DICHOTOMY AND H^∞ FUNCTIONAL CALCULI

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ABSTRACT. Dichotomy for the abstract Cauchy problem with any densely defined closed operator on a Banach space is studied. We give conditions under which an operator with an H^∞ functional calculus has dichotomy. For the operators with imaginary axis contained in the resolvent set and with polynomial growth of the resolvent along the axis we prove the existence of dichotomy on subspaces and superspaces. Applications to the dichotomy of operators on L_p-spaces are given. The principle of linearized instability for nonlinear equations is proved.

1. INTRODUCTION

In the present paper we use methods from [9, 10] to study dichotomies for solutions to the abstract Cauchy problem

$$\frac{d}{dt}u(t,x) = Au(t,x) \quad u(0,x) = x \in X, \quad t \geq 0 \tag{1}$$

with a closed densely defined operator A on a Banach space X. By a *solution* of (1) we will mean a classical solution, that is, $t \mapsto u(t,x) \in C([0,\infty), [\mathcal{D}(\mathcal{A})]) \cap \mathcal{C}^\infty([\prime,\infty),\mathcal{X})$. Dichotomy means the existence of a bounded projection, P, such that the solutions that start in Im (P) decay to zero and the solutions that start in Im $(I - P)$ are unbounded.

Dichotomy and, in particular, exponential dichotomy is one of the main tools in the study of linear differential equations in Banach spaces, linearized instability for nonlinear equations, existence of invariant and center manifolds, etc. Due to the importance of the subject the literature on dichotomy is vast; besides the classical books [6, 15, 16, 27], we mention here more recent papers [3, 5, 24, 30] and [31], where one can find further references.

Assume, for a moment, that (1) is well-posed; that is, A generates a strongly continuous semigroup $\{e^{tA}\}_{t\geq0}$ on X. The semigroup is called hyperbolic if $\sigma(e^{tA})\cap \mathbb{T}$ is empty, for $t \neq 0$, where we write $\sigma(\cdot)$ for the spectrum and $\mathbb{T}$ for the unit circle. Suppose we know that A generates a hyperbolic semigroup. Then (1) has dichotomy (and even uniform exponential dichotomy— see definitions below), and

P is the Riesz projection for e^{tA}, $t > 0$ that corresponds to the part of $\sigma(e^{tA})$ in the unit disk. Also, by the spectral inclusion theorem

$$\sigma(e^{tA}) \setminus \{0\} \supseteq \exp t\sigma(A), \quad t \neq 0$$

(see, e.g., [29, p. 45]), one has

$$\sigma(A) \cap i\mathbb{R} = \emptyset, \tag{2}$$

and, moreover,

$$\sigma(A) \cap \{z \in \mathbb{C} : |\operatorname{Re} F| \leq \epsilon\} = \emptyset \text{ for some } \epsilon > \nvdash. \tag{3}$$

However, it is more important to know under which additional condition on A either (2) or (3) imply dichotomy. If the spectral mapping theorem

$$\sigma(e^{tA}) \setminus \{0\} = \exp t\sigma(A), \quad t \neq 0$$

holds for the semigroup $\{e^{tA}\}$, then (2) implies the hyperbolicity of the semigroup. This is the case, for example, when A generates an analytic semigroup; see [28]. We note that the spectral mapping theorem holds, in fact, only provided some condition on the growth of the resolvent $R(z, A) = (z - A)^{-1}$ is fulfilled. If, for instance, X is a Hilbert space then, by the Gearhart-Herbst spectral mapping theorem (see [28]), condition (2) implies the hyperbolicity of the semigroup $\{e^{tA}\}$ provided $\|R(z, A)\|$ is bounded along $i\mathbb{R}$. For any Banach space by a spectral mapping theorem from [23] this implication is true also provided a certain condition on the boundedness of the resolvent holds.

Another way to obtain P under conditions (2) or (3) is to integrate $R(z, A)$ along $i\mathbb{R}$. If A is a bounded operator with (2), then the Riesz-Dunford functional calculus for A gives the dichotomy projector P. If A is unbounded this way does not work without additional conditions on the decay of $\|R(z, A)\|$ along $i\mathbb{R}$. The necessary and sufficient conditions for a semigroup with (3) to be hyperbolic are given in [19]. These conditions include, in particular, the integrability of $R(z, A)$ along $i\mathbb{R}$ in Cesàro sense.

The present paper has two goals. First, we would like to consider dichotomy for *non*-well-posed abstract Cauchy problems (1). That is, we do *not* assume that A generates a strongly continuous semigroup. Second, we study dichotomy under very mild conditions on $R(z, A)$, $z \in i\mathbb{R}$. We require only a polynomial growth of the resolvent. Our main technical tool is to use an H^∞ functional calculus for A to obtain the dichotomy projection P.

In the first part of the paper, similarly to the stability theory for semigroups, cf. [28], we define strong and uniform dichotomy for A in (1). We show that A has uniform dichotomy provided both $A|_{\operatorname{Im} P}$ and $-A|_{\operatorname{Im} (I-P)}$ generate uniformly stable semigroups. The operators that satisfy these assumptions are called the bigenerators and were studied in [3]. We show that A has strong dichotomy provided these semigroups are strongly stable and $\sigma(A) \cap i\mathbb{R}$ is finite. Next, we assume that A has an $H^\infty(\Omega)$ functional calculus and prove that A has strong (resp. uniform) dichotomy provided $\overline{\Omega}$ is disjoint from $i\mathbb{R}$ (resp. from a vertical strip, containing $i\mathbb{R}$). This corresponds to conditions (2) and (3), respectively. We apply these results for two classes of operators A on L^p-spaces having H^∞ calculi: when iA generates a bounded group [17] and when A is an elliptic differential operator [1].

In the second part of the paper we assume that (2) holds and $\|R(z, A)\|$ has no more than polynomial growth along $i\mathbb{R}$. Under these mild assumptions A, generally, does not have the dichotomy on the entire space X. We are able to prove, however,

the existence of the Banach spaces Z and W such that $Z \hookrightarrow X \hookrightarrow W$ and the restriction and extension of A on Z and W, respectively, have strong dichotomy.

To comment on the last result, let us assume, for a moment, that A generates a continuous semigroup, condition (2) holds and $\|R(z, A)\|$ is bounded for $z \in i\mathbb{R}$. If X is a Hilbert space, the Gearhart-Herbst spectral mapping theorem implies that the semigroup $\{e^{tA}\}$ is hyperbolic. This means that A has uniform dichotomy on the entire space X. If X is a Banach space then, generally, $\{e^{tA}\}$ is not hyperbolic, and, by our result, A has strong dichotomy only on a subspace $Z \hookrightarrow X$.

