Note on Gy. Elekes's conjectures concerning unavoidable patterns in proper colorings

Vera Rosta
Webster University Geneva
1293 Bellevue, Switzerland
rosta@masg22.epfl.ch

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Abstract

A counterexample is presented to Gy. Elekes's conjecture concerning the existence of long 2-colored paths in properly colored graphs. A modified version of the conjecture is given and its connections to a problem of Erdős - Gyárfás and to Szemerédi's theorem are examined.

The coloring of the edges of a simple undirected graph is considered *proper* if adjacent edges have different colors. To solve some combinatorial geometry questions, Elekes formulated the following conjectures:

Conjecture 1 *[2] Let the edges of the complete graph K_n be properly colored with cn colors, $c > 0$. If n is sufficiently large then it must contain a six cycle with opposite edges having the same color.*

Conjecture 2 *[3] If the edges of the complete bipartite graph $K(n, n)$ (or the edges of a complete graph K_n) are properly colored with cn colors where $c > 0$, $n > n_1(k, c)$ then there exists an alternating 2-colored path of length k.*

This last conjecture is closely related to the following well known theorem of Szemerédi:

Theorem 1 *[6] Any set $A = \{a_1, a_2, \ldots, a_n\} \subset N$ whith $a_n < cn$, and $n > n_2(k, c)$ contains an arithmetic sequence of length k.*

Szemerédi's theorem would follow easily from the last conjecture: Let $G = (A_1, A_2)$ be a complete bipartite graph where A_1, A_2 are identical copies of A and the color of the edge (x, y) is $x - y$ where $x \in A_1$. Then the edges of G are properly colored with $2cn$ colors. If Conjecture 2 were true, with $n > n_1(2k, 2c)$ an alternating 2 -colored path of length $2k$ would guarantee an arithmetic progression of size k.

Here we give a coloring disproving the complete graph version of Conjecture 2 for $k > 3$, which can be easily applied to the bipartite case.

Example 1 *Let $2^{m-1} < n \leq 2^m$ for some m. Label the vertices of the complete graph K_{2^m} by the 0-1 vectors of length m. Color the edges by $2^m - 1$ colors as follows. The color of edge (x, y) is the 0-1 vector $x + y \pmod 2$. Consider K_n as a subgraph of K_{2^m}.*

It is easy to see that in the example the union of any $t > 1$ colors consists of disjoint components of at most 2^t vertices. Also no open path can consist of edges colored (in this order) $a, b, c, \ldots, x, a, b, c, \ldots, x$ since such a sequence must always return to the starting point (i.e., it is a closed walk) by the mod 2 property . Therefore this example contradicts the second but not the first conjecture. In the special case of $n = 2^m$ this example uses $n-1$ colors, proper coloring is a 1-factorization and if there is no 2-colored path with 4 edges then this coloring is unique up to isomorphism [4].

Closely related to the topic of this note is the following question raised by P. Erdős and A. Gyárfás [5] : Is it possible to have a proper edge coloring of K_n with cn colors so that the union of any two color classes has no paths or cycles with 4 edges? M. Axenovich [1] has an example showing that it is possible with $2n^{1+c/\sqrt{logn}}$ colors.

The above bipartite graph version of Szemerédi's theorem has no 2-colored cycles at all. This suggests the following modification of Conjecture 2:

Conjecture 3 *Let (A, B) be a complete bipartite graph with $|A| = |B| = n$, $n > n_3(k, c)$ and the edges are properly colored with cn colors so that the union of any two color classes does not contain a cycle. Then there is an alternating 2-colored path of length k.*

Conjecture 3 and the Erdős-Gyárfás problem are closely related. For $k = 4$ the two are equivalent. For $k > 4$ a positive answer to either of them would mean a negative answer for the other but a negative answer for either of them would not solve the other (where a positive answer for Conjecture 3 means that it is true).

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