Toughness of $K_{a,t}$-minor-free graphs

Guantao Chen* Yoshimi Egawa† Ken-ichi Kawarabayashi‡
Bojan Mohar§ Katsuhiro Ota¶

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Abstract

The toughness of a non-complete graph G is the minimum value of $\frac{|S|}{\omega(G-S)}$ among all separating vertex sets $S \subset V(G)$, where $\omega(G-S) \geqslant 2$ is the number of components of $G-S$. It is well-known that every 3-connected planar graph has toughness greater than 1/2. Related to this property, every 3-connected planar graph has many good substructures, such as a spanning tree with maximum degree three, a 2-walk, etc. Realizing that 3-connected planar graphs are essentially the same as 3-connected $K_{3,3}$-minor-free graphs, we consider a generalization to a-connected $K_{a,t}$-minor-free graphs, where $3 \leqslant a \leqslant t$. We prove that there exists a positive constant $h(a,t)$ such that every a-connected $K_{a,t}$-minor-free graph G has toughness at least $h(a,t)$. For the case where $a=3$ and t is odd, we obtain the best possible value for $h(3,t)$. As a corollary it is proved that every such graph of order n contains a cycle of length $\Omega(\log_{h(a,t)} n)$.

1 Introduction

In this paper, all graphs are finite and simple. A graph H is a *minor* of a graph K if H can be obtained from a subgraph of K by contracting edges.

*Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303. Email: gchen@gsu.edu

†Department of Mathematics, Tokyo University of Science, Shinjuku-ku, Tokyo, 162-8601, Japan.

‡National Institute of Informatics, 2-1-2, Hitotsubashi, Chiyoda-ku, Tokyo, Japan. Research partly supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research, by C & C Foundation, by Kayamori Foundation and by Inoue Research Award for Young Scientists. Email: k_keniti@nii.ac.jp

§Department of Mathematics, Simon Fraser University, Burnaby, B.C. V5A 1S6. Research of this author was supported in part by the ARRS, Research Program P1-0297, by an NSERC Discovery Grant, and by the Canada Research Chair Program. On leave from IMFM & FMF, Department of Mathematics, University of Ljubljana, 1000 Ljubljana, Slovenia. Email: mohar@sfu.ca

¶Department of Mathematics, Keio University, Kohoku-ku, Yokohama, 223-8522, Japan. Research partly supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (B). Email: ohta@math.keio.ac.jp

In the core of the seminal Graph Minor Theory of Robertson and Seymour lies a powerful theorem capturing the "rough" structure of graphs excluding a fixed minor. This result was used to prove Wagner's Conjecture that finite graphs are well-quasi-ordered under the graph minor relation. From the theoretical point of view there are two classes of graphs for which one would like to understand the excluded minor structure in more detail. These are the complete graphs K_t ($t \geqslant 1$) and complete bipartite graphs $K_{a,t}$, where a is constant and $t \geqslant 1$.

The family of graphs containing no $K_{a,t}$-minors has attracted a lot of attention, even when a is small, say $a = 3$. Graphs containing no $K_{3,t}$-minor form an important class of graphs in the theory of graph minors (related to surface embeddings). In [5], it is shown that if G is a 3-connected graph with no $K_{3,t}$-minor, then it has a large wheel. More generally, it is shown in [2] that every sufficiently large $16a$-connected graph contains a $K_{a,t}$-minor. This means that large connectivity guarantees the existence of a $K_{a,t}$-minor with only finitely many exceptions for each a and t.

In this paper we set a different goal. We study the "toughness" of graphs without a $K_{a,t}$-minor.

We say that a vertex set $S \subset V(G)$ is *separating* if $G - S$ has at least two components. We define the *toughness* of G as the minimum value of $\frac{|S|}{\omega(G-S)}$ among all separating sets $S \subset V(G)$, where $\omega(G - S) \geqslant 2$ is the number of component of $G - S$. If G is a complete graph, then it has no separating vertex sets, and we define its toughness to be ∞. We say that G is *t-tough* if its toughness is at least t.

Our main aim is to find out how small the toughness can be in the family of all a-connected graphs without a $K_{a,t}$-minor. The graph $K_{a,t-1}$ is a-connected if $t > a$, has toughness $\frac{a}{t-1}$, and has no $K_{a,t}$-minor. As for the general case, it turns out that there is a positive lower bound on the toughness in terms of a and t.

For $t \geqslant a \geqslant 3$, we define $g(a,t) = \frac{1}{2}(a-1)!(t-1)$. We prove the following result.

