Nonexistence of graphs with cyclic defect

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Abstract

In this note we consider graphs of maximum degree Δ, diameter D and order $\mathrm{M}(\Delta, D) - 2$, where $\mathrm{M}(\Delta, D)$ is the *Moore bound*, that is, graphs of *defect* 2. In [1] Delorme and Pineda-Villavicencio conjectured that such graphs do not exist for $D \geq 3$ if they have the so called 'cyclic defect'. Here we prove that this conjecture holds.

Keywords: Graphs with cyclic defect, Moore bound, defect, repeat.

1 Nonexistence of graphs with cyclic defect

Let G be a graph of maximum degree Δ, diameter D and order $\mathrm{M}(\Delta, D) - 2$, where $\mathrm{M}(\Delta, D) = 1 + \Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 + \cdots + \Delta(\Delta - 1)^{D-1}$ is the *Moore bound*, that is, graphs of *defect* 2. In such a graph G any vertex v can reach within D steps either two vertices (called *repeats* of v) in two different ways each, or one vertex (called *double repeat* of v) in three different ways; all the other vertices of G are reached from v in at most D steps in exactly one way. The *repeat (multi)graph* of G, $R(G)$, consists of the vertex set $V(G)$ and there is an edge $\{u, v\}$ in $R(G)$ if and only if v is a repeat of u (and vice versa) in G. Clearly, when defect is 2, $R(G)$ is either one cycle of length $n = |V(G)|$ or a disjoint union of cycles whose sum of lengths is equal to n. If $R(G)$ is cycle of length n then we say that G has *cyclic defect*. Interest in such graphs is part of the general study of the

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degree/diameter problem. For a survey of this problem, see [4]. Graphs with cyclic defect were first studied by Fajtlowicz [2] who proved that when $D = 2$ the only graph with cyclic defect is the Mobius ladder on 8 vertices (with $\Delta = 3$). Subsequently, for $D \geq 3$, Delorme and Pineda-Villavicencio [1] proposed several ingenious algebraic techniques for dealing with graphs with cyclic defect and they proved the nonexistence of such graphs for many values of D and Δ. They conjectured that graphs with cyclic defect do not exist for $D \geq 3$. In this paper we use structural properties of graphs with cyclic defect to prove that this conjecture holds.

Observation 1.1 *For $\delta < 1 + (\Delta - 1) + (\Delta - 1)^2 + \ldots + (\Delta - 1)^{D-1}$, $\Delta \geq 3$ and $D \geq 2$, a graph of defect δ must be regular.*

It is also easy to see that there are no graphs with cyclic defect of degree $\Delta = 2$. Therefore, from now on we assume G to be a Δ-regular graph with cyclic defect, degree $\Delta \geq 3$, and diameter $D \geq 3$.

We say that $S \subset V(G)$ is a *closed set of repeats* if for every vertex of S none of its repeats is outside of S. Clearly, a graph with cyclic defect cannot contain a closed set of repeats that is of cardinality less that $|V(G)|$.

We denote by Θ_D the union of three independent paths of length D with common endvertices. Since the $3D - 1$ vertices of Θ_D comprise a closed set of repeats, while G contains $\Delta(1 + (\Delta - 1) + (\Delta - 1)^2 + \cdots + (\Delta - 1)^{D-1}) - 1$ vertices, we have

Observation 1.2 *Graph with cyclic defect does not contain Θ_D.*

Suppose G contains a cycle C of length $2D - m$, $m > 1$. Then for every vertex v on C, there are more than 2 vertices on C that are repeats of v. Since each vertex has at most two distinct repeats, we have immediately that $m \leq 1$. Moreover, if $m = 1$ then C is a closed set of repeats consisting of $2D - 1$ vertices, while G contains $\Delta(1 + (\Delta - 1) + (\Delta - 1)^2 + \cdots + (\Delta - 1)^{D-1}) - 1$ vertices, a contradiction for every $\Delta \geq 3$. Therefore, we have

Observation 1.3 *Graph with cyclic defect does not contain a cycle of length less than $2D$.*

This means that the girth of G is $2D$, and every vertex v is contained in exactly two $2D$-cycles, and no other cycle of length at most $2D$.

