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On the Regularity of Elliptic Differential Equations Using Symmetry Techniques and Suitable Discrete Spaces *

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Abstract

We present an elementary and short proof for regularity of second order elliptic differential equations with homogeneous Dirichlet boundary conditions. The proof uses a discrete function space with piecewise multilinear functions and symmetry techniques on the unit cube.

1 Introduction

In this paper, we prove the regularity of second order elliptic differential equations with homogeneous Dirichlet boundary conditions on domains whose boundary is locally a smooth deformation of the boundary of the unit cube. In the two dimensional case, this implies the regularity of Poisson's equation on every domain with a piecewise smooth boundary and with no reentrant corner.

The usual approach to prove regularity is to first prove the regularity on domains with a smooth boundary and then to study each corner (see [2]). Especially in more than two dimensions, this approach leads to very long proofs. Here, we will present an elementary and short proof for the regularity of elliptic equations for a certain class of domains with corners. However, we can not analyze every corner in more than two dimension, but we can show the W_2^2-regularity of the solution near a lot of corners which appear in application.

The proof of regularity in this paper consists of two ideas. First, we assume that the domain Ω is the d-dimensional unit cube $\Omega^d :=]0,1[^d$. Then, we can extend every function in a symmetric way to a function on a band or a torus. Then, with the help of finite difference operators, it is no problem to prove the regularity of the solution. This is an old technique. But this approach can be used only for a certain class of elliptic equations (see section 2). The crucial restriction is that the values of some coefficients have to be zero at the boundary

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of the domain. The main idea is now not to analyze the regularity of the continuous problem directly, but to analyze the discrete regularity of a suitable discrete problem. Now, the above restrictions to the coefficients at the boundary do not appear (see section 3). In section 4, we show that the discrete regularity of the discrete problem implies the regularity of the continuous problem. The last step is to generalize the regularity on the unit cube to the regularity on a certain class of domains with corners.

Now, let us describe an elliptic equation on the bounded domain $\Omega \subset \mathbb{R}^d$. Assume that

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1d} \\ \vdots & & \vdots \\ b_{d1} & \cdots & b_{dd} \end{pmatrix}$$

is a matrix contained in $(W^1_\infty(\Omega))^{d\times d}$, and that B is a uniformly elliptic matrix. Then, the bilinear form

$$\begin{aligned} a : \overset{\circ}{W}{}^1_2(\Omega) \times \overset{\circ}{W}{}^1_2(\Omega) &\to \mathbb{R}, \\ (u,v) &\mapsto \int_\Omega (\nabla u)^T B \nabla v \, d\lambda \end{aligned} \tag{1}$$

is $\overset{\circ}{W}{}^1_2(\Omega)$-elliptic. Furthermore, assume that $f \in L^2(\Omega)$ and let us write

$$f(v) := \int_\Omega fv \, d\lambda.$$

Then, there is a unique solution $u \in \overset{\circ}{W}{}^1_2(\Omega)$ of the weak equation

$$a(u,v) = f(v) \quad \text{for every} \quad v \in \overset{\circ}{W}{}^1_2(\Omega). \tag{2}$$

Our aim is to prove $u \in \overset{\circ}{W}{}^2_2(\Omega)$ under suitable assumptions on Ω.

2 Regularity Using Symmetry

Using symmetry, very simple regularity proofs can be obtained for a restricted class of equations. Therefore we first assume $\Omega = \Omega^d =]0,1[^d$.

The idea of using symmetry is to extend functions defined on the unit cube to functions on a band. This band is

$$B^d = S^1 \times]0,1[^{d-1}$$

where S^1 is the interval $]-1,1[$ identified at the points $\{-1,1\}$. Formally, we also can define S^1 by

$$S^1 = \mathbb{R}/(2\mathbb{Z}).$$

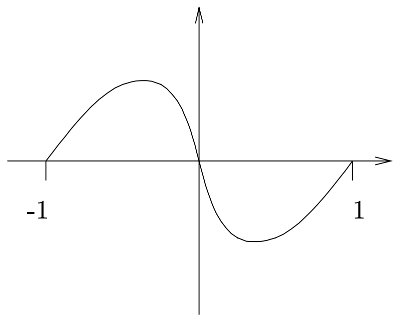

Figure 1: Extension Operator $\tilde{}$.

