

List-Distinguishing Colorings of Graphs

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Abstract

A coloring of the vertices of a graph G is said to be *distinguishing* provided that no nontrivial automorphism of G preserves all of the vertex colors. The *distinguishing number* of G , denoted $D(G)$, is the minimum number of colors in a distinguishing coloring of G . The distinguishing number, first introduced by Albertson and Collins in 1996, has been widely studied and a number of interesting results exist throughout the literature.

In this paper, the notion of distinguishing colorings is extended to that of list-distinguishing colorings. Given a family $L = \{L(v)\}_{v \in V(G)}$ of lists assigning available colors to the vertices of G , we say that G is L -distinguishable if there is a distinguishing coloring f of G such that $f(v) \in L(v)$ for all v . The *list-distinguishing number* of G , $D_\ell(G)$, is the minimum integer k such that G is L -distinguishable for any assignment L of lists with $|L(v)| = k$ for all v . Here, we determine the list-distinguishing number for several families of graphs and highlight a number of distinctions between the problems of distinguishing and list-distinguishing a graph.

Keywords: Distinguishing Coloring, List Coloring, List-Distinguishing Coloring

1 Introduction

A vertex coloring of a graph G , $f : V(G) \rightarrow \{1, \dots, r\}$ is said to be r -*distinguishing* if no nontrivial automorphism of the graph preserves all of the vertex colors. The *distinguishing*

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number of a graph G , denoted $D(G)$, is the minimum integer r such that G has a r -distinguishing coloring and was first introduced by Albertson and Collins in [AC96].

In this paper we introduce the list-coloring analogue to the distinguishing problem. Given a family $L = \{L(v)\}_{v \in V(G)}$ of lists assigning available colors to the vertices of G , we say that G is L -distinguishable if there is a distinguishing coloring f of G such that $f(v) \in L(v)$ for all v . The *list-distinguishing number* of G , written $D_\ell(G)$, is the smallest positive integer k such that G is L -distinguishable for any assignment L of lists with $|L(v)| = k$ for all v . Since all of the lists can be identical, we observe that $D(G) \leq D_\ell(G)$.

In some cases, the list-distinguishing number can easily be shown to equal the distinguishing number. For example, it is not difficult to see that $D(K_n) = n = D_\ell(K_n)$ and $D(K_{n,n}) = n + 1 = D_\ell(K_{n,n})$. In other cases, determining $D_\ell(G)$ is not nearly as simple, as the techniques needed to k -list-distinguish a graph can be significantly different from those used to distinguish a graph. This is especially apparent when considering those graphs G with distinguishing number exactly 2. Such a graph G necessarily has at least one nontrivial automorphism, and hence must have distinguishing number at least two. All that remains then is to demonstrate a 2-distinguishing coloring of G , which, to be clear, is often a highly nontrivial task. However, when one attempts to 2-list-distinguish the same G , it is not sufficient to simply demonstrate a coloring, as the lists assigned to $V(G)$ may be highly disparate in nature.

As an example, consider the problem of distinguishing C_n for $n \geq 6$, which was shown in [AC96] to be 2. The automorphism group of C_n is nontrivial, so clearly $D(C_n) \geq 2$, and the coloring in Figure 1, which generalizes to a coloring of C_n , suffices to complete the proof.

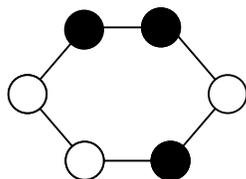


Figure 1: A distinguishing coloring of C_6 .

The next proposition gives some indication of the increased difficulty one might encounter when considering list-distinguishing colorings.

Proposition 1. *For $n \geq 6$, $D_\ell(C_n) = D(C_n) = 2$.*

Proof. Assign a list $L(v)$ of two colors to each $v \in V(C_n)$. If $|\bigcup L(v)| = 2$, then the lists are identical, so we can color the vertices in a manner identical to the traditional distinguishing coloring, and we are done. Assume then that $|\bigcup L(v)| \neq 2$ and let c be the color that appears in the fewest lists. Choose a vertex with c in its list, label it v_1 , and then continue labeling the vertices consecutively in numerical order. Color v_1 with c . Moving forward, no other vertex will receive color c , so that v_1 will be fixed by every color-preserving automorphism. Since c appears in the fewest lists, there are at least $\frac{n}{3}$

vertices that do not have a c in their list. Choose a vertex, v_i , such that $c \notin L(v_i)$ and is not antipodal to v_1 . Consider the vertex v_{n+2-i} , which is the image of v_i under the unique non-identity automorphism that fixes v_1 . There exists $x \in L(v_{n+2-i})$ such that $x \neq c$; color v_{n+2-i} with x . Now there is a $y \in L(v_i)$ such that $y \neq x$ and we know that $y \neq c$, so color v_i with y . Color the remaining vertices with any element from their list that is not c , as in Figure 2. Since any non-identity color-preserving automorphism must map v_1 to v_1 , v_i must map to v_{n+2-i} , but they are colored differently. Therefore the coloring is list-distinguishing, and $D_\ell(C_n) = D(C_n) = 2$.

□

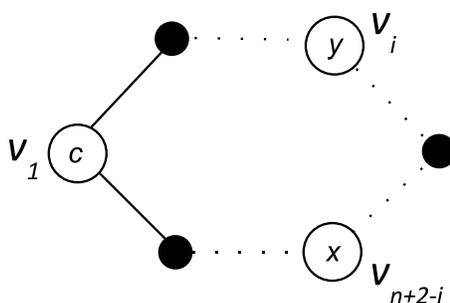


Figure 2: A 2-list-distinguishing coloring of C_n . The color c appears only on v_1 , and the colors x and y are distinct.

While Proposition 1 indicates how different the problems of determining $D(G)$ and $D_\ell(G)$ may be, at times it is possible to utilize existing techniques to obtain results on D_ℓ . For instance, Brooks-type theorems for the (traditional) distinguishing number were given independently by Klavžar, Wong, and Zhu [KWZ06] and Collins and Trenk [CT06]. We are able to modify the approach from [KWZ06] to give a Brooks-type theorem for list-distinguishing colorings. Interestingly, we are able to show that the traditional distinguishing problem is precisely what prevents the exceptional graphs K_n , $K_{n,n}$ and C_5 from being Δ -list-distinguishable.

Proposition 2. *Let G be a connected graph and let $\mathcal{L} = \{L(v)\}$ be an assignment of lists of size $\Delta(G)$ to $V(G)$. Then G can be \mathcal{L} -distinguished unless G is one of K_n , $K_{n,n}$, or C_5 and $|\bigcup L(v)| = \Delta(G)$. In these exceptional cases, G can be colored from any assignment of lists of length $\Delta(G) + 1$.*

Our proof relies on the following lemma from [KWZ06].