In the last section of the paper we consider a semilinear equation with a linear part that satisfies the condition of polynomial growth of the resolvent. Using the result on dichotomy on subspaces, we prove the "principle of linearized instability" for the equation. This generalizes some results from [16].

We use the following notation: $\sigma(A)$, $\rho(A)$, $R(z, A)$, $\mathcal{D}(\mathcal{A})$ - the spectrum, resolvent set, resolvent, domain of an operator A, $\mathcal{L}(\mathcal{X})$ - the set of bounded linear operators on a Banach space X.

2. DICHOTOMY AND SEMIGROUPS

In the theory of stable strongly continuous semigroups (see [28, p. 99]) the following terminology is used. A strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ is called *stable* if

$$\lim_{t\to\infty} T(t)x = 0 \text{ for all } x \in X.$$

The semigroup is called *uniformly exponentially stable* if there exists positive ϵ so that

$$\lim_{t\to\infty} \|e^{\epsilon t}T(t)\| = 0.$$

Similarly, we define dichotomy for a densely defined closed operator A in (1) as follows:

Definition 2.1. *We will say that an operator A has strong dichotomy if there exists a bounded projection, P, such that $PA \subseteq AP$, $A|_{Im\ P}$ generates a stable strongly continuous semigroup, and all nontrivial solutions of* (1) *such that $x \in Im\ (I - P)$ are unbounded.*

We will say that an operator A has uniform exponential dichotomy if the semigroup generated by $A|_{Im\ P}$ is uniformly exponentially stable and there exists positive ϵ such that

$$\underline{\lim}_{t\to\infty} \|e^{-\epsilon t}u(t, x)\| > 0 \tag{4}$$

for every solution u of (1) *with $x \in Im\ (I - P)$.*

The following proposition shows that (1) has uniform exponential dichotomy provided A is, in the terminology of [3], a bigenerator.

Proposition 2.2. *Suppose there exists a bounded projection P such that $PA \subseteq AP$ and both $A|_{Im\ P}$ and $-A|_{Im\ (I-P)}$ generate strongly continuous uniformly exponentially stable semigroups. Then A has uniform exponential dichotomy.*

Proof. Suppose u is a nontrivial solution of (1), with $x \in Im\ (I - P)$. We must show that u satisfies (4). Let $G \equiv A|_{Im\ (I-P)}$. Since $t \mapsto (I-P)u(t, x)$ is a solution of (1), it follows by the uniqueness of the solutions of (1) that $u(t, x) \in Im\ (I - P)$, for all $t \geq 0$. Thus we may define

$$w(t) \equiv e^{-tG}u(t, x) \ (t \geq 0).$$

Since $\frac{d}{dt}w(t) = 0$, for all $t \geq 0$, it follows that $w(t) = w(0) = x$, for all $t \geq 0$. Thus

$$\|x\| \leq \|e^{-tG}\|\|u(t,x)\|, \ \forall t \geq 0,$$

so that

$$\|u(t,x)\| \geq \|e^{-tG}\|^{-1}\|x\|, \ \forall t \geq 0,$$

as desired.   □

In order to characterize strong dichotomy in terms of strong stability of the semigroups generated by $A|_{\text{Im } P}$ and $-A|_{\text{Im } (I-P)}$, we need to introduce the Hille-Yosida space (see [22, 20, 11], or [10, Chapter V]).

Definition 2.3. *Suppose A is a closed operator, such that the only solution of* (1), *with $x = 0$, is trivial. The Hille-Yosida space, $Z(A)$, for A, is defined to be the set of all x for which a bounded uniformly continuous mild solution of* (1) *exists.*

We define a norm on $Z(A)$ by

$$\|x\|_{Z(A)} \equiv \sup_{t \geq 0} \|u(t,x)\|.$$

In the following lemma the Hille-Yosida spaces for A and $-A$ were used to find a maximal subspace on which A generates a bounded group (see [20] and [10, Chapter V] for the proof).

Lemma 2.4. *Suppose that A is as in Definition 2.3 and define $Z \equiv Z(A) \cap Z(-A)$. Then the following holds:*

(1) *Z is the maximal continuously embedded Banach subspace of X such that $A|_Z$ generates a bounded strongly continuous group;*
(2) *$\sigma(A|_Z) \subseteq \sigma(A)$.*

It is clear that Z, from Lemma 2.4, is the set of all bounded, uniformly continuous mild solutions of the reversible abstract Cauchy problem

$$\frac{d}{dt}u(t,x) = Au(t,x), \quad u(0,x) = x, \quad t \in \mathbb{R}. \tag{5}$$

Under natural conditions on $\sigma(A)$, this abstract Cauchy problem cannot have solutions bounded on the entire line:

Lemma 2.5. *Suppose that A is as in Definition 2.3, $\sigma_p(A) \cap i\mathbb{R}$ is empty, and*

$$\sigma(A) \cap i\mathbb{R} \quad \text{is countable.}$$

Then all nontrivial solutions of (5) *are unbounded.*

Proof. Suppose u is a bounded solution of (5). Fix $\lambda \in \rho(A)$. Then

$$(\lambda - A)^{-1}u(0) \in Z \equiv Z(A) \cap Z(-A),$$

since $t \mapsto (\lambda - A)^{-1}u(t)$ has a bounded derivative, hence is uniformly continuous. By Lemma 2.4(2), $\sigma(A|_Z) \cap i\mathbb{R}$ is countable. But since $A|_Z$ generates a bounded strongly continuous group, $\sigma(A|_Z) \subseteq i\mathbb{R}$. Thus $\sigma(A|_Z)$ is a countable subset of $i\mathbb{R}$. If $\sigma(A|_Z)$ is nonempty, then it follows that it must contain an isolated point. This isolated point is an imaginary eigenvalue for $A|_Z$ (see [7, Chapter 8]), hence for A. Since $\sigma_p(A) \cap i\mathbb{R}$ is empty, this would be a contradiction. Thus $\sigma(A|_Z)$ is empty, which implies that Z is trivial (see [7, Chapter 8]). Thus $(\lambda - A)^{-1}u(0) = 0$, so that u is trivial.   □

When $\sigma(A) \cap i\mathbb{R}$ is empty, Lemma 2.5 may be found in [12] and [18].

The following proposition is the analogue of Proposition 2.2 for the case of strong dichotomy.