Theorem 1 *Let $t \geqslant a \geqslant 3$ be integers. If G is an a-connected $K_{a,t}$-minor-free graph, then for every separating vertex set S of the vertices, the number of components of $G - S$ is at most $g(a,t)(|S| - a + 1)$. Consequently, the toughness of G is at least $1/g(a,t)$.*

Combining Theorem 1 with Win's theorem in [7], which states that every $\frac{1}{k-2}$-tough graph contains a spanning k-tree (recall that a *k-tree* is a tree with maximum degree at most k), we obtain the following corollary.

Corollary 2 *If G is an a-connected $K_{a,t}$-minor-free graph, then G contains a spanning $(g(a,t) + 2)$-tree.*

In particular for $a = 3$, we obtain the following result.

Corollary 3 *If G is a 3-connected $K_{3,t}$-minor-free graph, then G contains a spanning $(t + 1)$-tree.*

Corollaries 2 and 3 imply the following results that are of independent interest.

Corollary 4 *If G is an a-connected $K_{a,t}$-minor-free graph of order n, then G contains a path of length at least $2\log_{g(a,t)+1} n$, and a cycle of length at least $\frac{4}{5}\log_{g(a,t)+1} n$.*

Furthermore, if $a = 3$, then G contains a path of length at least $2\log_t n$, and a cycle of length at least $\frac{4}{5}\log_t n$.

Proof. By Corollaries 2 and 3, there is a spanning tree of maximum degree at most $l+1$ in G, where $l = g(a,t)+1$, and $l = t$ for $a = 3$. This implies that G has a path of length $2\log_l n$. Bondy and Locke [3] proved that, if a 3-connected graph has a path of length k, then it has a cycle of length at least $2k/5$. This easily completes the proof. ■

The second conclusion in Corollary 4 is much weaker than the result by Chen et al. [4]. They proved that such a graph has a cycle of length at least $n^{f(t)}$ for some value $f(t) > 0$. But as far as we know, when $a \geqslant 4$, this is the first result that shows the existence of a long path and a long cycle for $a \geqslant 4$.

A 3-connected $K_{3,3}$-minor-free graph is nothing but a 3-connected planar graph, or K_5. Barnette [1] proved in 1966 that every 3-connected planar graph contains a spanning 3-tree. Thus, Corollary 3 is not best possible for $t = 3$. However, in Section 3, we show that Theorem 1 is best possible when t is odd.

Very recently, Ota and Ozeki [6] proved that if $t \geqslant 4$ is even, then every 3-connected $K_{3,t}$-minor-free graph contains a spanning $(t-1)$-tree. This implies that for each odd integer $t \geqslant 3$, every 3-connected $K_{3,t}$-minor-free graph contains a spanning t-tree.

2 Bipartite Minors in Bipartite Graphs

In this section, we prove our main theorem.

For $x \in V(G)$, we write $N(x)$ for the neighbourhood of x in G. Suppose that a graph G has an H-minor. Then G contains pairwise vertex-disjoint connected subgraphs A_v, $v \in V(H)$ such that if u and v are adjacent in H then G has an edge joining A_u and A_v. For $v \in V(H)$, the subgraph A_v (or its vertex set) is called a *branch set* of the H-minor in G.

The following theorem is an essential part of our proof of Theorem 1. Recall that $g(a,t) = \frac{1}{2}(a-1)!(t-1)$.

Theorem 5 *Let $t \geqslant a \geqslant 3$ be integers. Let G be a bipartite graph with partite sets X and Y. Suppose that each vertex $x \in X$ has degree at least a, and that*

$$|X| > g(a,t)(|Y| - a + 1).$$

Then G has a $K_{a,t}$-minor, in which each of the branch sets corresponding to the vertices in the partite set of order t of $K_{a,t}$ is a singleton of X.

In order to prove Theorem 5, we first settle the case $a = 3$. Namely, we first prove the following theorem.

Theorem 6 *Let G be a bipartite graph with partite sets X and Y satisfying $|X| > (t-1)(|Y|-2)$. If every vertex of X has degree at least three, then G has a $K_{3,t}$-minor, in which each of the branch sets corresponding to the vertices in the partite set of order t of $K_{3,t}$ is a singleton of X.*

Proof. The proof is by induction on $|Y|$. If $|Y| = 3$, then the condition implies that G contains a $K_{3,t}$ as a subgraph. Thus the result follows immediately.