Lemma 1. *Suppose (G, ℓ) is a connected, vertex-colored graph such that $\ell(v)$ is the color for $v \in V(G)$. Let every vertex of the set $X \subseteq V(G)$ be fixed by every automorphism of (G, ℓ) . Let $x \in X$ and set $S = N_G(x) \setminus X$. If $\ell(u) \neq \ell(v)$ for any pair of distinct vertices u and v in S , then every vertex of S is fixed by every automorphism of (G, ℓ) .*

Proof. (of Proposition 2) Assign list $L(v)$ to each vertex $v \in V(G)$ such that $|L(v)| = \Delta(G)$, and assume $|\bigcup L(v)| \neq \Delta(G)$. Since G is connected, there exist two vertices, x and y , such that $L(x) \neq L(y)$ and $xy \in E(G)$. Let $c_x \in L(x) - L(y)$, and color x with c_x . Going forward, no other vertex but x will receive color c_x , assuring that x will be fixed by every color-preserving automorphism of G . Construct a breadth first search spanning tree of G rooted at x . Since $|N_G(x)| \leq \Delta(G)$, we can color each vertex $w \in N_G(x) - y$ with a unique color, $c_w \in L(w)$, such that $c_w \neq c_x$. Since $|L(y)| = \Delta(G)$ and $c_x \notin L(y)$, there exists a color, $c_y \in L(y)$ such that $c_y \neq c_w$ for all $w \in N_G(x) - y$; color y with c_y . From here each vertex has at most $\Delta(G) - 1$ children in the spanning tree. Therefore we can color each sibling of the spanning tree uniquely from its list, never using the color c_x . By an inductive application of Lemma 1, the coloring so constructed is a list-distinguishing coloring of G .

It remains to consider when $|\bigcup L(v)| = \Delta(G)$, which is the same as distinguishing coloring. Therefore from [CT06] and [KWZ06], we have that if G is K_n , $K_{n,n}$, or C_5 then $D_\ell(G) = \Delta(G) + 1$ and if not then $D_\ell(G) = \Delta(G)$. \square

Proposition 2 immediately yields the following Brooks-type result.

Theorem 1. *Let G be a graph with maximum degree Δ . Then $D_\ell(G) \leq \Delta$ unless G is one of C_5, K_n or $K_{n,n}$, in which case $D_\ell(G) = \Delta + 1$.*

2 Dihedral Groups

Let G be a graph and let Γ be a group. If $Aut(G)$, the automorphism group of G , is isomorphic to Γ then we will say that G realizes Γ . Given a group element $g \in Aut(G)$ and a vertex v in G , we let vg denote the result of the action of g on v . For a vertex v in G , we let $St(v) = \{h \in \Gamma \mid vh = v\}$ and $O(v) = \{b \in V(G) \mid v = bh \text{ for some } h \in \Gamma\}$ be the *stabilizer* and *orbit* of v under the action of Γ on G . The order of the orbit of v under the action of Γ on G is $|Aut(G)|/|St(v)|$. Also, we let $N_G(x)$ denote the set of vertices adjacent to x .

We let D_n denote the dihedral group of order $2n$, which is the group of symmetries of a regular n -gon. Throughout this section, we use the standard presentation $D_n = \langle \sigma_n, \tau_n \mid \sigma_n^{2n} = \tau_n^2 = e, \sigma_n \tau_n = \tau_n \sigma_n^{-1} \rangle$ where σ_n and τ_n denote the appropriate rotation and reflection of the n -gon, respectively and e denotes the identity element. We will frequently write $\tau_n = \tau$ and $\sigma_n = \sigma$ if the context is clear. When there is no danger of ambiguity, we will also let e denote the identity element of an arbitrary group Γ .

In this section, we study the list-distinguishing number of graphs realizing D_n for some $n \geq 3$. It is clear that C_n realizes D_n , but there are many other graphs that realize the dihedral group (see Figure 3).

Albertson and Collins [AC96] proved that if G realizes D_n then $D(G) = 2$ unless $n = \{3, 4, 5, 6, 10\}$ in which case $D(G)$ is either 2 or 3. The main result of this section is as follows.

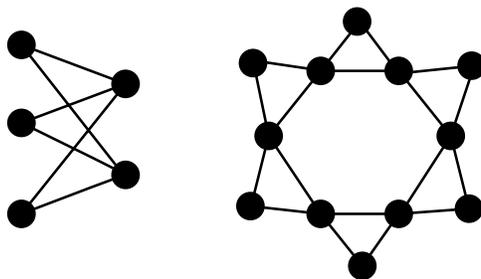


Figure 3: Two graphs that realize D_6 .

Theorem 2. *Let G be a graph realizing D_n . Then $D(G) = D_\ell(G)$.*

One interesting, and useful, consequence of our proof of Theorem 2 is that we determine precisely those graphs that realize D_n and have (traditional) distinguishing number 3.

We also point out at this time that the lemmata developed here to prove Theorem 2 appear nearly identical to those utilized to prove the corresponding theorem in [AC96]. However, the techniques used here frequently vary greatly from those in [AC96], illustrating further the distinctions between the distinguishing and list-distinguishing numbers.

Lemma 2. *Let G realize group Γ and suppose u_1, \dots, u_t are vertices from different vertex orbits of G . If $\bigcap_{i=1}^t St(u_i) = \{e\}$, then $D_\ell(G) = 2$.*

Proof. For each u_i , select a color $c_i \in L(u_i)$ and then color each other vertex in $O(u_i)$ with any color other than c_i . Let g be any nonidentity element in Γ . Since $\bigcap_{i=1}^t St(u_i) = \{e\}$, at least one u_i is not fixed by G . Since $u_i g$ is not colored c_i , this is a 2-list-distinguishing coloring. \square

The following lemmas, the first of which appears in [AC96], will be useful as we proceed.

Lemma 3. *Let G realize D_n , and suppose that G has t vertex orbits. If u_1, \dots, u_t are vertices from the t different vertex orbits of G , then*

$$\langle \sigma \rangle \cap St(u_1) \cap \dots \cap St(u_t) = \{e\}.$$

Lemma 4. *Let G realize D_n . If there is a vertex u in G such that $St(u) = \langle \sigma^j \rangle$ then $D_\ell(G) = 2$.*

Proof. The proof of this lemma is identical to the proof of the corresponding lemma in [AC96]. Let u_1, \dots, u_t be vertices from each of the different orbits of G . Then $\bigcap_i St(u_i) \subseteq St(u)$, which implies that $\bigcap_i St(u_i) = \{e\}$, completing the proof by Lemma 2. \square

At this point we begin to more seriously modify the techniques from [AC96] in order to better fit our list-coloring framework.