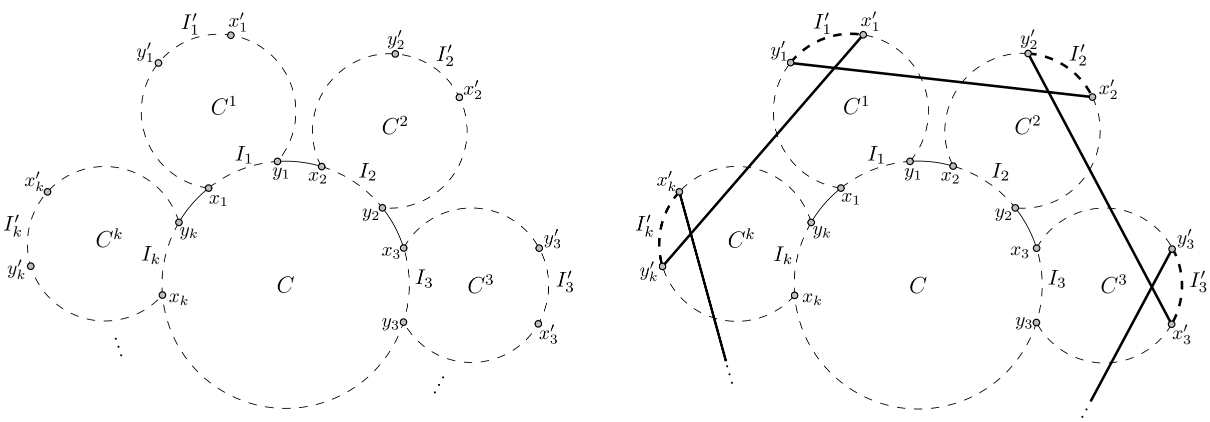
Let S be a set of vertices in G and H a subgraph of G. We denote by $S' = \mathrm{rep}^H(S)$ the set of repeats of S that occur in H. Furthermore, two $2D$-cycles C^1 and C^2 are called *neighbouring cycles* if they have non-empty intersection. The following lemma was proved in [3]; it will be used to prove the main result of this paper.

Lemma 1.1 (Repeat Cycle Lemma) *[3] Let G be a graph with $D \geq 4$ and $D \geq 2$, and defect 2. Let C be a $2D$-cycle in G. Let $\{C^1, C^2, \ldots, C^k\}$ be the set of neighbouring cycles of C, and $I_i = C^i \cap C$ for $1 \leq i \leq k$. Suppose at least one I_j, for $j \in \{1, \ldots, k\}$, is a path of length smaller than $D-1$. Then, there is an additional $2D$-cycle C' in G, called* repeat cycle, *intersecting C^i at $I'_i = \mathrm{rep}^{C^i}(I_i)$, where $1 \leq i \leq k$.*

For an illustration, see Fig. 1

Corollary 1.1 *If C and C' are repeat cycles of each other then they comprise a closed set of $4D$ repeats.*

Proof. Consider an arbitrary vertex $x \in C \cap I_i,\ i \in 1, \ldots, k$. The vertex x has two repeats: one of them is the vertex on C that is at distance D from x. The second repeat of x is on the intersection of the repeat cycle C' and I'_i. Since C and C' are repeat cycles of each other, we have $R(C) = C \cup C' = R(C')$ and so $C \cup C'$ is a closed set of repeats. □



(a) (b)

Figure 1: Illustration for Lemma 1.1 [3].

We are now ready to prove the main result.

Theorem 1.1 *Graphs with cyclic defect do not exist for $\Delta \geq 3$ and $D \geq 3$.*

Proof. Let G be a graph with cyclic defect. Let C be a cycle of length $2D$ in G. We need to consider two cases.

Case 1. There exist two $2D$-cycles, say C_1 and C_2, with intersection that is a path of length smaller than $D-1$. Then, by Corollary 1.1, cycle C_1 has a repeat cycle C'_1 and the two cycles C_1 and C'_1 comprise a closed set of $4D$ repeats, a contradiction since G is a graph with cyclic defect and $\Delta(1 + (\Delta-1) + (\Delta-1)^2 + \cdots + (\Delta-1)^{D-1}) - 1$ vertices.