Suppose now that $|Y| \geqslant 4$. We may assume that $d(x) = 3$ for every $x \in X$.

Let y_1 and y_2 be distinct vertices in Y that have a common neighbor in X. Let $A = N(y_1) \cap N(y_2) \neq \emptyset$. If $|A| \leqslant t-1$, then we contract $\{y_1, y_2\} \cup A$ into a single vertex y', and delete all edges between y' and $Y \setminus \{y_1, y_2\}$ to obtain a bipartite graph G' with partite sets $X' = X \setminus A$ and $Y' = (Y \setminus \{y_1, y_2\}) \cup \{y'\}$. It is easy to see that $d_{G'}(x) = 3$ for each $x \in X'$, and

$$|X'| \;\geqslant\; |X| - (t-1) \;>\; (t-1)(|Y|-3) \;=\; (t-1)(|Y'|-2).$$

By the induction hypothesis, G' has a $K_{3,t}$-minor with the desired condition, and so does G.

Thus we may assume that $|A| \geqslant t$. If $G - \{y_1, y_2\}$ is connected, then, since each vertex in A has degree one in $G - \{y_1, y_2\}$, $G - (\{y_1, y_2\} \cup A)$ is connected. Let us contract $G - (\{y_1, y_2\} \cup A)$ into a vertex. Then we obtain a $K_{3,t}$ as a minor of G with its t vertices of degree three corresponding to a subset of A of cardinality t.

If $G - \{y_1, y_2\}$ is disconnected, then, since $|X| > (t-1)|Y \setminus \{y_1, y_2\}|$, there is a component H of $G - \{y_1, y_2\}$ satisfying

$$|V(H) \cap X| \;>\; (t-1)|V(H) \cap Y|.$$

Let G'' be the subgraph of G induced by $V(H) \cup \{y_1, y_2\}$. The partite sets of G'' are $X'' = V(H) \cap X$ and $Y'' = \{y_1, y_2\} \cup (V(H) \cap Y)$, and

$$|X''| \;>\; (t-1)|V(H) \cap Y| \;=\; (t-1)(|Y''|-2).$$

By the induction hypothesis, G'' has a $K_{3,t}$-minor with the required properties, and so does G. ■

Proof of Theorem 5. The proof is by induction on a and $|Y|$. The case when $a = 3$ was settled in Theorem 6, so we may assume that $a \geqslant 4$. If $|Y| = a$, then the condition implies that G is the complete bipartite graph $K_{a,|X|}$, which contains $K_{a,t}$ as a subgraph.

Suppose now that $|Y| \geqslant a+1$. We may assume that $d(x) = a$ for every $x \in X$. Let y_1 and y_2 be distinct vertices in Y that have a common neighbor in X. Let $A = N(y_1) \cap N(y_2)$. If $|A| \leqslant g(a,t)$, then we contract $\{y_1, y_2\} \cup A$ into a single vertex y', and delete all edges between y' and $Y \setminus \{y_1, y_2\}$ to obtain a bipartite graph G' with partite sets $X' = X \setminus A$ and $Y' = (Y \setminus \{y_1, y_2\}) \cup \{y'\}$. It is easy to see that $d_{G'}(x) = a$ for each $x \in X'$, and

$$|X'| \;\geqslant\; |X| - g(a,t) \;>\; g(a,t)(|Y|-a) \;=\; g(a,t)(|Y'|-a+1).$$

By the induction hypothesis, G' has a $K_{a,t}$-minor, and so does G.

Thus we may assume that

(*) $|N(y) \cap N(y')| > g(a,t)$ for every pair of vertices $y, y' \in Y$ with $N(y) \cap N(y') \neq \emptyset$.

We take a vertex $y_0 \in Y$. Let $X_0 = N(y_0)$ and let $Y_0 = N(X_0) \setminus \{y_0\}$. Let G_0 be the subgraph induced by $X_0 \cup Y_0$. Then, $d_{G_0}(x) = a - 1$ for every $x \in X_0$. By (*), for each vertex $y \in Y_0$, $d_{G_0}(y) > g(a,t)$. This implies that

$$|X_0| \;=\; \frac{|E(G_0)|}{a-1} \;>\; \frac{g(a,t)|Y_0|}{a-1} \;=\; g(a-1,t)|Y_0|.$$

By the induction hypothesis, G_0 has a $K_{a-1,t}$-minor, in which each of the branch sets corresponding to the vertices in the partite set of order t in $K_{a-1,t}$ is a singleton of X_0. Since y_0 is adjacent to all vertices of X_0, we find a $K_{a,t}$-minor in G with the desired condition. ■