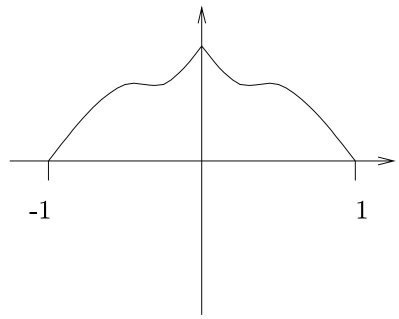

Figure 2: Extension Operator $\hat{}$.

Therefore S^1 is a circle. Obviously, there is a natural embedding

$$\Omega^d \hookrightarrow B^d.$$

This shows that we can restrict every function defined on B^d to a function on Ω^d. But usually, we will omit the restriction operator.

Now, we define two extension operators:

$$\begin{array}{rcl} \tilde{}: L^2(\Omega^d) & \to & L^2(B^d) \quad \text{and} \\ \hat{}: L^2(\Omega^d) & \to & L^2(B^d). \end{array}$$

The operator $\tilde{}$ is the anti-symmetric operator extension in the direction of the first coordinate. This means that

$$\tilde{v}(x_1, x_2, \cdots, x_d) = -v(-x_1, x_2, \cdots, x_d)$$

for $(x_1, x_2, \cdots, x_d) \in \Omega^d$ (see Figure 1).
The operator $\hat{}$ is the symmetric extension operator in the direction of the first coordinate. This means that

$$\hat{v}(x_1, x_2, \cdots, x_d) = v(-x_1, x_2, \cdots, x_d)$$

for $(x_1, x_2, \cdots, x_d) \in \Omega^d$ (see Figure 2).

Furthermore, denote $\tilde{W}_2^1(B^d)$ the Sobolev space of the symmetric functions

$$\tilde{W}_2^1(B^d) := \left\{ \tilde{u} \;\middle|\; u \in \overset{\circ}{W}{}_2^1(\Omega^d) \right\}.$$

Observe that $\tilde{W}_2^1(B^d)$ is a subspace of $W_2^1(B^d)$. The extended bilinear form $\tilde{a}$ is defined by

$$\begin{array}{rcl} \tilde{a} : \tilde{W}_2^1(B^d) \times \tilde{W}_2^1(B^d) & \to & \mathbb{R} \\ (\tilde{w}, \tilde{v}) & \mapsto & \displaystyle \frac{1}{2} \int\limits_{B^d} (\nabla u)^T \bar{B} \nabla v \; d\lambda \end{array}$$

where

$$\bar{B} := \begin{pmatrix} \hat{b}_{11} & \tilde{b}_{12} & \cdots & \tilde{b}_{1d} \\ \tilde{b}_{21} & \hat{b}_{22} & \cdots & \hat{b}_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{b}_{d1} & \hat{b}_{d2} & \cdots & \hat{b}_{dd} \end{pmatrix}.$$

Observe that the elements of the matrix $\bar{B}$ are the symmetric extended elements of the matrix B with the exception of the elements $\tilde{b}_{1i}$ and $\tilde{b}_{i1}$ for $i \neq 1$. For example, this implies that

$$\hat{b}_{kk} \in W^1_\infty(B^d) \quad \text{for} \quad 1 \leq k \leq d.$$

But the elements $\tilde{b}_{1i}$ and $\tilde{b}_{i1}$ do not have such a property in general. A simple calculation shows

$$\tilde{a}(\tilde{u}, \tilde{v}) = a(u, v) \quad \text{for every} \quad u, v \in \overset{\circ}{W}{}^1_2(\Omega^d).$$

At last we need some difference operators. Let $0 < h < 1$. Then define

$$\delta^1_h(w)(x_1) := \frac{w(x_1 + \frac{h}{2}) - w(x_1 - \frac{h}{2})}{h} \quad \text{and} \quad \delta^2_h := \delta^1_h \circ \delta^1_h.$$

So, these operators act in the direction of the first coordinate.