Proposition 2.6. *Suppose there exists a bounded projection P such that $PA \subseteq AP$, both $A|_{\mathrm{Im}\ P}$ and $-A|_{\mathrm{Im}\ (I-P)}$ generate strongly continuous stable semigroups, and*

$$\sigma(A) \cap i\mathbb{R} \text{ is countable.} \tag{6}$$

Then A has strong dichotomy.

Proof. Note first that $\sigma_p(A) \cap i\mathbb{R}$ is empty. Indeed, if $Ax = i\lambda x$ for some real λ, then $APx = i\lambda Px$, so that, since $A|_{\mathrm{Im}\ P}$ generates a stable strongly continuous semigroup, $Px = 0$; similarly, $(I - P)x = 0$.

Suppose u is a bounded solution of (1), with $x \in \mathrm{Im}\ (I - P)$. We must show that u is trivial. Clearly u extends to a bounded solution of (1.6), by defining

$$u(t) \equiv e^{tA|\ \mathrm{Im}\ (I-P)} u(0) \ (t \leq 0).$$

By Lemma 2.5, u is trivial. $\square$

The following example shows that hypothesis (6) in Proposition 2.6 is necessary. That is, it is not sufficient, for A to have strong dichotomy, to have both $A|_{\mathrm{Im}\ P}$ and $-A|_{\mathrm{Im}\ (I-P)}$ generate strongly continuous stable semigroups.

Example 2.7. *Take $X \equiv L^p(\mathbb{R}, \eth(\sim) \sim)$, $1 \leq p < \infty$, where g is a nondecreasing positive function on $\mathbb{R}$, and take A to be $-\frac{d}{ds}$. That is,*

$$\|f\|^p \equiv \int_{\mathbb{R}} |f(s)|^p g(s) \, ds,$$

and $-A$ is the generator of the strongly continuous contracting semigroup of left-translations

$$e^{-tA} f(s) \equiv f(s + t), \quad s \in \mathbb{R}, \approx \ \geq\ \nvdash, \ \mho \in \mathbb{X}.$$

It is not hard to see that, for f bounded and of compact support,

$$\lim_{t \to \infty} \|e^{-tA} f\|^p = g(-\infty) \int_{\mathbb{R}} |f(s)|^p \, ds.$$

Thus, if we choose g such that $g(-\infty) = 0$, then $-A$ generates a stable strongly continuous semigroup. Except for condition (6), we have the hypotheses in Proposition 2.6, with $P \equiv 0$. Strong dichotomy is thus equivalent to (1) having no nontrivial bounded solutions.

If we assume that g is exponentially bounded, then translation becomes a strongly continuous group,

$$e^{tA} f(s) \equiv f(s - t), \quad s, t \in \mathbb{R}, \ \mho \in \mathbb{X}.$$

It is again clear that

$$\lim_{t \to \infty} \|e^{tA} f\|^p = g(\infty) \int_{\mathbb{R}} |f(s)|^p \, ds$$

for any $f \in X$. Thus, if g is bounded, we do not have strong dichotomy; in fact, (1) has a bounded solution for all initial data in the domain of A.

An analogue of this example, for incomplete second-order Cauchy problems, is in [8, Example 2.15]. See [8, Section II] for the relationship between different versions of such Cauchy problems and stable or bounded strongly continuous semigroups. In the language of [8, Definition 2.7], the operator $-A|_{\text{Im }(I-P)}$, from Proposition 2.6, generates a bounded, nowhere-reversible strongly continuous semigroup.

3. DICHOTOMY AND H^∞ FUNCTIONAL CALCULI

In this section we will study dichotomy for (1) for operators that have an H^∞ functional calculus. Examples of operators with this property and applications of our dichotomy results are given in the next section.

Definition 3.1. *If Ω is an open subset of the complex plane, not equal to the entire plane, we will say that an operator A has an $H^\infty(\Omega)$ functional calculus if $\sigma(A) \subseteq \overline{\Omega}$ and there exists a continuous algebra homomorphism, $f \mapsto f(A)$, from $H^\infty(\Omega)$ into $\mathcal{L}(\mathcal{X})$, such that $f_0(A) = I$ and $g_\lambda(A) = (\lambda - A)^{-1}$, for all $\lambda \notin \overline{\Omega}$, where $f_0(z) \equiv 1, g_\lambda(z) \equiv (\lambda - z)^{-1}$.*

The main tool in the proof of the next proposition is the ABLV-Theorem (Arendt-Batty-Lyubich-Vũ; see [26] and [2]), that gives the best available condition for a strongly continuous semigroup to be stable.

Proposition 3.2. *Suppose Ω is an open set contained in the left half-plane, such that $\overline{\Omega} \cap i\mathbb{R}$ is countable, $\sigma_p(A) \cap i\mathbb{R}$ is empty and A is densely defined and has an $H^\infty(\Omega)$ functional calculus. Then A generates a stable strongly continuous semigroup, if either*

(1) *X is reflexive, or*
(2) *$\overline{\Omega} \cap i\mathbb{R}$ is empty.*

If $\Omega \subseteq \{z \in \mathbf{C} \mid \text{Re}(\mathbf{z}) < -\epsilon\}$, for some positive ϵ, then the semigroup is uniformly exponentially stable.

Proof. Since A has an $H^\infty(\Omega)$ functional calculus, and Ω is contained in the left half-plane, a short calculation shows that $\{\|\lambda^n(\lambda - A)^{-n}\| \mid \lambda > 0, n \in \mathbf{N}\}$ is bounded. By the Hille-Yosida theorem, since $\mathcal{D}(\mathcal{A})$ is dense, A generates a bounded strongly continuous semigroup.

Since $\sigma(A) \cap i\mathbb{R}$ is contained in $\overline{\Omega} \cap i\mathbb{R}$, the ABVL-Theorem ([26] and [2]) guarantees that either (1) or (2) above implies that the semigroup generated by A is stable.

If there exists positive ϵ such that $\Omega \subseteq \{z \in \mathbf{C} \mid \text{Re}(\mathbf{z}) < -\epsilon\}$, then, exactly as argued at the beginning of the proof, $(A + \epsilon)$ generates a bounded strongly continuous semigroup, so that the semigroup generated by A is uniformly exponentially stable. $\square$

To obtain dichotomy, we need to apply this result for both "stable" and "unstable" parts of A as follows.

Corollary 3.3. *Suppose $\sigma_p(A) \cap i\mathbb{R}$ is empty and A is densely defined and has an $H^\infty(\Omega)$ functional calculus, where Ω is an open subset of the complex plane such that $\overline{\Omega} \cap i\mathbb{R}$ is countable. Then there exists a bounded projection P such that $PA \subseteq AP$ and $A|_{\text{Im }P}$ and $-A|_{\text{Im }(I-P)}$ generate stable strongly continuous semigroups, if either*

(1) *X is reflexive, or*

(2) $\overline{\Omega} \cap i\mathbb{R}$ is empty.