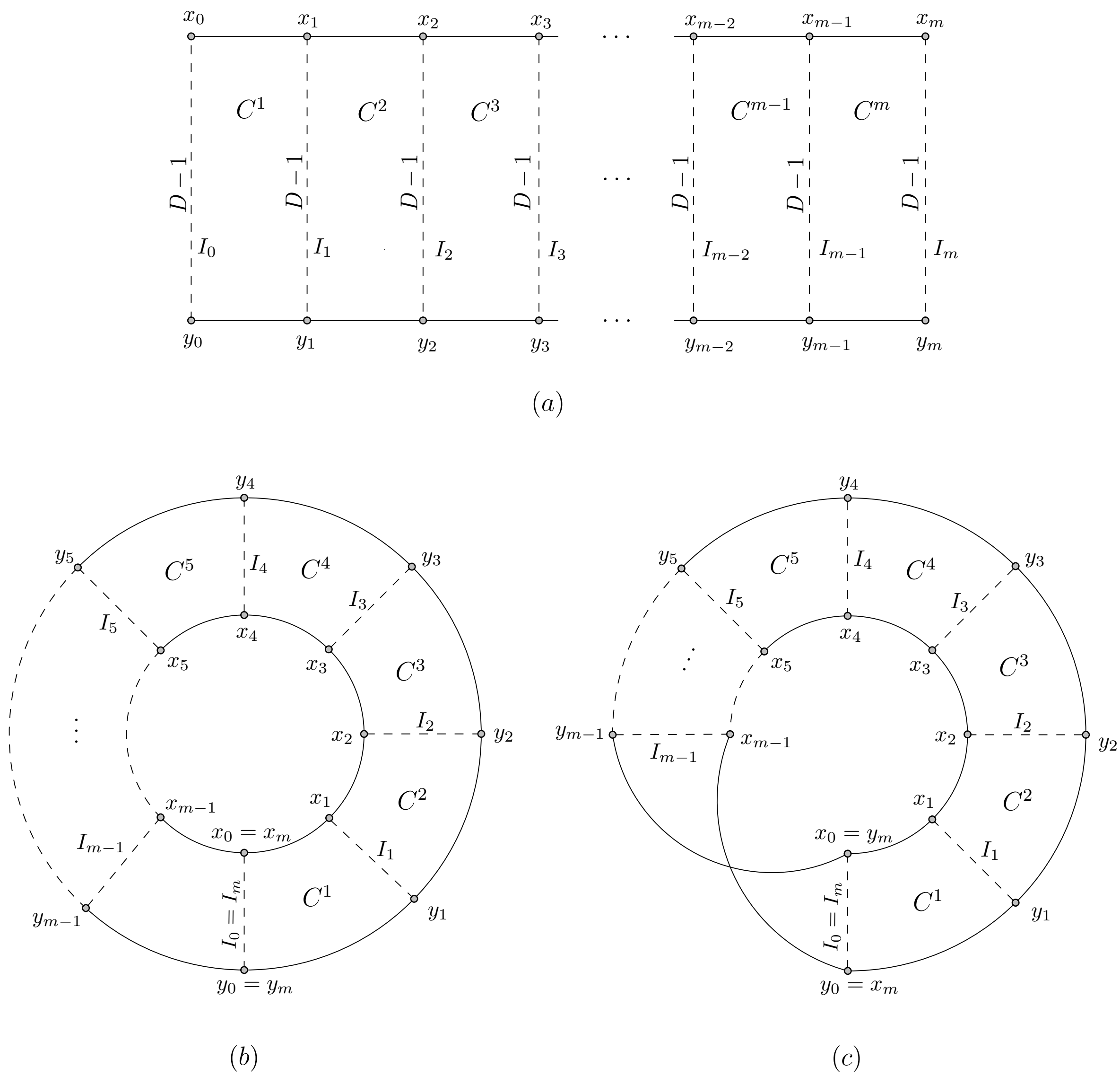


Figure 2: Illustration for Case 2 of the proof of Theorem 1.1 [3].

Case 2. There do not exist two cycles with intersection that is a path of length smaller than $D-1$. That is, any two $2D$-cycles have either empty intersection or they intersect in a path of length exactly $D-1$. Recall that the length of the path cannot be more since there are no Θ_D. Then G contains as a subgraph a succession of $2D$-cycles C_m, C_1, $C_2, \ldots, C_{m-1}$ such that any two consecutive cycles have intersection a path of length $D-1$ (that is, they share D vertices). Assume that the value of m is maximum possible. Refer to Fig 2(a). Since G is finite, C_1 and C_m must also intersect in a path of length $D-1$.

There are two possibilities, depicted in Fig. 2(b) and (c). Clearly, in the first case the vertices $x_1, x_2, \ldots, x_m$ form a closed set of repeats for any $\Delta \geq 3$, and this set does not include the vertices $y_1, y_2, \ldots, y_m$ so that G does not have cyclic defect.

In the second case, for any $\Delta \geq 3$, the vertices $x_1, x_2, \ldots, x_m$ and the vertices $y_1, y_2, \ldots, y_m$ together form a closed set of repeats consisting of $2m$ vertices which however does not include all the vertices of G if $D \geq 3$, a contradiction.

References

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