With these preliminaries, we obtain the following regularity result:

Theorem 1 (Regularity in Case of Diagonal Matrices B) *Assume that B is a diagonal matrix. Then, the solution u of the equation (2) on the unit cube Ω^d satisfies the inequality*

$$\|u\|_{W^2_2(\Omega^d)} \leq C\|f\|_{L^2(\Omega^d)} \tag{3}$$

where C is a constant independent of u.

Proof: By a symmetry argument we obtain (see Figure 1)

$$\delta^2_h(\tilde{v}) \in \tilde{W}^1_2(B^d) \tag{4}$$

for every $\tilde{v} \in \tilde{W}^1_2(B^d)$. We have to prove that

$$\tilde{a}(\delta^1_h(\tilde{u}), \delta^1_h(\tilde{v})) = -\tilde{a}(\tilde{u}, \delta^2_h(\tilde{v})) + S(\tilde{u}, \delta^1_h(\tilde{v})) \tag{5}$$

where S is a bilinear form with the property

$$\left|S(\tilde{u}, \delta^1_h(\tilde{v})\right| \leq C\|\tilde{u}\|_{W^1_2(B^d)}\|\delta^1_h(\tilde{v})\|_{W^1_2(B^d)}.$$

Equation (5) can be treated as a substitute for integration by parts. For the proof of equation (5) we need the following simple formula

$$\int_{S^1} \delta^1_h(w) bv dx = -\int_{S^1} wb\delta^1_h(v) dx - \int_{S^1} \mathcal{M}_h\left(w, \delta^1_{\frac{h}{2}}(b)\right) v dx \tag{6}$$

for every $w, v \in L^2(S^1)$ and $b \in L^\infty(S^1)$, where $\mathcal{M}_h$ is the Operator

$$\mathcal{M}_h(w,v)(x) := \frac{1}{2}\left(w\left(x+\frac{h}{2}\right)v\left(x+\frac{h}{4}\right) + w\left(x-\frac{h}{2}\right)v\left(x-\frac{h}{4}\right)\right).$$

Now, we obtain (5) by the fact that the non-diagonal elements of $\bar{B}$ are zero and that the diagonal elements are contained in $W^1_\infty(B^d)$.

Equations (2), (4), and (5) imply

$$\begin{aligned}
\|\delta_h^1(\tilde{u})\|^2_{W_2^1(\Omega^d)} &\leq C_1\|\delta_h^1(\tilde{u})\|^2_{W_2^1(B^d)} \\
&\leq C_2\tilde{a}(\delta_h^1(\tilde{u}), \delta_h^1(\tilde{u})) \\
&= -C_2\tilde{a}(\tilde{u}, \delta_h^2(\tilde{u})) + C_2 S(\tilde{u}, \delta_h^1(\tilde{u})) \\
&= -C_2 a(u, \delta_h^2(\tilde{u})) + C_2 S(\tilde{u}, \delta_h^1(\tilde{u})) \\
&= -C_2 f(\delta_h^2(\tilde{u})) + C_2 S(\tilde{u}, \delta_h^1(\tilde{u})) \\
&\leq C_3\left(\|f\|_{L^2(\Omega^d)} + \|\tilde{u}\|_{W_2^1(B^d)}\right)\|\delta_h^1(\tilde{u})\|_{W_2^1(B^d)} \\
&\leq C_4\|f\|_{L^2(\Omega^d)}\|\delta_h^1(\tilde{u})\|_{W_2^1(\Omega^d)}
\end{aligned}$$

where C_1, C_2, C_3, C_4 are suitable constants. Thus, we obtain

$$\|\delta_h^1(\tilde{u})\|_{W_2^1(\Omega^d)} \leq C_4\|f\|_{L^2(\Omega^d)}.$$

The limit $h \to 0$ shows (see Satz 9.5 in [4])

$$\left\|\frac{\partial}{\partial x_1}(\tilde{u})\right\|_{W_2^1(\Omega^d)} \leq C_5\|f\|_{L^2(\Omega^d)}.$$

A symmetry argument completes the proof.

Q.E.D.

3 Discrete Regularity on Discrete Spaces

The proof of Theorem 1 can not be extended to general matrices B, because the extended matrix elements $\tilde{b}_{1i}$ and $\tilde{b}_{i1}$ are not very smooth at the boundary of Ω^d. A different situation appears, if we study the discrete regularity in a suitable discrete space. Then, the idea of the proof of Theorem 1 can be used for general matrices B. This we will show now.