If there exists positive ϵ such that $\Omega \cap \{z \in \mathbf{C}\,|\,|\operatorname{Re}(\mathbf{z})| < \epsilon\}$ is empty, then "stable" may be replaced by "uniformly exponentially stable."

Proof. Let

$$\Omega_1 \equiv \Omega \cap \{z \in \mathbf{C}\,|\,\mathbf{Re}(\mathbf{z}) < \mathbf{0}\}, \ \Omega_\mathbf{2} \equiv \Omega \cap \{\mathbf{z} \in \mathbf{C}\,|\,\mathbf{Re}(\mathbf{z}) > \mathbf{0}\}.$$

Let $P \equiv 1_{\Omega_1}(A)$ for the characteristic function $1_{\Omega_1}(\cdot)$ of Ω_1. Then $I - P = 1_{\Omega_2}(A)$, thus we may apply Proposition 3.2 to both $A|_{\operatorname{Im} P}$ and $-A|_{\operatorname{Im} (I-P)}$.

If there exists positive ϵ such that $\Omega \cap \{z \in \mathbf{C}\,|\,|\operatorname{Re}(\mathbf{z})| < \epsilon\}$ is empty, then replace $Re(z) < 0$ with $Re(z) < -\epsilon$ and $Re(z) > 0$ with $Re(z) > \epsilon$, and again use Proposition 3.2. $\qquad\square$

We are ready to prove the main result of this section. For $0 < \theta \leq \pi$ let $S_\theta \equiv \{re^{i\phi}\,|\,r > 0, |\phi| < \theta\}$ denote a sector of angle θ.

Theorem 3.4. *Suppose Ω is an open subset of the complex plane such that $\overline{\Omega} \cap i\mathbb{R}$ is countable, $\sigma_p(A) \cap i\mathbb{R}$ is empty and A is densely defined and has an $H^\infty(\Omega)$ functional calculus. Then A has strong dichotomy, if either*

(1) *X is reflexive, or*
(2) *$\overline{\Omega} \cap i\mathbb{R}$ is empty.*

If, in addition to (2), either

(3) *there exists $\epsilon > 0$ such that Ω is disjoint from $\{z \in \mathbb{C}\,|\,|\operatorname{Re}(F)| < \epsilon\}$, or*
(4) *$0 \in \rho(A)$ and Ω is contained in a cone $(S_\theta \cup -S_\theta)$, for some $\theta < \frac{\pi}{2}$,*

then A has uniform exponential dichotomy.

Proof. The assertion about strong dichotomy follows from Corollary 3.3 and Proposition 2.6, since $\sigma(A)$ is contained in $\overline{\Omega}$.

Under hypothesis (3), uniform exponential dichotomy follows from Corollary 3.3 and Proposition 2.2.

Under hypothesis (4), it is straightforward to show, analogously to the proof of Proposition 3.2, that, for P as in Corollary 3.3, both $A|_{\operatorname{Im} P}$ and $-A|_{\operatorname{Im} (I-P)}$ generate bounded holomorphic strongly continuous semigroups. Since $0 \in \rho(A)$, so that $0 \in \rho(A|_{\operatorname{Im} P})$ and $\rho(-A|_{\operatorname{Im} (I-P)})$, these semigroups are both uniformly exponentially stable (see [29, Theorem 4.4.3]). Thus we may again apply Proposition 2.2. $\qquad\square$

Remark 3.5. Let us stress, that under hypothesis (3) both $A|_{\operatorname{Im} P}$ and $-A|_{\operatorname{Im} (I-P)}$ generate uniformly stable strongly continuous semigroups. We will use this fact in the last section.

4. EXPONENTIAL DICHOTOMY ON L^P SPACES

In this section we will apply Theorem 3.4 for two classes of operators on L^p-spaces having an H^∞ functional calculus.

1. Bounded groups. We cite the following result from [17]. Let $X = L^p(\Omega, \mu)$, for $1 < p < \infty$, (Ω, μ) be a measure space.

Lemma 4.1. *If iA generates a bounded strongly continuous group, A is injective and $0 < \theta < \frac{\pi}{2}$, then A has an $H^\infty(S_\theta \cup -S_\theta)$ functional calculus.*

Theorem 3.4 now implies the following.

Corollary 4.2. *If iA generates a bounded strongly continuous group and A is injective, then A has strong dichotomy.*

If, in addition, $0 \in \rho(A)$, then A has uniform exponential dichotomy.

2. Differential operators. Our next goal is to combine Theorem 3.4 and results from [1] to study the dichotomy of elliptic differential operators acting on vector valued L^p-functions over $\mathbb{R}^n$ with sufficiently large zero order term and certain regularity conditions on the coefficients.

To formulate the results from [1] we will need some notations. Let

$$\mathcal{A} = \sum_{|\alpha| \leq m} a_\alpha \mathcal{D}^\alpha$$

be a linear differential operator of order m on $X = L^p(\mathbb{R}^n, \mathbb{R}^N)$, $1 < p < \infty$, with $\mathcal{L}(\mathbb{R}^N)$-valued coefficients:

$$a_\alpha : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^N), \quad \alpha \in \mathbb{N}^n, \quad |\alpha| \leq m.$$

Fix any $M > 0$ and $\theta_0 \in [0, \pi/2]$. We will say that $\mathcal{A}$ is *uniformly (M, θ_0)-elliptic* if $\max_{|\alpha|=m} \|a_\alpha\|_\infty \leq M$ and for its principal symbol

$$\mathcal{A}_\pi(x, \xi) \equiv \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$$

the following conditions hold:

$$\sigma(\mathcal{A}_\pi(x,\xi)) \subset \mathcal{S}_{\theta_0} \setminus \{0\}, \quad \|[\mathcal{A}_\pi(x,\xi)]^{-1}\| \leq M, \quad x \in \mathbb{R}^n, \|\xi\| = 1.$$

To formulate the regularity conditions on the coefficients, for fixed $p \in (1, \infty)$ and $m \in \mathbb{N}$ choose any q_α such that

$$q_\alpha = p \text{ if } |\alpha| < m - n/p \text{ and } q_\alpha > n/(m - |\alpha|) \text{ if } m - n/p \leq |\alpha| \leq m.$$

