of conservation law.*

Remark 3.1 *Similarly, it is easy to check that the system of isentropic flow in two space dimensions is not diagonalizable. However, it is well known that the system of isentropic flow in one space dimension is always diagonalizable ([CH]).*

System of multicomponent chromatography

The following system arises in multicomponent chromatography (see [RA] or [T] for the case of one space dimension)

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{m} \frac{\partial}{\partial x_j}\left(\frac{a_i^j u_i}{1+\sum_{k=1}^{n} u_k}\right) = 0 \quad (i = 1, \cdots, n), \tag{3.7}$$

where $u_i = u_i(t, x)$ $(i = 1, \dots, n)$ are the non-negative unknown functions, a_i^j $(i = 1, \dots, n;\ j = 1, \dots, m)$ are positive constants satisfying

$$0 < a_1^j < a_2^j < \cdots < a_n^j \quad (j = 1, \dots, m). \tag{3.8}$$

Define nonlinear functions w_k^j by

$$\sum_{i=1}^{n} \frac{u_i}{w_k^j - a_i^j} + \frac{1}{w_k^j} = 0, \quad a_{k-1}^j < w_k^j < a_k^j \quad (k = 1, \dots, n;\ j = 1, \dots, m), \tag{3.9}$$

where $a_0^j \stackrel{\triangle}{=} 0$. Moreover, we may rewrite (3.7) as

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{m} A_j(u) \frac{\partial u}{\partial x_j} = 0, \tag{3.10}$$

where $u = (u_1, \dots, u_n)^T$.

It is easy to check that the eigenvalues of $A_j(u)$ are

$$\lambda_k^j(u) = \frac{w_k^j(u)}{1+\sum_{i=1}^{n} u_i} \quad (k = 1, \dots, n) \tag{3.11}$$

and the left eigenvector corresponding to $\lambda_k^j(u)$ can be chosen as

$$l_k^j(u) = \left(\frac{1}{w_k^j(u) - a_1^j}, \cdots, \frac{1}{w_k^j(u) - a_n^j}\right). \tag{3.12}$$

Obviously,

$$0 < \lambda_1^j(u) < \lambda_2^j(u) < \cdots < \lambda_n^j(u) \quad (j = 1, \dots, m). \tag{3.13}$$

Noting (3.12) and the definitions of $w_k^j(u)$, we see that $A_j(u)$ $(j = 1, \dots, m)$ share the same set of n linearly independent left eigenvectors, say, $l_1(u), \cdots, l_n(u)$, if and only if

$$a_i^j = c_j a_i \quad (i = 1, \dots, n;\ j = 1, \dots, m), \tag{3.14}$$

where a_i and c_j are positive constants with

$$0 < a_1 < \cdots < a_n.$$

Moreover, it is easy to check that (2.8) always hold. Therefore, by Theorem 2.1 we have

Theorem 3.2 *System (3.7) is diagonalizable if and only if (3.14) holds.*

In fact, if (3.14) holds, then system (3.7) can be reduced to

$$\frac{\partial w_i}{\partial t}+\sum_{j=1}^{m}c_j\lambda_j(w)\frac{\partial w_i}{\partial x_j}=0 \quad (i=1,\cdots,n), \tag{3.15}$$

where w_i is defined by

$$\sum_{k=1}^{n}\frac{u_k}{w_i-a_k}+\frac{1}{w_i}=0, \quad a_{i-1}<w_i<a_i \quad (a_0\overset{\triangle}{=}0) \tag{3.16}$$

and

$$\lambda_j(w)=\frac{w_j}{1+\sum_{k=1}^{n}u_k}=\left(\prod_{k=1}^{n}a_k\right)^{-1}w_j\prod_{k=1}^{n}w_k. \tag{3.17}$$

Remark 3.2 *In system (3.7), u_i stands for the fluid-phase concentration of the solute species S_i, and a_i^j denotes the limiting value of the solid-phase concentration of S_i in the x_j-direction. For any $j\in\{1,\dots,m\}$, let $(a_1^j,\cdots,a_n^j)$ be the vector composed of the limiting values a_i^j of the solid-phase concentrations of S_i $(i=1,\dots,n)$ in the x_j-direction. (3.14) implies that $(a_1^j,\cdots,a_n^j)=c_j(a_1,\cdots,a_n)$ $(j=1,\dots,m)$, that is to say, the vectors $(a_1^j,\cdots,a_n^j)$ $(j=1,\dots,m)$ parallel each other. Therefore, Theorem 3.2 shows that system (3.7) is diagonalizable if and only if $(a_1^j,\cdots,a_n^j)$ $(j=1,\dots,m)$ parallel each other. In the case of (3.14), system (3.7) possesses certain symmetry.*