For $h = \frac{1}{N}$ and $N \in \mathbb{N}$ let V_h be the finite element space with the following properties:

- every function in V_h has homogeneous Dirichlet boundary conditions,
- every function in V_h is a piecewise multilinear function on the uniform tensor product grid of mesh size h.

Let $u_h \in V_h$ be the solution of the weak equation

$$a(u_h, v_h) = f(v_h) \quad \text{for every} \quad v_h \in V_h. \tag{7}$$

Theorem 2 (Discrete Regularity) *The solution u_h of the equation (7) satisfies the inequality*

$$\|\delta_h^1(\tilde{u}_h)\|_{W_2^1(\Omega^d)} \leq C\|f\|_{L^2(\Omega^d)}$$

where C is a constant independent of u_h.

Proof: The proof of Theorem 1 shows, that we only have to prove

$$\tilde{a}(\delta_h^1(\tilde{u}), \delta_h^1(\tilde{v})) = -\tilde{a}(\tilde{u}, \delta_h^2(\tilde{v})) + S(\tilde{u}, \delta_h^1(\tilde{v}))$$

where S is a bilinear form with the property

$$\left|S(\tilde{u}, \delta_h^1(\tilde{v})\right| \leq C\|\tilde{u}\|_{W_2^1(B^d)}\|\delta_h^1(\tilde{v})\|_{W_2^1(B^d)}.$$

Furthermore observe that we only have to study the following terms of $\tilde{a}(\delta_h^1(\tilde{u}), \delta_h^1(\tilde{v}))$:

$$\int_{B^d} \tilde{b}_{1i}\frac{\partial\delta_h^1(\tilde{u})}{\partial x_1}\frac{\partial\delta_h^1(\tilde{v})}{\partial x_i}d\lambda \quad \text{and} \quad \int_{B^d} \tilde{b}_{i1}\frac{\partial\delta_h^1(\tilde{u})}{\partial x_i}\frac{\partial\delta_h^1(\tilde{v})}{\partial x_1}d\lambda.$$

for $i \neq 1$. The other terms can be treated like in the proof of Theorem 1. By integration by parts we obtain

$$\begin{aligned}
&\int_{B^d} \tilde{b}_{1i}\frac{\partial\delta_h^1(\tilde{u})}{\partial x_1}\frac{\partial\delta_h^1(\tilde{v})}{\partial x_i}d\lambda = \\
&= \int_{B^d} \tilde{b}_{1i}\frac{\partial\delta_h^1(\tilde{u})}{\partial x_i}\frac{\partial\delta_h^1(\tilde{v})}{\partial x_1}d\lambda - \int_{B^d}\frac{\partial\tilde{b}_{1i}}{\partial x_1}\delta_h^1(\tilde{u})\frac{\partial\delta_h^1(\tilde{v})}{\partial x_i}d\lambda + \int_{B^d}\frac{\partial\tilde{b}_{1i}}{\partial x_i}\delta_h^1(\tilde{u})\frac{\partial\delta_h^1(\tilde{v})}{\partial x_1}d\lambda.
\end{aligned}$$

The last two terms of this sum are of low order. This shows that we only have to prove for a function $b \in W_\infty^1(\Omega^d)$

$$\int_{B^d} \tilde{b}\frac{\partial\delta_h^1(\tilde{u})}{\partial x_i}\frac{\partial\delta_h^1(\tilde{v})}{\partial x_1}d\lambda = -\int_{B^d} \tilde{b}\frac{\partial\tilde{u}}{\partial x_i}\frac{\partial\delta_h^2(\tilde{v})}{\partial x_1}d\lambda + S'(\tilde{u}, \delta_h^1(\tilde{v}))$$

where S' is a bilinear form with the property

$$\left|S'(\tilde{u}, \delta_h^1(\tilde{v})\right| \leq C\|\tilde{u}\|_{W_2^1(B^d)}\|\delta_h^1(\tilde{v})\|_{W_2^1(B^d)}.$$