Let $\omega : \mathbb{R} \to \mathbb{R}$ be a modulus of continuity that satisfies the condition

$$\int_0^1 \frac{\omega^{1/3}(t)}{t}\, dt < \infty.$$

Let $BUC(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^N); \omega)$ denote the set of bounded uniformly continuous functions with the finite norm

$$\|a\|_{C(\omega)} \equiv \|a\|_\infty + \sup_{x \neq y} \frac{|a(x) - a(y)|}{\omega(|x - y|)}.$$

Also, let

$$L^q_{\text{unif}}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^N)) \equiv$$

$$\left\{ a \in L^1_{\text{loc}}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^N)) : \|a\|_{q, \text{unif}} \equiv \sup_{n \in \mathbb{Z}^n} \|a(\cdot - n)\|_{L^q((-1,1)^n, \mathcal{L}(\mathbb{R}^N))} < \infty \right\}.$$

We impose the following regularity conditions on the coefficients:

$$a_\alpha \in BUC(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^N); \omega) \text{ if } |\alpha| = m,$$

$$a_\alpha \in L^q_{\text{unif}}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^N)) \text{ if } |\alpha| \leq m - 1, \tag{7}$$

and

$$\max_{|\alpha| \leq m-1} \|a_\alpha\|_{q_\alpha, \text{unif}} + \max_{|\alpha|=m} \|a_\alpha\|_{C(\omega)} \leq M. \tag{8}$$

The following result was proved in [1].

Lemma 4.3. *There exists a constant $\mu > 0$ such that for each (M, θ_0)-elliptic operator $\mathcal{A}$ on $\mathbb{R}^\varkappa$, satisfying (7)–(8), the operator $\mu + \mathcal{A}$ has $H^\infty(S_\theta \backslash \{0\})$ functional calculus for $0 \leq \theta_0 < \theta < \pi/2$.*

Theorem 3.4 now gives the following fact.

Corollary 4.4. *Assume $\mathcal{A}$ and μ are as in Lemma 4.3. If $\mu + \mathcal{A}$ is injective, then $\mu + \mathcal{A}$ has strong dichotomy. If $\mu + \mathcal{A}$ is invertible, then $\mu + \mathcal{A}$ has uniform exponential dichotomy.*

5. EXPONENTIAL DICHOTOMY ON SUBSPACES AND SUPERSPACES

In this section we will assume that $\sigma(A) \cap i\mathbb{R} = \emptyset$ and the resolvent of A grows no faster than a polynomial along $i\mathbb{R}$. Under these conditions A, generally, does not have dichotomy on X. However, we will identify Banach spaces Z and W such that $Z \hookrightarrow X \hookrightarrow W$ and the restriction and extension of A on Z and W, respectively, have dichotomy. Our main tool is the existence of an H^∞ functional calculus on Z and W.

Lemma 5.1. *Suppose Ω is an open subset of the complex plane whose complement contains a half-line and whose boundary is a positively oriented countable system of piecewise smooth, mutually nonintersecting (possibly unbounded) arcs, $\sigma(A) \subseteq \Omega$, A is densely defined and $\|(w - A)^{-1}\|$ is $O((1 + |w|)^N)$, for $w \notin \Omega$.*
   Then there exist Banach spaces Z, W, and an operator B, on W, such that

$$[\mathcal{D}(\mathcal{A}^{\mathcal{N}+\in})] \hookrightarrow \mathcal{Z} = [\mathcal{D}(\mathcal{B}^{\mathcal{N}+\in})] \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{W},$$

$A|_Z$ and B are densely defined and have $H^\infty(\Omega)$ functional calculi and $A = B|_X$.

Proof. The existence of Z is proven in [9, Theorem 7.1], except that the density of $\mathcal{D}(A|_\mathcal{Z})$ is not addressed. This density follows by observing that, since $\mathcal{D}(\mathcal{A})$ is dense, it follows that $\mathcal{D}(\mathcal{A}^{\mathcal{N}+\ni})$ is dense in $[\mathcal{D}(\mathcal{A}^{\mathcal{N}+\in})]$, hence is dense in Z; it is clear that $\mathcal{D}(\mathcal{A}^{\mathcal{N}+\ni})$ is contained in $\mathcal{D}(A|_\mathcal{Z})$.
   Define W to be the completion of Z with respect to the norm

$$\|x\|_W \equiv \|A^{-(N+2)}x\|_Z.$$

We construct a functional calculus as follows. For any $f \in H^\infty(\Omega)$, $x \in W$, define

$$(\Lambda f)x \equiv \lim_{n \to \infty} f(A|_Z)x_n,$$

where the limit is taken in W, and $\{x_n\}$ is any sequence in Z converging to x in W. Note that the existence and uniqueness of $\lim_{n \to \infty} f(A|_Z)x_n$ follows from the boundedness of $f(A|_Z)$ and the fact that $A^{-(N+2)}$ commutes with $f(A|_Z)$.
   It is clear that $f \mapsto \Lambda f$ is a continuous algebra homomorphism from $H^\infty(\Omega)$ into $\mathcal{L}(\mathcal{W})$. Let us show that this homomorphism is as in Definition 3.1.
   For $\lambda \notin \overline{\Omega}$, we claim that Λg_λ is injective. To see this, suppose $x \in W$ and $\Lambda g_\lambda x = 0$. Choose $\{x_n\} \subset Z$ such that $x_n \to x$ in W. Then $y_n \equiv g_\lambda(A|_Z)x_n \to 0$ in W. Since $g_\lambda(A|_Z) = (\lambda - A|_Z)^{-1}$, this means that

$$(\lambda - A|_Z)A^{-(N+2)}y_n = A^{-(N+2)}x_n \to A^{-(N+2)}x \quad \text{and } A^{-(N+2)}y_n \to 0,$$

both in Z. Since $A|_Z$ is closed, this implies that $A^{-(N+2)}x$, hence x, must equal 0, proving the claim.

Since $f \mapsto \Lambda f$ is an algebra homomorphism, $\{\Lambda g_\lambda \,|\, \lambda \notin \overline{\Omega}\}$ is a pseudoresolvent family. Thus $\{\Lambda g_\lambda \,|\, \lambda \notin \overline{\Omega}\}$ is a pseudoresolvent family of injective operators. This means that there exists an operator B, on W, such that $(\lambda - B)^{-1} = \Lambda g_\lambda$, for all λ not in $\overline{\Omega}$. It is clear that $f \mapsto \Lambda f$ is now an $H^\infty(\Omega)$ functional calculus for B.

There exists a constant M such that

$$\|A^{-(N+2)}y\|_Z \le M\|A^{-(N+2)}y\|_{[\mathcal{D}(\mathcal{A}^{(\mathcal{N}+\in)})]} \equiv M\|y\|,$$

for all $y \in X$. This implies that

$$\|x\|_W \equiv \|A^{-(N+2)}x\|_Z \le M\|x\|,$$

for all $x \in Z$; that is, $X \hookrightarrow W$.