Lemma 5. *Let G realize D_n and let u be a vertex in G such that $St(u)$ is either $\langle \sigma^j, \tau \sigma^i \rangle$ or $\langle \tau \sigma^i \rangle$. If $|O(u)| \geq 6$, then $O(u)$ is 2-list-distinguishable.*

Proof. Regardless of whether $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$ or $St(u) = \langle \tau\sigma^i \rangle$, we have that $O(u) = \{u, u\sigma, u\sigma^2, \dots, u\sigma^{j-1}\}$ for some j , and hence we assume that $j \geq 6$. Consider the set $A = \{u, u\sigma^2, u\sigma^3\}$ and select a color $c_u \in L(u)$. If possible, color $u\sigma^2$ and $u\sigma^3$ with c_u as well. We will demonstrate that it is possible to extend this to a 2-distinguishing coloring without using c_u on any vertex in $V(G) - A$.

Case 1: All three vertices in A are colored with c_u .

We proceed by coloring the vertices in $V(G) - A$ using any color in their respective lists except c_u . Any automorphism g in D_n that fixes this coloring of $O(u)$ must permute the vertices in A , and specifically must map u to some element of A . Therefore g must lie in $St(u)$, $St(u)\sigma^2$ or $St(u)\sigma^3$. As was demonstrated in [AC96], the only automorphisms from these sets that permute A actually fix all of $O(u)$. Thus this is a 2-list-distinguishing coloring of $O(u)$.

Case 2: The vertex u is the only one in A colored with c_u .

Suppose that $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$. Then j divides n , so the assumption that $j \geq 6$ implies that $u\sigma^2 \neq u\sigma^{n-2}$, which also holds when $St(u) = \langle \tau\sigma^i \rangle$. In either case, we extend the coloring of A by first assigning the vertex $u\sigma^{n-2}$ any color $c \neq c_u$ in $L(u\sigma^{n-2})$. Since we utilize the color c_u on the vertices in A wherever possible, and u is the only vertex of G that receives color c_u , we conclude that $c_u \notin L(u\sigma^2)$. Therefore there is a color c' in $L(u\sigma^2)$, different from both c and c_u . Color $u\sigma^2$ using color c' and color the remaining vertices of G with any color aside from c_u in their respective lists.

We now show that this coloring distinguishes $O(u)$. Since u is the unique vertex of color c_u , any color-preserving automorphism $g \in D_n$ must lie in $St(u)$. Then either $g = \sigma^{dj}$ or $g = \tau\sigma^{i+dj}$, with d being necessarily zero when $St(u) = \langle \tau\sigma^i \rangle$. Note that σ^{dj} fixes $O(u)$ and that, for any d , $\tau\sigma^{i+dj}$ takes $u\sigma^2$ to $u\sigma^{n-2}$. Since we have constructed our coloring so that $u\sigma^2$ and $u\sigma^{n-2}$ have different colors, this is a 2-list-distinguishing coloring of $O(u)$.

Case 3: The vertices u and $u\sigma^2$ are the only ones in A colored with c_u .

As above, if $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$, then j divides n . Consequently, the assumption that $j \geq 6$ implies that $u\sigma^3 \neq u\sigma^{n-1}$, which again also holds when $St(u) = \langle \tau\sigma^i \rangle$. In this case, we extend the coloring of u and $u\sigma^2$ in a similar manner to Case 1, with two exceptions. We make no special color assignment to $u\sigma^{n-2}$, save the standard assumption that it does not receive color c_u . Instead, we assign different colors to the vertices $u\sigma^3$ and $u\sigma^{n-1}$ such that neither vertex is colored with c_u . As above, this is possible since $c_u \notin L(u\sigma^3)$, or else we would have used it to color $u\sigma^3$.

Now, every automorphism $g \in D_n$ that preserves the colors on $O(u)$ must either fix or interchange u and $u\sigma^2$. As discussed in the previous case, any element of $St(u)$ will either fix all of $O(u)$ or will map $u\sigma^2$ to $u\sigma^{n-2}$. Thus as $u\sigma^{n-2}$ is not colored with c_u , any color preserving automorphism g must interchange u and $u\sigma^2$. This implies that g has the form $g'\sigma^2$, where g' is an element of $St(u)$. Specifically, either $g' = \sigma^{dj}$ or $g' = \tau\sigma^{i+dj}$ for some integer d , with $d = 0$ if $St(u) = \langle \tau\sigma^i \rangle$. In either case, $g'\sigma^2$ takes $u\sigma^3$ to $u\sigma^{n-1}$, implying that this coloring 2-list-distinguishes $O(u)$.

Case 4: The vertices u and $u\sigma^3$ are the only ones in A colored with c_u .

Suppose first that $St(u) = \langle \tau\sigma^i \rangle$. We wish, as above, to extend our coloring of A to a list-distinguishing coloring of $O(u)$ in which no vertices aside from u and $u\sigma^3$ receive color c_u . Any element $g \in D_n$ that fixes such a coloring must either stabilize u or exchange u and $u\sigma^3$. Consequently, either $g = \tau\sigma^i$ or $g = \tau\sigma^{i+3}$, so consider the outcomes when these elements are applied to A :

$$\begin{aligned} A\tau\sigma^i &= \{u, u\sigma^{n-2}, u\sigma^{n-3}\} \\ A\tau\sigma^{i+3} &= \{u\sigma^3, u\sigma, u\}. \end{aligned}$$

Therefore, we would like to choose colors $x \in L(u\sigma)$ and $y \in L(u\sigma^{n-2})$, $x, y \neq c_u$ and choose a color $z \neq x, y$ from $L(u\sigma^2)$. We would then extend our coloring of A by assigning these colors to their respective vertices and then coloring the remaining vertices in G using any color except c_u . This is possible unless all of the following hold: $L(u\sigma) = \{c_u, x\}$, $L(u\sigma^{n-2}) = \{c_u, y\}$ and $L(u\sigma^2) = \{x, y\}$. In this case, suppose $L(u) = \{c_u, c'_u\}$. We will then recolor u with c'_u and again color $u\sigma^2$ and $u\sigma^3$ with these colors if possible. Then, the analysis conducted thus far assures that we can construct a 2-list-distinguishing coloring of $O(u)$, as we cannot have $L(u\sigma) = \{c'_u, x\}$, $L(u\sigma^{n-2}) = \{c'_u, y\}$ and $L(u\sigma^2) = \{x, y\}$ for $x = y$, and this was the only obstacle preventing us from constructing the desired coloring when $St(u) = \langle \tau\sigma^i \rangle$.