By the formula (6), we obtain

$$\begin{aligned}
&\int_{B^d} \tilde{b}\frac{\partial\delta_h^1(\tilde{u})}{\partial x_i}\frac{\partial\delta_h^1(\tilde{v})}{\partial x_1}d\lambda = \\
&= -\int_{B^d} \tilde{b}\frac{\partial\tilde{u}}{\partial x_i}\frac{\partial\delta_h^2(\tilde{v})}{\partial x_1}d\lambda - \int_{B^d}\mathcal{M}_h\left(\frac{\partial\tilde{u}}{\partial x_i}, \delta_{\frac{h}{2}}^1(\tilde{b})\right)\frac{\partial\delta_h^1(\tilde{v})}{\partial x_1}d\lambda = \\
&= -\int_{B^d} \tilde{b}\frac{\partial\tilde{u}}{\partial x_i}\frac{\partial\delta_h^2(\tilde{v})}{\partial x_1}d\lambda - 2\int_{\Omega^d}\mathcal{M}_h\left(\frac{\partial\tilde{u}}{\partial x_i}, \delta_{\frac{h}{2}}^1(\tilde{b})\right)\frac{\partial\delta_h^1(\tilde{v})}{\partial x_1}d\lambda.
\end{aligned}$$

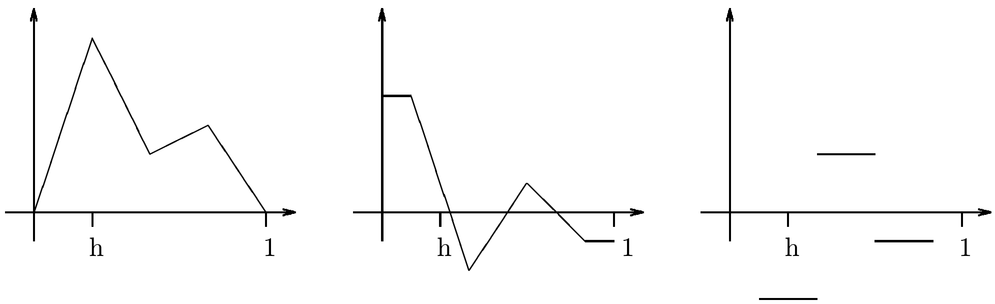

Figure 3: Functions $\tilde{v}$, $\delta_h^1(\tilde{v})$, and $\frac{\partial}{\partial x_1}\delta_h^1(\tilde{v})$ in x-direction and restricted on the interval $[0,1]$.

Thus, it is enough to prove

$$\left|\int_{\Omega^d} \mathcal{M}_h\left(\frac{\partial \tilde{u}}{\partial x_i}, \delta^1_{\frac{h}{2}}(\tilde{b})\right)\frac{\partial \delta_h^1(\tilde{v})}{\partial x_1} d\lambda\right| \leq C\|\tilde{u}\|_{W_2^1(\Omega^d)}\|\delta_h^1(\tilde{v})\|_{W_2^1(\Omega^d)}.$$

Now, look to Figure 3. The function $\frac{\partial \delta_h^1(\tilde{v})}{\partial x_1}$ is zero on the domain

$$\Phi_h :=]0,\tfrac{h}{2}[\times]0,1[^{d-1} \bigcup \,]1-\tfrac{h}{2},1[\times]0,1[^{d-1}.$$

Therefore, we get

$$\begin{aligned}
&\left|\int_{\Omega^d} \mathcal{M}_h\left(\frac{\partial \tilde{u}}{\partial x_i}, \delta^1_{\frac{h}{2}}(\tilde{b})\right)\frac{\partial \delta_h^1(\tilde{v})}{\partial x_1} d\lambda\right| = \\
&= \quad \left|\int_{\Omega^d\setminus\Phi_h} \mathcal{M}_h\left(\frac{\partial \tilde{u}}{\partial x_i}, \delta^1_{\frac{h}{2}}(\tilde{b})\right)\frac{\partial \delta_h^1(\tilde{v})}{\partial x_1} d\lambda\right| \leq \\
&\leq \quad \left\|\delta^1_{\frac{h}{2}}(\tilde{b})\right\|_{L^\infty\left(\Omega^d\setminus\Phi_{\frac{h}{2}}\right)} \|\tilde{u}\|_{W_2^1(\Omega^d)}\|\delta_h^1(\tilde{v})\|_{W_2^1(\Omega^d)} \leq \\
&\leq \quad \|b\|_{W^1_\infty(\Omega^d)}\|\tilde{u}\|_{W_2^1(\Omega^d)}\|\delta_h^1(\tilde{v})\|_{W_2^1(\Omega^d)}.
\end{aligned}$$

This completes the proof.