To show that $A = B|_X$, it is sufficient to show that $A^{-1} = B^{-1}|_X$. Suppose $x \in X$. Then $x \in W$, so there exists $\{x_n\} \subseteq Z$ such that $x_n \to x$ and $A^{-1}x_n \to B^{-1}x$, both in W. This means that $A^{-(N+2)}x_n \to A^{-(N+2)}x$ and

$$A^{-(N+2)}A^{-1}x_n \to A^{-(N+2)}B^{-1}x \quad \text{in } Z,$$

hence in X. Thus $A^{-(N+2)}A^{-1}x = A^{-(N+2)}B^{-1}x$, so that $A^{-1}x = B^{-1}x$, as desired.

Since $\mathcal{D}(\mathcal{A}|_{\mathcal{Z}})$ is dense in Z, it is dense in W; since $\mathcal{D}(\mathcal{A}|_{\mathcal{Z}})$ is contained in $\mathcal{D}(\mathcal{B})$, it follows that B is densely defined.

All that remains is to show that $[\mathcal{D}(\mathcal{B}^{\mathcal{N}+\in})] = \mathcal{Z}$. For $x \in \mathcal{D}((\mathcal{A}|_{\mathcal{Z}})^{\mathcal{N}+\in})$,

$$\|x\|_Z = \|x\|_{[\mathcal{D}(\mathcal{B}^{\mathcal{N}+\in})]},$$

thus, since $\mathcal{D}((\mathcal{A}|_{\mathcal{Z}})^{\mathcal{N}+\in})$ is dense in Z, it follows that

$$[\mathcal{D}(\mathcal{B}^{\mathcal{N}+\in})] = \mathcal{Z}.$$

$\square$

The following lemma shows that the polynomial growth of the resolvent along $i\mathbb{R}$ automatically implies the same growth outside some Ω as in Lemma 5.1

Lemma 5.2. *Suppose $i\mathbb{R} \subseteq \rho(\mathbb{A})$ and $\|(iy - A)^{-1}\|$ is $O(1 + |y|^N)$, for y real.*

Then there exists Ω, as in Lemma 5.1, such that $\overline{\Omega} \cap i\mathbb{R}$ is empty, $\sigma(A) \subseteq \Omega$ and $\|(z - A)^{-1}\|$ is $O(1 + |z|^N)$, for z outside Ω.

Proof. This follows from a power series expansion of the resolvent. There exists a constant M so that

$$\|(iy - A)^{-1}\| \le M(1 + |y|^N), \forall y \in \mathbb{R}.$$

For $\frac{1}{\epsilon} > 2M(1 + |y|^N)$ one has $(\epsilon + iy) \in \rho(A)$ with

$$(\epsilon + iy - A)^{-1} = \sum_{k=0}^{\infty} (-\epsilon)^k (iy - A)^{-(k+1)},$$

so that

$$\|(\epsilon + iy - A)^{-1}\| \le \sum_{k=0}^{\infty} (\epsilon)^k (M(1 + |y|^N))^{k+1}$$

$$= \frac{M(1 + |y|^N)}{1 - \epsilon M(1 + |y|^N)} \le 2M(1 + |y|^N),$$

as required.

$\square$

Remark 5.3. The proof of Lemma 5.2 also shows that the resolvent of A is bounded in a vertical strip around $i\mathbb{R}$ provided $i\mathbb{R} \subset \rho(\mathbb{A})$ and $\|(iy - A)^{-1}\|$, $y \in \mathbb{R}$, is bounded.

The following theorem is an immediate consequence of Theorem 3.4 and Lemmas 5.1 and 5.2.

Theorem 5.4. *Suppose A is densely defined, $i\mathbb{R} \subseteq \rho(\mathbb{A})$ and $\|(iy - A)^{-1}\|$ is $O(1 + |y|^N)$, for y real. Then*

(1) *there exists a Banach space Z such that*
$$[\mathcal{D}(\mathcal{A}^{\mathcal{N}+\in})] \hookrightarrow \mathcal{Z} \hookrightarrow \mathcal{X}$$
and $A|_Z$ has strong dichotomy, and

(2) *there exists a Banach space W and an operator B, on W, such that*
$$[\mathcal{D}(\mathcal{B}^{\mathcal{N}+\in})] \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{W},$$
$A = B|_X$, and B has strong dichotomy.

A similar result for uniform exponential dichotomy also follows from Theorem 3.4 and Lemma 5.2.

Theorem 5.5. *Suppose A is densely defined, there exists positive ϵ such that $\{z \in \mathbb{C} : |\operatorname{Re} F| < \epsilon\} \subseteq \rho(\mathbb{A})$, and $\|(z - A)^{-1}\|$ is $O(1 + |z|^N)$, for $|\operatorname{Re} z| < \epsilon$. Then*

(1) *there exists a Banach space Z such that*
$$[\mathcal{D}(\mathcal{A}^{\mathcal{N}+\in})] \hookrightarrow \mathcal{Z} \hookrightarrow \mathcal{X}$$
and $A|_Z$ has uniform exponential dichotomy, and

(2) *there exists a Banach space W and an operator B, on W, such that*
$$[\mathcal{D}(\mathcal{B}^{\mathcal{N}+\in})] \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{W},$$
$A = B|_X$, and B has uniform exponential dichotomy.

Since $\|(iy - A)^{-1}\| = O(1/|y|)$ provided iA generates a bounded strongly continuous group, the following result holds.

Corollary 5.6. *If iA generates a bounded strongly continuous group and $0 \in \rho(A)$, then*

(1) *there exists a Banach space Z such that*
$$[\mathcal{D}(\mathcal{A})] \hookrightarrow \mathcal{Z} \hookrightarrow \mathcal{X}$$
and $A|_Z$ has uniform exponential dichotomy, and

(2) *there exists a Banach space W and an operator B, on W, such that*
$$[\mathcal{D}(\mathcal{B})] \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{W},$$
$A = B|_X$, and B has uniform exponential dichotomy.

Example 5.7. *To illustrate the effect of "dichotomy on subspaces" in Theorems 5.4–5.5, let us consider the operator*
$$A \equiv i\frac{d}{dx} \quad on \quad X \equiv \{f \in L^p[0,1] : \int_0^1 f(x)\, dx = 0\}, \quad 1 \le p < \infty.$$

For $1 < p < \infty$ the operator A has the uniform dichotomy with the bounded projector
$$P : f \sim \sum_{k \ne 0} a_k e^{ikx} \mapsto \sum_{k > 0} a_k e^{ikx}.$$

For $p = 1$ this projector is unbounded, and A does not have dichotomy on the entire space X. Note that iA generates a bounded strongly continuous group and $\|R(iy, A)\| = O(1/|y|)$. Theorem 5.4 gives a dense subspace Z in X such that $A|_Z$ has strong dichotomy.