Hence we assume that $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$. In this case, j divides n , and we may assume that $n > j \geq 6$. Again, we wish to extend our coloring of A to a distinguishing coloring of G in which no vertices aside from u and $u\sigma^3$ receive color c_u . If $g \in D_n$ fixes such a coloring then either $g \in St(u)$ or $g = g'\sigma^3$ for some $g' \in St(u)$. Thus for some $d \geq 0$, $g = \sigma^{dj}$, either σ^{dj+3} , $\tau\sigma^{i+dj}$ or $\tau\sigma^{i+dj+3}$. Applying each of these elements to A , we obtain

$$\begin{aligned} A\sigma^{dj} &= A \\ A\sigma^{dj+3} &= \{u\sigma^3, u\sigma^5, u\sigma^6\} \\ A\tau\sigma^{dj+i} &= \{u, u\sigma^{n-2}, u\sigma^{n-3}\} \\ A\tau\sigma^{dj+i+3} &= \{u\sigma^3, u\sigma, u\}. \end{aligned}$$

Since j divides n and n is at least seven, $u\sigma^{n-3} \neq u\sigma^3$. Therefore $g = \tau\sigma^{dj+i}$ cannot preserve our proposed coloring. Additionally, σ^{dj} fixes all of $O(u)$, so we need only consider the cases where $g = \sigma^{dj+3}$ or $g = \tau\sigma^{dj+i+3}$.

If $j > 6$, then since σ^{dj+3} takes $u\sigma^3$ to $u\sigma^6$ this choice of G cannot distinguish u . Thus when $j > 6$ it suffices to assume that $g = \tau\sigma^{i+dj}$ and thus we need only to distinguish $u\sigma$ from $u\sigma^2$ without assigning c_u to $u\sigma$, which is clearly possible as we have assumed that $c_u \notin L(u\sigma^2)$.

Thus we assume that $j = 6$ and that either $g = \sigma^{dj}$ or $g = \tau\sigma^{i+dj+3}$. We would like to choose colors $x \in L(u\sigma)$ and $y \in L(u\sigma^5)$, $x, y \neq c_u$ and choose a color $z \neq x, y$ from $L(u\sigma^2)$. We would then extend our coloring of A by assigning these colors to their respective vertices and then coloring the remaining vertices in G using any color except c_u . As above, this is possible unless $L(u\sigma) = \{c_u, x\}$, $L(u\sigma^5) = \{c_u, y\}$ and $L(u\sigma^2) = \{x, y\}$. However, in this case we proceed by changing our initial coloring of A by using the color

$c'_u \neq c_u$ in $L(u)$. As when $St(u) = \langle \tau\sigma^i \rangle$, this allows us to construct a coloring that will distinguish $O(u)$, and therefore completes the proof. \square

The following lemma is a slight modification of the corresponding lemma in [AC96].

Lemma 6. *Let G realize D_n and let u be a vertex such that $u \in V(G)$ and $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$ or $\langle \tau\sigma^i \rangle$. If $|O(u)| \geq 6$, then $D_\ell(G) = 2$.*

Proof. Assign lists of length two to each vertex in G . If $St(u) = \langle \tau\sigma^i \rangle$, then the intersection of the subgroups conjugate to $St(u)$ is the identity. Applying Lemma 5, $O(u)$ is 2-list-distinguishable and thus by Lemma 2, $D_\ell(G) = 2$.

Therefore, assume $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$. Since $O(u)$ is 2-list-distinguishable, we need only consider the automorphisms that act trivially on $O(u)$. These are the intersection of the stabilizers of vertices of $O(u)$, which is $\Lambda = \langle \sigma^j \rangle$. The group action of Λ on G creates vertex orbits U_1, U_2, \dots, U_t . From each orbit U_i such that $|U_i| > 1$, select a vertex u_i and color it with any color $c_i \in L(u_i)$. Then color the remaining vertices of U_i with any color other than c_i , construct a distinguishing coloring of $O(u)$ from its assigned lists and color all other uncolored vertices in G with any color from their respective lists. If a nontrivial automorphism in Λ fixes u_i , then it must fix all of U_i . Thus each $g \neq e$ in Λ must move some u_i to another vertex in its orbit, implying that the only color preserving automorphism is the identity. Consequently, this is a 2-list-distinguishing coloring of G , so $D_\ell(G) = 2$. \square

Lemmas 2–5 provide the necessary machinery to complete the proof of Theorem 2, and we do so now.

Proof. (of Theorem 2) When $n > 10$ there must be an orbit of order at least 6, so it remains to show that the theorem holds when $n = 3, 4, 5, 6$ or 10. Suppose first that G realizes D_n , where $n = 3$ or 5 and select a vertex u with nontrivial orbit in G . By the above lemmas, we may assume that every vertex u in G has $St(u) = \langle \tau\sigma^i \rangle$ or $\langle \sigma^j, \tau\sigma^i \rangle$. However, as n is prime, if σ^j fixes u , then $j = 0$ or $j = 1$. Consequently, we may assume that $j = 0$, as $\langle \sigma, \tau\sigma^i \rangle = D_n$, and by assumption u has a nontrivial orbit.

Note as well that if $\tau\sigma^i$ is in $St(u)$, then τ is in $St(u\sigma^x)$ where $2x \equiv n - i \pmod{n}$. Such an x exists for all i when $n = 3$ or 5, so we may assume that $St(u) = \langle \tau \rangle$. Then, since the orbit of u is by assumption nontrivial, $O(u) = \{u, u\sigma, \dots, u\sigma^{n-1}\}$ and the only element of D_n that fixes all of $O(u)$ is e .

If G has exactly one nontrivial orbit, then necessarily $D(G) = 3$, and since this orbit is precisely $O(u) = \{u, u\sigma, \dots, u\sigma^{n-1}\}$, it is straightforward to show that $D_\ell(G) = 3$ as well. Suppose then that G has vertices u and v with distinct nontrivial orbits and that we have also assigned lists of length 2 to the vertices of G . We may furthermore assume, via the above discussion, that $St(u) = St(v) = \langle \tau \rangle$. If it is not possible to color each vertex in $O(u)$ with a unique color, then each vertex in $O(v)$ must have list $L(u\sigma^i) = \{c_1, c_2\}$. We then color u with color c_1 and color both $u\sigma$ and $u\sigma^2$ with color c_2 .

If g , a nonidentity element of D_n , fixes this coloring of $O(u)$, then $g \in St(u)$. However, since $O(v)$ is nontrivial, g must exchange two elements in $O(v)$, say $v\sigma^i$ and $v\sigma^j$. Assigning these vertices distinct colors from their respective lists serves to 2-list-distinguish G , and thus implies that $D(G) = D_\ell(G) = 2$.

Next, let G be a graph that realizes D_4 and furthermore assume that there is no vertex u in G such that $St(u) = \langle \sigma^j \rangle$ or $|O(u)| \geq 6$. Lemmas 2 and 3 imply that if there are vertices u and v in distinct orbits of G such that $\tau\sigma^i \in St(u)$ but $\tau\sigma^i \notin St(v)$, then $D(G) = D_\ell(G) = 2$.