Q.E.D.

4 General Regularity

Theorem 3 (Regularity in Case of General Matrices B) *The solution u of the equation (2) on the unit cube Ω^d satisfies the inequality*

$$\|u\|_{W_2^2(\Omega^d)} \leq C\|f\|_{L^2(\Omega^d)}$$

where C is a constant independent of u.

PROOF: Theorem 2 and the limit $h \to 0$ imply that

$$\left\| \frac{\partial^2 u_h}{\partial x_k \partial x_l} \right\|_{L^2(\Omega^d)} \leq C \|f\|_{L^2(\Omega^d)}$$

for $k \neq l$. Furthermore V^h is dense in $\overset{\circ}{W}{}_2^1(\Omega^d)$. Thus, we get

$$\lim_{h \to 0} \|u - u_h\|_{W_2^1(\Omega^d)} = 0.$$

Therefore, we obtain

$$\begin{aligned} \left| \int_{\Omega^d} u \frac{\partial^2 \varphi}{\partial x_k \partial x_l} \, d\lambda \right| &= \left| \lim_{h \to 0} \int_{\Omega^d} u_h \frac{\partial^2 \varphi}{\partial x_k \partial x_l} \, d\lambda \right| = \\ &= \left| \lim_{h \to 0} \int_{\Omega^d} \frac{\partial^2 u_h}{\partial x_k \partial x_l} \varphi \, d\lambda \right| \leq C \|f\|_{L^2(\Omega^d)} \|\varphi\|_{L^2(\Omega^d)} \end{aligned}$$

for every $\varphi \in \mathcal{C}^\infty(\Omega^d)$. This shows that

$$\left\| \frac{\partial^2 u}{\partial x_k \partial x_l} \right\|_{L^2(\Omega^d)} \leq C \|f\|_{L^2(\Omega^d)}. \tag{8}$$

Now, we write $B = B^{diag} + B^{rest}$, where B^{diag} is the diagonal matrix of B. Let us define

$$F(v) := f(v) - \int_{\Omega^d} (\nabla u)^T B^{rest} \nabla v \, d\lambda \quad \text{and} \quad a^{diag}(u, v) := \int_{\Omega^d} (\nabla u)^T B^{diag} \nabla v \, d\lambda.$$

By (2) and (8), we obtain

$$\begin{aligned} a^{diag}(u, v) &= F(v) \quad \text{for every} \quad v \in \overset{\circ}{W}{}_2^1(\Omega^d) \quad \text{and} \\ |F(v)| &\leq C \|v\|_{L^2(\Omega^d)}. \end{aligned}$$

Theorem 1 implies that

$$\|u\|_{W_2^2(\Omega^d)} \leq C \|f\|_{L^2(\Omega^d)}.$$

Q.E.D.

Theorem 3 shows the regularity of elliptic equations on the unit cube Ω^d. Now, we generalize this result to more general domains. Let us assume that Ω is a domain with the following properties:

- $\Omega \subset \mathbb{R}^d$ is open and $\bar{\Omega}$ is compact,
- for every $x \in \Omega$ exists an open neighborhood U_x of x in Ω, a point $x' \in \Omega^d$, an open neighborhood $U'_{x'}$ of x' in Ω^d, and a $\mathcal{C}^2$-diffeomorphism

  $$\Phi_x : \bar{U}_x \mapsto \bar{U}'_{x'}$$

  such that

  $$\Phi_x \left(\partial U_x \cap \partial \Omega \right) = \partial \bar{U}'_{x'} \cap \partial \Omega^d.$$

Now, we obtain the following Corollary:

Corollary 1 (Regularity in Case of General Domains Ω) *Let us assume that the domain Ω satisfies the above assumptions. Let u be the solution of equation (2). Then, there is a constant C independent of u such that*

$$\|u\|_{W_2^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

The proof of this Corollary is analogous to the proof of Satz 9.1.4 in [3] or Theorem 8.12. in [1].

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