Proof of Theorem 1. Let $S \subset V(G)$, and let $C_1, \ldots, C_m$ be the components of $G - S$. Contract each C_i into a single vertex, and delete all edges in S. Then we obtain a bipartite graph G' with partite sets S and X, where the vertices of X correspond to the components $C_1, \ldots, C_m$. Since G is a-connected, $d_{G'}(x) \geqslant a$ for each $x \in X$. Moreover, since G is $K_{a,t}$-minor-free, its minor G' is also $K_{a,t}$-minor-free. By Theorem 5, we have $m = |X| \leqslant g(a,t)(|S| - a + 1)$, which proves the assertion of the theorem. ■

3 Sharpness

In this section, we discuss the sharpness of Theorems 1 and 5 for $a = 3$, while we do not think that Theorem 1 is sharp when $a \geqslant 4$. The following proposition shows that Theorem 1 is sharp when $a = 3$ and t is odd. Note that $g(3,t) = t - 1$.

Proposition 7 *For each odd integer $t \geqslant 3$, there exist infinitely many 3-connected $K_{3,t}$-minor-free graphs G containing a subset $S \subset V(G)$ such that the number of components of $G - S$ is $(t-1)(|S| - 2)$.*

Proof. Let $t = 2r + 1$. Let H be a planar triangulation, and let $n = |V(H)|$. For each face f of H, we add a set X_f of r new vertices, each of which is adjacent to the three vertices on the boundary of f. Let G be the resulting graph on $V(H) \cup \bigcup_{f \in F(H)} X_f$.

If we set $S = V(H) \subset V(G)$, then the number of components of $G - S$ is

$$r|F(H)| \;=\; r(2n-4) \;=\; 2r(n-2) \;=\; (t-1)(|S|-2).$$

On the other hand, we can show that G is $K_{3,2r+1}$-minor-free. If $r = 1$, then the result is obvious because G is planar. So we assume $r \geqslant 2$. Suppose that G contains a $K_{3,2r+1}$-minor. Let A_1, A_2, A_3 and $B_1, B_2, \ldots, B_{2r+1}$ be the branch sets of the $K_{3,2r+1}$-minor in G such that each A_i is adjacent to every B_j. We may assume that A_i and B_j are chosen to be minimal.

Claim. If a vertex x in $\bigcup_{f \in F(H)} X_f$ is in some branch set, then $\{x\} = B_j$ for some j.

Suppose $x \in X_f$ and x is contained in a branch set B with $|B| \geqslant 2$. Since $N_G(x)$ is a triangle, it is easy to see that $B - x$ is connected, and that $N_G(B - x) \setminus B = N_G(B) \setminus B$. Thus, we can replace B with $B - x$ as a branch set of a $K_{3,2r+1}$-minor, which contradicts the minimality of the branch sets. Thus $B = \{x\}$. Since $d(x) = 3$, the branch set B corresponds to a vertex of degree three in $K_{3,2r+1}$. This proves the claim.

Now, let G' be the graph obtained from G by identifying X_f into a single vertex v_f for each $f \in F(H)$. Consider the identification image of the $K_{3,2r+1}$-minor in G. Since $|X_f| = r$, $B_1, B_2, \ldots, B_{2r+1}$ are identified into at least $\lceil (2r+1)/r \rceil = 3$ sets, and we obtain a $K_{3,3}$-minor in G', which contradicts that G' is a planar graph. This proves that G does not contain a $K_{3,2r+1}$-minor. ■

On the other hand, Theorem 6 is best possible for every integer $t \geqslant 3$.

Proposition 8 *There exists a $K_{3,t}$-minor-free bipartite graph G having partite sets X and Y with $|X| = (t-1)(|Y| - 2)$ such that each vertex in X has degree three.*

Proof. Let $X = \{x_{ij} : 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant t-1\}$ and $Y = \{z_1, z_2, y_1, \ldots, y_m\}$. Let G be a bipartite graph with partite sets X and Y such that $N(x_{ij}) = \{z_1, z_2, y_i\}$. Then, $|X| = (t-1)m = (t-1)(|Y|-2)$, and it is not difficult to see that G is $K_{3,t}$-minor-free. ■

The graph in Proposition 8 is not 3-connected. We do not know whether Theorem 1 is sharp or not when $a = 3$ and t is even.

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