6. NONLINEAR ABSTRACT CAUCHY PROBLEM

In this section we assume that A generates a strongly continuous semigroup on X. Let g be a nonlinear function,

$$g : \mathbb{R} \times \mathbb{U} \to \mathcal{D}(\mathcal{A}^{\mathcal{N}+\in}) \quad \text{for an open set} \quad \mathcal{U} \subset \mathcal{D}(\mathcal{A}^{\mathcal{N}+\in}), \quad \prime \in \mathcal{U},$$

such that $g(t, 0) = 0$. Assume that g is Hölder:

$$\|g(t, x_1) - g(t, x_2)\|_{\mathcal{D}(\mathcal{A}^{\mathcal{N}+\in})} \le k(r)\|x_1 - x_2\|_X, \tag{9}$$
$$\text{for } x_i \in U, \|x_i\|_X \le r, \ i = 1, 2, \text{ and } k(r) \to 0 \text{ as } r \to 0.$$

For $t_0 \in \mathbb{R}$ consider the following semilinear abstract Cauchy problem:

$$\frac{d}{dt}u(t, x) = Au(t, x) + g(t, u(t, x)), \quad u(t_0, x) = x \in X. \tag{10}$$

We say, that $u(\cdot, x)$ is a mild solution of (10) on (t_0, τ), if it satisfies the integral equation

$$u(t, x) = e^{A(t-t_0)}x + \int_{t_0}^{t} e^{A(t-s)}g(s, u(s, x)) \, ds, \quad t \in (t_0, \tau). \tag{11}$$

For A as in Theorem 5.5 we will prove the following "principle of linearized instability" (see [16, Th. 5.1.3], [21, 25] and references therein for similar results on sectorial operators A). Recall (see Remark 5.3) that, for instance, the condition $\|(iy - A)^{-1}\| = O(1)$, $y \in \mathbb{R}$ implies the hypothesis of Theorem 5.5 with $N = 0$.

Theorem 6.1. *Suppose there exists positive ϵ such that $\{z \in \mathbb{C} : |\operatorname{Re} F| < \epsilon\} \subseteq \rho(\mathbb{A})$, and $\|(z - A)^{-1}\|$ is $O(1 + |z|^N)$, for $|\operatorname{Re} z| < \epsilon$. Assume $\sigma(A) \cap \{z : \operatorname{Re} z > 0\} \ne \emptyset$. Then the zero solution of (10) is unstable in X. That is, for some positive ϵ and a sequence $x_n \in X$ such that $\|x_n\|_X \to 0$ there exist solutions $u_*(\cdot) = u_*(\cdot, x_n)$ of (11) with $x = x_n$ so that $\|u_*(t_n, x_n)\|_X \ge \epsilon$ for some $t_n \ge t_0$.*

Proof. By Theorem 5.5 there exists a Banach space Z with $\mathcal{D}(\mathcal{A}^{\mathcal{N}+\in}) \hookrightarrow \mathcal{Z} \hookrightarrow \mathcal{X}$ such that, for constants $M, M_1 > 0$,

$$\|x\|_X \le M\|x\|_Z, \ x \in Z, \quad \text{and} \quad \|x\|_Z \le M_1\|x\|_{\mathcal{D}(\mathcal{A}^{\mathcal{N}+\in})}, \ x \in \mathcal{D}(\mathcal{A}^{\mathcal{N}+\in}), \tag{12}$$

and $A|_Z$ generates a hyperbolic strongly continuous semigroup on Z. This means that for a projection P bounded on Z and some $\beta > 0$ and $C > 1$ one has

$$\|e^{tA_-}x\|_Z \le Ce^{-t\beta}\|x\|_Z, \quad \|e^{-tA_+}x\|_Z \le Ce^{-t\beta}\|x\|_Z, \quad t > 0. \tag{13}$$

Here A_- and A_+ denote the restrictions of A on $\Im(P)$ and $\Im(Q)$, respectively, where $Q \equiv I - P$.

For C and β from (13) choose $\tilde{x}_0 \in \operatorname{Im}(I - P)$, $\tilde{x}_0 \in Z$, so that

$$0 < \|\tilde{x}_0\|_Z \le \frac{1}{2CM}, \tag{14}$$

and r small enough, so that

$$\frac{2}{\beta}CMM_1k(r)(\|P\|_{\mathcal{L}(\mathcal{Z})} + \|I - P\|_{\mathcal{L}(\mathcal{Z})}) \le \frac{\|\tilde{x}_0\|_X}{8}. \tag{15}$$

By (14) and (12) with $C > 1$ one has

$$\frac{2}{\beta}CMM_1k(r)(\|P\|_{\mathcal{L}(\mathcal{Z})} + \|I - P\|_{\mathcal{L}(\mathcal{Z})}) \le \frac{1}{16C} \le \frac{1}{2}. \tag{16}$$

Denote $x_0 = r\tilde{x}_0$. Then (14) gives:

$$\|x_0\|_Z = \|r\tilde{x}_0\|_Z \le \frac{r}{2CM}. \tag{17}$$

Fix $\tau \ge t_0$. Denote $\mathcal{C} = \mathcal{C}((-\infty, \tau], \mathcal{Z})$ the space of Z-valued continuous functions with sup-norm. Consider a subset $\mathcal{B} = \mathcal{B}_{\tau,\S}$, of $\mathcal{C}$, defined as follows:

$$\mathcal{B} = \{\sqcap \in \mathcal{C} : \|\sqcap(\sqcup)\|_{\mathcal{Z}} \le \frac{\nabla}{\mathcal{M}} \cdot \rceil^{\frac{\beta}{\in}(\sqcup - \tau)}, \quad \sqcup \le \tau, \quad (\mathcal{I} - \mathcal{P})\sqcap(\tau) = \S_{\prime}\}. \tag{18}$$

Define a nonlinear operator $T = T_{\tau,x_0}$ in $\mathcal{C}$ as follows:

$$(Tu)(t) \equiv e^{-A_+(\tau-t)}x_0 \quad - \quad \int_t^{\tau} e^{-A_+(s-t)}(I - P)g(s, u(s))\, ds$$

$$+ \quad \int_{-\infty}^t e^{A_-(t-s)}Pg(s, u(s))\, ds, \quad t \le \tau.$$

Claim 1. T preserves $\mathcal{B}$.