Suppose that τ stabilizes some element in every orbit of G (the case where some other $\tau\sigma^i$ stabilizes an element in every orbit is handled similarly). Under our assumptions, every nontrivial orbit in G must have order either two or four. If every nontrivial orbit has two elements, then τ stabilizes every vertex in G , a contradiction. Therefore, there is some vertex u such that $St(u) = \langle \tau \rangle$ and $O(u) = \{u, u\sigma, u\sigma^2, u\sigma^3\}$. If possible, color the orbit of u in a manner consistent with the traditional distinguishing coloring of C_4 , specifically, for some i , color $u\sigma^i$ and $u\sigma^{i+1}$ with the same color c and then color the other two vertices in $O(u)$ with distinct colors other than c . Since no element of D_4 fixes all of $O(u)$, this would suffice to 2-list-distinguish G and also shows that $D_\ell(G) \leq 3$. Thus we may assume that $L(u\sigma^i) = \{c_1, c_2\}$ for all i implying that if $O(u)$ is the only nontrivial orbit of G , $D(G) = D_\ell(G) = 3$.

Next, assume that there is some vertex v , not in $O(u)$, such that $O(v)$ is nontrivial and assign lists of length two to each vertex in v . We claim that it is possible to construct a distinguishing coloring of v from these lists. If $|O(v)| = 4$, then $O(v) = \{v, v\sigma, v\sigma^2, v\sigma^3\}$ and we may assume that $L(v\sigma^i) = \{c'_1, c'_2\}$ for all i . Without loss of generality, suppose that τ stabilizes v (and therefore $v\sigma^2$). Color u and $u\sigma$ with color c_1 , $u\sigma^2$ and $u\sigma^3$ with color c_2 , v and $v\sigma^2$ with color c'_1 and, finally, $v\sigma$ and $v\sigma^3$ with color c'_2 . Then the only automorphism that fixes the colors in both $O(u)$ and $O(v)$ is e , so this coloring 2-list-distinguishes G .

We may therefore suppose that $O(v)$, and every nontrivial orbit of G aside from $O(u)$ has exactly two elements, and hence that $St(v) = \langle \sigma^2, \tau \rangle$ (as, again, the case where $St(v) = \langle \sigma^2, \tau\sigma^j \rangle$ is similar) and $O(v) = \{v, v\sigma\}$. We will color u and $u\sigma$ with color c_1 , $u\sigma^2$ and $u\sigma^3$ with color c_2 and color v and $v\sigma$ with distinct colors. The only nonidentity automorphism of G that preserves this coloring of $O(u)$ is $\tau\sigma^3$. However, this interchanges the elements of $O(v)$, which received different colors, so this coloring 2-list-distinguishes G .

Next we consider the penultimate case that G realizes D_6 . By the above lemmas, we may assume that there is no vertex u in G has $St(u) = \langle \sigma^j \rangle$ or $|O(u)| \geq 6$. If there exists a $u \in V(G)$ such that $St(u) = \langle \tau\sigma^i \rangle$, then $|O(u)| = 6$, therefore we assume that every vertex u in G has $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$. Given this stabilizer, it is not hard to show that for every vertex u in G , $|O(u)| = 1, 2$ or 3 . This implies that G can easily be 3-list-distinguished, as it is possible to all of the vertices in a given orbit with distinct colors. Furthermore, if there is no orbit of order 3, then it is not difficult to 2-list-distinguish G .

Therefore, let us first assume there is exactly one orbit of order 3. Specifically, let u be

a vertex such that $|O(u)| = 3$ and observe that necessarily $O(u) = \{u, u\sigma, u\sigma^2\}$. Suppose first that for each $x \in O(u)$ there is some $\phi_x \in \text{Aut}(G) \cap \text{St}(x)$ such that the following hold:

1. ϕ_x interchanges the two vertices in $O(u) - x$, and
2. ϕ_x fixes all of $V(G) - O(u)$.

In this case, we cannot 2-distinguish G with any 2-coloring, as without loss of generality both $u\sigma$ and $u\sigma^2$ will receive the same color. However, then the automorphism ϕ_u described above is nontrivial and color-preserving, regardless of how the remainder of G is colored.

If there is some $x \in O(u)$ for which no such ϕ_x exists, then we claim that G is 2-list-distinguishable (and hence 2-distinguishable). In this case, we assign distinct colors to all pairs of vertices lying in orbits of order two and also color all vertices of $O(u)$ with distinct colors, if this is possible. If it is not possible to color $O(u)$ in this way, then each vertex in $O(u)$ must be assigned identical lists, say $\{c_1, c_2\}$. We then color x with c_1 and the vertices in $O(u) - x$ with c_2 . Due to the assumptions that $O(u)$ is the unique orbit of order three and that there is no ϕ_x as described above, any nontrivial automorphism $g \in \text{Aut}(G)$ that preserves this coloring of $O(u)$ must interchange the elements of an orbit of order two. However, all such orbits have been colored with distinct colors, so g is not color-preserving. It follows that $D_\ell(G) = 2$.

Suppose then that G has more than one orbit of order 3, and assign lists of order 2 to the vertices of G . Choose vertices u and v in distinct orbits such that $|O(u)| = |O(v)| = 3$. If it is not possible to color each vertex in $O(u)$ with a distinct color, then each vertex in $O(u)$ must have the same list, $\{c_1, c_2\}$. We then color u with c_1 and the other two vertices with color c_2 . If g , a nonidentity element of D_6 , fixes this coloring of $O(u)$, then $g \in \text{St}(u)$. However, since $|O(v)| = 3$, g must exchange two elements in $O(v)$. Assigning these vertices distinct colors from their respective lists serves to 2-list-distinguish G . Therefore, $D(G) = D_\ell(G) = 2$.

Suppose lastly that G realizes D_{10} and that there is no vertex u in G such that $\text{St}(u) = \langle \sigma^j \rangle$ or $|O(u)| \geq 6$. If there exists a $u \in V(G)$ such that $\text{St}(u) = \langle \tau\sigma^i \rangle$, then $|O(u)| = 10$, therefore we may assume that every vertex u in G has $\text{St}(u) = \langle \sigma^j, \tau\sigma^i \rangle$. Given this stabilizer, it is not hard to show that for every vertex u in G , $|O(u)| = 1, 2$ or 5 .

Furthermore, as every orbit of order 5 in $V(G)$ has the form $O(u) = \{u, u\sigma, \dots, u\sigma^4\}$ and stabilizer $\langle \sigma^5, \tau\sigma^j \rangle$ for some j , it is not difficult to see that the action of D_{10} on any such orbit of order five can be viewed precisely as the action of D_5 on the vertices of C_5 . We observe that this implies that G is 3-list-distinguishable, as then we may color each orbit of order five such that three vertices receive distinct colors and also color the orbits of order two so that both vertices receive distinct colors. Keeping in mind the action of $\text{Aut}(G)$ on $O(u)$, we can see that this is a distinguishing coloring.