Remark 3.3 *We have a similar result for the following system arising in electrophoresis (see [S] or [T] for the case of one space dimension)*

$$\frac{\partial u_i}{\partial t}+\sum_{j=1}^{m}\frac{\partial}{\partial x_j}\left(\frac{a_i^j u_i}{\sum_{k=1}^{n}u_k}\right)=0 \quad (i=1,\cdots,n), \tag{3.18}$$

where $u_i=u_i(t,x)$ $(i=1,\cdots,n)$ are the non-negative unknown functions satisfying

$$\sum_{k=1}^{n}u_k>0,$$

and a_i^j $(i=1,\cdots,n;\ j=1,\cdots,m)$ are positive constants satisfying

$$0<a_1^j<a_2^j<\cdots<a_n^j \quad (j=1,\dots,m).$$

Quasilinear hyperbolic system of conservation laws with rotational invariance

Consider the following quasilinear hyperbolic system

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left(f_i(|u|)u\right)=0, \tag{3.19}$$

where $u = (u_1, \dots, u_n)^T$, $f_i \in C^2(\mathbb{R}^+, \mathbb{R})$ and satisfies

$$f_i'(r) > 0, \quad \forall\, r > 0, \tag{3.20}$$

m is an integer ≥ 1. System (3.19) can be used to describe the propagation of waves in various situations in mechanics (such as the reactive flows, magneto-hydrodynamics and elasticity theory, etc.) at least for the case that $m = 1$ ([B], [F1], [KK], [LW]). It is no longer strictly hyperbolic, and possesses the eigenvalues with constant multiplicity even for the case that $m = 1$. When $m = 1$ and $n = 2$, system (3.19) was first studied by [KK] and [LW]. Freistühler [F1]-[F2] considered the Riemann problem and the Cauchy problem for system (3.19) with $m = 1$ and $n \geq 1$.

Rewrite (3.19) as

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{m} A_i(u) \frac{\partial u}{\partial x_i} = 0, \tag{3.21}$$

where

$$A_i(u) = \begin{pmatrix} f_i(r) + \frac{f_i'(r)}{r} u_1^2 & \frac{f_i'(r)}{r} u_1 u_2 & \cdots & \frac{f_i'(r)}{r} u_1 u_n \\ \frac{f_i'(r)}{r} u_1 u_2 & f_i(r) + \frac{f_i'(r)}{r} u_2^2 & \cdots & \frac{f_i'(r)}{r} u_2 u_n \\ \vdots & \vdots & \ddots & \vdots \\ \frac{f_i'(r)}{r} u_1 u_n & \frac{f_i'(r)}{r} u_2 u_n & \cdots & f_i(r) + \frac{f_i'(r)}{r} u_n^2 \end{pmatrix}, \tag{3.22}$$

in which $r = |u| > 0$.

In what follows, we consider the case that $r > 0$. Without loss of generality, we may suppose that $u_1 \neq 0$. It is easy to calculate that

$$\lambda^i(u) \stackrel{\triangle}{=} \lambda_1^i(u) \equiv \cdots \equiv \lambda_{n-1}^i(u) = f_i(r) \tag{3.23}$$

is an eigenvalue of $A_i(u)$ with constant multiplicity $n-1$. $A_i(u)$ $(i = 1, \dots, m)$ have $n-1$ independent common left eigenvectors corresponding to the eigenvalue $\lambda^i(u)$:

$$\begin{gathered} l_1(u) = (-u_2, u_1, 0, \cdots, 0), \qquad l_2(u) = (-u_3, 0, u_1, 0, \cdots, 0), \\ \cdots\cdots, \\ l_{n-2}(u) = (-u_{n-1}, 0, \dots, u_1, 0),\ l_{n-1}(u) = (-u_n, 0, \cdots, 0, u_1). \end{gathered} \tag{3.24}$$

Moreover,

$$\lambda_n^i(u) \stackrel{\triangle}{=} f_i(r) + r f_i'(r) \tag{3.25}$$

is another eigenvalue of $A_i(u)$, and

$$l_n(u) = (u_1, \dots, u_n) \tag{3.26}$$

is a common left eigenvector of $A_i(u)$ corresponding to the eigenvalue $\lambda_n^i(u)$. Obviously, $l_j(u)$ $(j = 1, \dots, n)$ given by (3.24) and (3.26) are independent. On the other hand, when $r > 0$, we have

$$\lambda_n^i(u) > \lambda^i(u) \quad (i = 1, \dots, m). \tag{3.27}$$

It is easy to check that system (3.19) satisfies all conditions required by Theorem 2.2. Hence, by Theorem 2.2 we have

Theorem 3.3 *Consider system (3.19) on the domain of $r > 0$. If (3.20) holds, then system (3.19) is always diagonalizable.*

In fact, let

$$u = rs, \tag{3.28}$$

where $r = |u|$, $s = (s_1, \cdots, s_n)^T \in S^{n-1}$. Then system (3.19) can be rewritten as

$$\frac{\partial s}{\partial t} + \sum_{i=1}^{m} f_i(r) \frac{\partial s}{\partial x_i} = 0, \tag{3.29}$$

$$\frac{\partial r}{\partial t} + \sum_{i=1}^{m} \frac{\partial}{\partial x_i} (r f_i(r)) = 0. \tag{3.30}$$

Remark 3.4 *A method for finding exact solutions to system (3.29)-(3.30) was given by [KN] at least for the case that $m = 1$.*

Remark 3.5 *In this paper, we introduce the multi-dimensional system (3.7) of multicomponent chromatography, the multi-dimensional system (3.18) of electrophoresis and the multi-dimensional system (3.19) of conservation laws with rotational invariance. When $m = 1$, they go back to the classical one-dimensional systems. Therefore, systems (3.7), (3.18) and (3.19) can be regarded as the generalization of the classical one-dimensional systems. The further study for systems (3.7), (3.18) and (3.19) remains to be done.*

Acknowledgments This work was partially supported by the Grant-in-Aid for Scientific Research for JSPS Postdoctoral Fellowship for Foreign Researcher in Japan, provided by the Japan Ministry of Education, Science and Culture. The author thanks the referee for his/her pertinent comments and valuable suggestions.

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