Proof. By (13) and (17) one has:

$$\|e^{-A_+(\tau-t)}x_0\|_Z \le Ce^{-\beta(\tau-t)}\|x_0\|_Z \le \frac{1}{2}\frac{r}{M}e^{\frac{\beta}{2}(t-\tau)}. \tag{19}$$

Fix $u \in \mathcal{B}$. Then (13) gives:

$$\left\|\int_{-\infty}^t e^{A_-(t-s)}Pg(s, u(s))ds\right\|_Z \le C\|P\|_{\mathcal{L}(\mathcal{Z})}\int_{-\infty}^t e^{-\beta(t-s)}\|g(s, u(s))\|_Z ds. \tag{20}$$

Since $u \in \mathcal{B}$, one has from (12):

$$\|u(s)\|_X \le M\|u(s)\|_Z \le re^{\frac{\beta}{2}(s-\tau)} \le r, \quad s \le \tau.$$

Then (9) can be applied in (20), and we use (12) to continue the estimate in (20):

$$\le C\|P\|_{\mathcal{L}(\mathcal{Z})}M_1k(r)r\int_{-\infty}^t e^{\frac{\beta}{2}(s-\tau)}e^{-\beta(t-s)}\, ds = \frac{2}{3\beta}CMM_1k(r)\|P\| \cdot \frac{r}{M}e^{\frac{\beta}{2}(t-\tau)}. \tag{21}$$

Similarly,

$$\left\|\int_t^{\tau} e^{-A_+(s-t)}(I - P)g(s, u(s))\, ds\right\| \le \frac{2}{\beta}CMM_1k(r)\|I - P\| \cdot \frac{r}{M}e^{\frac{\beta}{2}(t-\tau)}. \tag{22}$$

Adding (19),(21) and (22), and taking into account the inequality (16), we get the desired estimate, as in (18). $\qquad\square$

Claim 2. T is a strict contraction on $\mathcal{B}$.

Proof. Indeed, similarly to Claim 1, for $u_1, u_2 \in \mathcal{B}$ one has :

$$\|Tu_1 - Tu_2\|_\mathcal{C} \le$$

$$\max_t \left\{ \quad \left\| \int_t^\tau e^{-A_+(s-t)}(I-P)[g(s,u_1(s)) - g(s,u_2(s))]\,ds \right\|_Z + \right.$$

$$\left\| \int_{-\infty}^t e^{A_-(t-s)}P[g(s,u_1(s)) - g(s,u_2(s))]\,ds \right\|_Z \right\}$$

$$\le \frac{1}{\beta}CMM_1k(r)(\|P\| + \|I-P\|)\|u_1 - u_2\|_\mathcal{C} \le \frac{1}{4}\|u_1 - u_2\|_\mathcal{C}$$

by (13) and (16). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Therefore, the equation $u = Tu$ has a unique solution $u_*(\cdot) \equiv u_{\tau,x_0}(\cdot)$ in $\mathcal{B}$.

Claim 3. u_* is a solution of (11) with $x \equiv u_{\tau,x_0}(t_0)$.

Proof. Indeed, we project $u_* = Tu_*$ on $\mathrm{Im}\, P$ to obtain:

$$Pu_*(t) = \int_{-\infty}^t e^{A_-(t-s)}Pg(s,u_*(s))\,ds = \qquad\qquad\qquad (23)$$

$$e^{A_-(t-t_0)}\left[\int_{-\infty}^{t_0} e^{A_-(t_0-s)}Pg(s,u_*(s))\,ds \right] + \int_{t_0}^t e^{A_-(t-s)}Pg(s,u_*(s))\,ds.$$

Similarly,

$$(I-P)u_*(t) \;=\; e^{-A_+(\tau-t)}x_0 - \int_t^\tau e^{-A_+(s-t)}(I-P)g(s,u_*(s))\,ds \qquad (24)$$

$$\;=\; e^{A_+(t-t_0)}\left[e^{A_+(t_0-\tau)}x_0 - \int_{t_0}^\tau e^{A_+(t-s)}(I-P)g(s,u_*(s))\,ds \right]$$

$$+\; \int_{t_0}^t e^{A_+(t-s)}(I-P)g(s,u_*(s))\,ds.$$

Since

$$x = u_{\tau,x_0}(t_0) \;=\; e^{A_+(t_0-\tau)}x_0 - \int_{t_0}^\tau e^{A_+(t-s)}(I-P)g(s,u_*(s))\,ds$$

$$+\; \int_{-\infty}^{t_0} e^{A_-(t_0-s)}Pg(s,u_*(s))\,ds,$$

we see that u_* satisfies (11) just by adding (23) and (24). $\qquad\qquad\qquad$ $\square$

To finish the proof of the theorem, let $\epsilon = \frac{7}{8}\|x_0\|_X$. For $n \in \mathbb{N}$ and $\tau = t_0 + n$ construct $u_*(\cdot) \equiv u_{t_0+n,x_0}(\cdot)$ as above and denote $x_n = u_{t_0+n,x_0}(t_0)$. Since $u_* \in \mathcal{B}_{\sqcup,+\backslash,\S'}$, one has

$$\|x_n\|_X \leq M\|x_n\|_Z = M\|u_{t_0+n,x_0}(t_0)\|_Z \leq re^{-\frac{\beta}{2}n} \to 0 \text{ as } n \to \infty.$$

By Claim 3, $u_* = u_*(\cdot, x_n)$ is a mild solution for (10) with $x = x_n$.

It remains to show that, for $t_n \equiv \tau$,

$$\|u_*(t_n, x_n)\|_X = \|u_{\tau,x_0}(\tau)\|_X \geq \epsilon = \frac{7}{8}\|x_0\|_X. \tag{25}$$

Indeed, as in Claim 1, one has:

$$
\begin{aligned}
\|u_*(\tau) - x_0\|_Z &= \left\| \int_{-\infty}^{\tau} e^{A_-(t_0-s)} Pg(s, u_*(s))\, ds \right\|_Z \quad \text{(using (15))} \\
&\leq \frac{\|\tilde{x}_0\|_X}{8} \cdot \frac{r}{M} = \frac{\|x_0\|_X}{8M}.
\end{aligned}
$$

Now the estimate

$$\|x_0\|_X - \|u_*(\tau)\|_X \leq \|u_*(\tau) - x_0\|_X \leq M\|u_*(\tau) - x_0\|_Z \leq \frac{\|x_0\|_X}{8}$$

gives (25). $\qquad\qquad\square$

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