Again we can easily 2-list-distinguish G if there are no orbits of order 5, so we first assume there is exactly one orbit of order 5. Specifically, let u be a vertex such that $|O(u)| = 5$ and recall that necessarily $O(u) = \{u, u\sigma, \dots, u\sigma^4\}$. Suppose as well for every $x \in O(u)$ there is some $\phi_x \in St(x)$ such that the following hold:

1. ϕ_x interchanges the pairs $(x\sigma, x\sigma^4)$ and $(x\sigma^2, x\sigma^3)$, and
2. ϕ_x fixes all of $V(G) - O(u)$.

As in the previous case, we claim that the existence of these ϕ_x implies that G is not 2-distinguishable. In any 2-coloring of $O(u)$, there is some vertex x such that the pairs $(x\sigma, x\sigma^4)$ and $(x\sigma^2, x\sigma^3)$ are monochromatic. However, then the automorphism ϕ_x described above is nontrivial and color-preserving, regardless of how the remainder of G is colored. It follows that G is not 2-list-distinguishable and, by our above observation, that G must therefore be 3-list-distinguishable.

Suppose therefore, without loss of generality, that there is no such $\phi_u \in Aut(G)$ and assign lists of length two to $V(G)$. We color each orbit of length two with distinct colors, and if possible assign distinct colors to three vertices in $O(u)$. If it is not possible to color three vertices with distinct colors, then each vertex in $O(u)$ must have the same list, $\{c_1, c_2\}$. We will then color u with color c_1 and color $u\sigma, u\sigma^2, u\sigma^3$ and $u\sigma^4$ with color c_2 . Given the action of D_{10} on $O(u)$, the only nontrivial automorphisms that would fix this coloring of $O(u)$ necessarily fixes u and exchanges the pairs $(u\sigma, u\sigma^4)$ and $(u\sigma^2, u\sigma^3)$. By the assumption that there is no ϕ_u satisfying conditions (1) and (2), such an automorphism must exchange the vertices in an orbit of order two and is therefore not color-preserving. It follows that G is 2-list distinguishable.

Now assume there is more than one orbit of order 5 and assign lists of length 2 to the vertices of G . Choose vertices u and v in distinct orbits such that $|O(u)| = |O(v)| = 5$. If it is possible to color three vertices in $O(u)$ with distinct colors, then such a coloring can be extended to a 2-list-coloring of G . If it is not possible to color three vertices with distinct colors, then again each vertex in $O(u)$ must have the same list, $\{c_1, c_2\}$. We then color two vertices, v and w , with c_1 and the rest with color c_2 . If g , a nonidentity element of D_{10} , fixes this coloring of $O(u)$, then either $g \in St(v) \cap St(w)$ or g interchanges v and w . However, since $|O(v)| = 5$, g must exchange two elements in $O(v)$. Assigning these vertices distinct colors from their respective lists serves to 2-list-distinguish G . Therefore, $D(G) = D_\ell(G) = 2$. \square

As a consequence of the proof of Theorem 2, we obtain the following.

Theorem 3. *Let G be a graph realizing D_n such that $V(G)$ has no orbit of order greater than five, and also such that G does not satisfy the hypotheses of Lemma 2 or Lemma 4. Then $D(G) = 3$ if and only if G has exactly one nontrivial orbit O satisfying one of the following:*

1. $n \in \{3, 4, 5\}$ and $|O| = n$, or

2. $n = 6$, $|O| = 3$, and for each $x \in O$ there is some $\phi_x \in St(x)$ such that the following hold:

(a) ϕ_x interchanges the two vertices in $O(u) - x$, and

(b) ϕ_x fixes all of $V(G) - O(u)$, or

3. $n = 10$, $|O| = 5$, and for each $x \in O$ there is some $\phi_x \in St(x)$ such that the following hold:

(a) ϕ_x interchanges the pairs $(x\sigma, x\sigma^4)$ and $(x\sigma^2, x\sigma^3)$, and

(b) ϕ_x fixes all of $V(G) - O(u)$.

3 Cartesian Products of Graphs

We use the standard definition for Cartesian products of graphs G and H , (for example see [Wes01]) which we denote by $G \square H$. A fiber $G_i^{(g_1, \dots, g_k)}$ of $G_1 \square \dots \square G_k$ is the subgraph induced by the vertex set $\{(g_1, \dots, x, \dots, g_k) \mid x \in G_i\}$.

The automorphism group of Cartesian products is well understood. If $G = (G_1 \square G_2)$ and $\phi \in Aut(G_1)$, then $\phi' : V(G) \rightarrow V(G)$ defined by $\phi' : (u_i, v_j) \mapsto (\phi u_i, v_j)$ is an automorphism of G . If $G_1 \cong G_2$, then $\alpha : (u_i, v_j) \mapsto (v_j, u_i)$ is also an automorphism of G . In fact, the automorphisms of a factor and the exchange of isomorphic factors generate $Aut(G)$ [Imr67].

We now turn our attention to the Cartesian product of two cycles, also known as the toroidal grid. We label a vertex of $C_n \square C_m$ as (u_i, v_j) if it is in fibers C_n^i and C_m^j , but for simplicity, we denote $L((u, v))$ as $L(u, v)$. We also define $S_c(G) = \{v \in V(G) \mid c \in L(v)\}$ and will write $S_c(G) = S_c$ if the context is clear. If $n \neq m$, the automorphism group of $C_n \square C_m$ is generated by the Cartesian product of the generators of C_n and C_m . This leads us to the following elementary lemma, given without proof, about the action of an automorphism of a cycle factor C_n , which will be used in subsequent proofs.

Lemma 7. *Let $G = (C_n \square C_m)$ such that $n \neq m$. Let $\phi \in Aut(C_n)$, $\psi \in Aut(C_m)$ and $\phi' \in Aut(C_n \square C_m)$ such that $\phi' : (u_i, v_j) \mapsto (\phi u_i, \psi v_j)$. If $\phi' : (u_i, v_j) \mapsto (u_k, \psi v_j)$, then $\phi' : (u_i, v_p) \mapsto (u_k, \psi v_p)$ for all $p \in \{1, \dots, m\}$.*

The distinguishing number of Cartesian products has been extensively studied. In [BC04] Bogstad and Cowen determined that $D(Q_d) = 2$ if $d \geq 4$ for the d -dimensional hypercube, Q_d . Using the motion lemma, a probabilistic result of Russell and Sundaram [RS98], Albertson extended Bogstad and Cowen's result by showing that for an arbitrary graph G , $D(G^r) = 2$ if $r \geq 4$ [Alb05]. This result was improved by Klavžar and Zhu in [KZ07] and was further strengthened by Imrich and Klavžar who demonstrated that $D(G^r) = 2$ when $r \geq 2$ and $G \neq K_2, K_3$ [IK06]. Other interesting results about distinguishing Cartesian products of graphs appear in [Bou09] and [CHK10].

To prove that $D(C_n \square C_m) = D_\ell(C_n \square C_m)$ for all n and m , we use the following lemma, which says that if we can 2-list-distinguish any fiber in a specific way then the entire graph can be 2-list-distinguished.

Lemma 8. *Let $G = C_n \square C_m$, and assign a list $L(u_i, v_j)$ of size two to each $(u_i, v_j) \in V(C_n \square C_m)$. If there exists a fiber that can be L -list-distinguished with a vertex that has a unique color in that fiber, then G can be L -list-distinguished.*

Proof. Let $G = C_n \square C_m$, and assign a list $L(u_i, v_j)$ such that $|L(u_i, v_j)| = 2$ to each $(u_i, v_j) \in V(C_n \square C_m)$. Assume we can color C_n^1 so that color c_1 is used on the vertex (u_1, v_1) but on no other vertex of C_n^1 .

Case 1: Suppose $V(C_n^2) \subseteq S_{c_1}$.

Color each vertex $(u_i, v_2) \in C_n^2$ with c_1 . Assign list $L'(u_x, v_y) = L(u_x, v_y) - c_1$ to all uncolored vertices, and L' -list-color these vertices arbitrarily.

Case 2: There exist vertices $(u_i, v_2), i \neq 1$ and $(u_1, v_j), j \neq 1$ such that $(u_i, v_2), (u_1, v_j) \notin S_{c_1}$.

For all uncolored vertices, assign list $L'(u_x, v_y) = L(u_x, v_y) - c_1$. Color (u_i, v_m) with some $c_2 \in L'(u_i, v_m)$, color (u_i, v_2) with some $c_3 \in L'(u_i, v_2)$ such that $c_3 \neq c_2$, and L' -list-color the rest of the vertices. Since they are the only two fibers with a vertex colored c_1 , C_n^1 must map to C_m^1 in any non-trivial color-preserving automorphism; furthermore, C_n^1 has been list-distinguished, so there is at most one automorphism, α , that could map C_n^1 to C_m^1 . Assume that $\alpha(u_i, v_1) \mapsto (u_1, v_j)$, and let the vertex (u_i, v_1) be colored c_k . The vertex $(u_1, v_j) \notin S_{c_1}$, so recolor (u_1, v_j) with $c_a \in L'(u_1, v_j)$ such that $c_a \neq c_k$.

Case 3: There exists a vertex $(u_i, v_2), i \neq 1$, such that $(u_i, v_2) \notin S_{c_1}$, and $(u_1, v_j) \in S_{c_1}$ for all $j \neq 1$.

Color (u_1, v_2) with c_1 , and assign list $L'(u_x, v_y) = L(u_x, v_y) - c_1$ to all uncolored vertices. Let vertex (u_i, v_1) be colored c_p , and color (u_i, v_2) with $c_b \in L'(u_i, v_2)$ such that $c_b \neq c_p$. Now L' -list-color the rest of the vertices.

We claim that in each of the three cases above the coloring is list-distinguishing. First consider an automorphism of the form $(\sigma_n^a \tau_n^b, e)$, $\sigma_n^a \tau_n^b \neq e$, which maps (u_t, v_1) to (u_r, v_1) for some $t \neq r$ by Lemma 7. The graph induced by the vertices of C_n^1 has been assigned a distinguishing coloring. Thus an automorphism of this form does not preserve colors.

Also by Lemma 7, the automorphism $(e, \sigma_m^c \tau_m^d)$, $\sigma_m^c \tau_m^d \neq e$, maps either (u_i, v_2) to $(u_i, v_j), 2 \neq j$ for all $i \in \{1, \dots, n\}$ or (u_1, v_1) to $(u_1, v_j), 1 \neq j$. However in case 1, $n - 1$ vertices of C_n^2 are colored c_1 , and this is the only fiber with such a coloring. The vertex (u_1, v_1) is colored c_1 while the rest of the fiber C_n^1 is not; again this is the only fiber with such a coloring. In Case 2, (u_1, v_1) is the only vertex colored c_1 , and we know that (u_i, v_2) must map to (u_i, v_m) , which are colored differently. In Case 3, (u_1, v_1) and (u_1, v_2) are the only two vertices colored c_1 , so any non-trivial automorphism must interchange (u_1, v_1) and (u_1, v_2) , which will also interchange (u_i, v_1) and (u_i, v_2) , which have different colors. Therefore in each of these cases, automorphisms of this type are not color-preserving.

The automorphism $(\sigma_n^a \tau_n^b, \sigma_m^c \tau_m^d)$ is equal to $(e, \sigma_m^c \tau_m^d) \circ (\sigma_n^a \tau_n^b, e)$. In Case 1, the automorphism $(\sigma_n^a \tau_n^b, e)$ will map (u_1, v_1) to (u_j, v_1) for some $j \in \{1 \dots n\}$ and (u_i, v_2) to (u_k, v_2) for all $i \in \{1, \dots, n\}$. But $n - 1$ vertices of C_n^2 are colored with c_1 and the vertex (u_1, v_1) is in fiber C_n^1 , so we are in the previous case when performing $(\sigma_n^a \tau_n^b, e)$. Cases 2 and 3 will similarly result in the previous case when performing $(\sigma_n^a \tau_n^b, e)$. Therefore in each of these cases, automorphisms of this type are not color-preserving.

If $n = m$ consider the automorphism α , such that $\alpha(u_i, v_1) \mapsto (u_1, v_j)$ for some i and j , which would map C_m^1 to C_n^1 . In Case 1, C_n^2 has $n - 1$ vertices colored c_1 and there is no C_m fiber with this number of vertices colored c_1 . In Case 2, we have broken α by coloring (u_i, v_1) and (u_1, v_j) differently. In Case 3, C_m^1 has two vertices colored c_1 and no C_n fiber has more than one vertex colored c_1 . Therefore in each of these cases, automorphisms of this type are not color-preserving.

In every case, colors can be chosen from the list of size 2 to list-distinguish $C_n \square C_m$. If there exists a fiber that can be L -list-distinguished, $|L(v)| = 2$ for all $v \in V(G)$, with a vertex that has a unique color for that fiber, then G can be L -list-distinguished. \square

In Figure 4, $\sim c$ refers to any color that is not c .

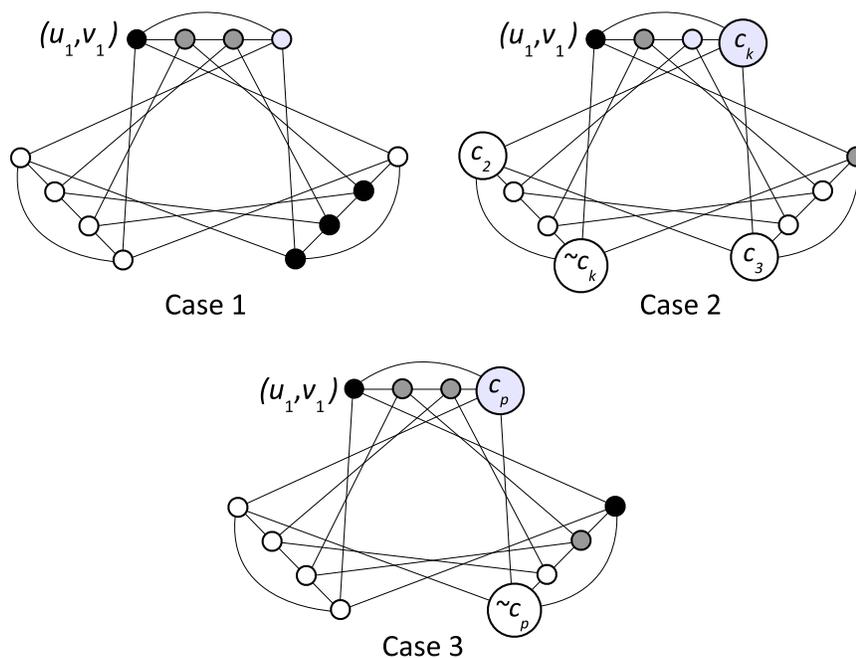


Figure 4: Examples of colorings described in Lemma 8. Vertices represented by open circles cannot be colored black.

Lemma 9. *Let $n \geq 3$ and $m \geq 3$ such that at most one of n and m is 3. Then $D_\ell(C_n \square C_m) = 2$.*

Proof. Assign a list $L(u_i, v_j)$ such that $|L(u_i, v_j)| = 2$ to each $(u_i, v_j) \in V(C_n \square C_m)$. If $|\cup L(u_i, v_j)| = 2$, then color the vertices as in [BC04] if $n = m = 4$ or as in [IK06] otherwise. If not, then there must be two vertices whose lists are different. Since $C_n \square C_m$ is connected, there exist two adjacent vertices with different lists, and these two vertices are either on the same C_n or C_m fiber. Without loss of generality, assume they are the vertices (u_1, v_1) and (u_2, v_1) on C_n^1 . Color the vertex (u_n, v_1) with $c_3 \in L(u_n, v_1)$. Since their lists are not the same, color vertices (u_1, v_1) and (u_2, v_1) from their list such that the colors are different and both not c_3 , say c_1 and c_2 respectively. Color the rest of the vertices of C_n^1 anything from their list other than c_1 .

Since (u_1, v_1) is the only vertex with color c_1 , any color-preserving automorphism on the fiber C_n^1 must map (u_1, v_1) to itself. Therefore, either $(u_n, v_1) \mapsto (u_2, v_1)$ or $(u_n, v_1) \mapsto (u_n, v_1)$. In the former case, the automorphism does not preserve the colors. This leaves us with only the latter, which is the trivial automorphism, so this is an L -list-distinguishing coloring on C_n^1 that uses a unique color c_1 . By Lemma 8, we know that we can extend this coloring to a L -list-distinguishing coloring of $C_n \square C_m$. Therefore, $D_\ell(C_n \square C_m) = 2$ if at most one of n and m is 3. \square

Lemma 10. $D_\ell(C_3 \square C_3) = D(C_3 \square C_3) = 3$.

Proof. Assign list $L(u_i, v_j)$ such that $|L(u_i, v_j)| = 3$ to each $u_i, v_j \in V(C_3 \square C_3)$. Color the vertices (u_i, v_1) , $i \in \{1, 2, 3\}$ from their list such that each vertex received a unique color. Let vertex (u_i, v_1) for $i \in \{1, 2, 3\}$ be colored c_i , and we will not use the color c_1 on any other vertex. If one of the vertices (u_1, v_2) and (u_1, v_3) has a color that is not c_1, c_2 , or c_3 in its list, say c_4 , then use that color. Then color the other anything from its list that is not c_1 . If both of their lists are $\{c_1, c_2, c_3\}$, then color them both c_2 . Lastly, color (u_3, v_2) and (u_3, v_3) from their list such that the colors are different and both not c_1 , and then color the remaining vertices anything from their list that is not c_1 .

By Lemma 7 and since C_3^1 has a list-distinguishing coloring, there is no color-preserving automorphism of the form $(\sigma_n^a \tau_n^b, e)$. Since (u_1, v_1) is the only vertex with color c_1 , the only non-trivial automorphism of the form $(e, \sigma_m^c \tau_m^d)$ or $(\sigma_n^a \tau_n^b, \sigma_m^c \tau_m^d)$ would map (u_3, v_2) to (u_3, v_3) . However, these vertices are colored differently. Lastly, the automorphism $\alpha : (u_i, v_j) \mapsto (v_j, u_i)$ must map (u_1, v_1) to itself. However, (u_2, v_1) and (u_3, v_1) are colored differently than (u_1, v_2) and (u_1, v_3) . Thus the only color preserving automorphism is the identity, and $D_\ell(C_3 \square C_3) = D(C_3 \square C_3) = 3$. \square

The proof of the next theorem follows immediately from Lemma 9 and Lemma 10.

Theorem 4. $D_\ell(C_n \square C_m) = D(C_n \square C_m)$ for all $n, m \geq 3$.

Klavžar and Zhu [KZ07] and Fischer and Isaak [FI08] independently determined the distinguishing number of Cartesian products of complete graphs. Again, list-distinguishing this class of graphs has proved more difficult. For instance, a tedious (yet straightforward) case analysis yields $D_\ell(K_3 \square K_4) = 2$. As we feel that the development of more broadly

applicable techniques may increase the overall understanding of the list-distinguishing number, we pose the following problem.

Problem 1. Determine $D_\ell(K_n \square K_m)$.

4 Conclusion

When presenting a list variant p_ℓ of a graph parameter p , it is natural to ask whether p and p_ℓ differ significantly. For instance, it is well known that the list chromatic number and the chromatic number can differ by an arbitrarily large amount [Alo93, ERT80]. On the other hand, the list coloring conjecture states that the edge chromatic number and edge-list-chromatic number are always equal and remains one of the foremost open problems in chromatic graph theory. With this in mind, we pose the following question.

Question 1. Does there exist a graph G such that $D(G) < D_\ell(G)$?

Amusingly, the first author feels that no such graph G exists, while the third author believes that Question 1 can be answered in the affirmative. As such, we make no formal conjecture related to Question 1 here.

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